Problem Solving in Math (Math 43900) Fall 2013

Week two (September 3) solutions

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1. Let $f(n)$ be the number of regions which are formed by n lines in the plane, where no two lines are parallel and no three meet in a point (e.g. $f(1) = 2$, $f(2) = 4$ and $f(3) = 7$). Find a formula for $f(n)$, and prove that it is correct.

   Solution: Suppose you have n lines down already, so $f(n)$ regions. The $(n + 1)$st line, not being parallel to any other will, will meet all n, and all at different places (since no three lines meet in a point). Without loss of generality we can assume that the $(n + 1)$st line is the x-axis, and along the line we can mark, in order, the n meeting points with other lines, $-\infty < x_1 < x_2 < \ldots < x_n < \infty$. The segment on the $(n + 1)$st line from $-\infty$ to x_1 form the boundary of two regions (above and below it) that were previously one region; so this segment adds one region to the total. Similarly all the other segments add one region. There are $n + 1$ segments in all, so we get the relation

   $$f(n + 1) = f(n) + n + 1 \quad ((\text{for } n \geq 1)), \quad f(1) = 2.$$

   Computing the first few values, it seems clear that $f(n)$ grows quadratically, and that in fact $f(n) = (n^2 + n + 2)/2$. We prove this by induction on n, with $P(n)$ the statement "$f(n) = (n^2 + n + 2)/2$". $P(1)$ asserts "$f(1) = (1^2 + 1 + 2)/2 = 2$, which is true. Suppose $P(n)$ is true for some $n \geq 1$. Let's look at $P(n + 1)$, which is the assertion "$f(n + 1) = ((n + 1)^2 + (n + 1) + 2)/2 = (n^2 + 3n + 4)/2$". Since $P(n)$ is assumed true, we know

   $$f(n) = \frac{n^2 + n + 2}{2}.$$

   We also know $f(n + 1) = f(n) + n + 1$, so

   $$f(n + 1) = \frac{n^2 + n + 2}{2} + n + 1 = \frac{n^2 + 3n + 4}{2},$$

   and so indeed $P(n+1)$ is true. That $P(n)$ is true for all $n \geq 1$, i.e., that $f(n) = (n^2+n+2)/2$ for all $n \geq 1$, has now been proved by induction.

   Source: A classic.

2. Prove the following inequalities:

   (a) $2(\sqrt{n+1} - 1) \leq 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}$.

   Solution: Let $S(n) = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}}$. First we'll sketch a proof by induction that $S(n) < 2\sqrt{n}$ for all $n \geq 1$. The inequality for $n = 1$ is certainly true. For $n \geq 1$,

$S(n+1) = S(n) + \frac{1}{\sqrt{n+1}}$. To show that this is less than $2\sqrt{n+1}$ is equivalent to showing $S(n) < 2\sqrt{n+1} - \frac{1}{\sqrt{n+1}}$. Since we already know (by induction) that $S(n) < 2\sqrt{n}$ it is now sufficient to show that

$$2\sqrt{n} \leq 2\sqrt{n+1} - \frac{1}{\sqrt{n+1}}.$$

After a little algebra (square both sides, isolate the last remaining square root on one side, square both sides again, simplify), this is seen to be true for all n.

Proving $2(\sqrt{n+1} - 1) \leq S(n)$ goes the same way; its true for $n = 1$. For $n > 1$, induction allows us to reduce the problem to verifying

$$2(\sqrt{n+2} - 1) - \frac{1}{\sqrt{n+1}} \leq 2(\sqrt{n+1} - 1),$$

which just requires a little algebra along the same lines as before.

Notice that for large n, the two sides ($2(\sqrt{n+1} - 1)$ and $2(\sqrt{n})$) are very close!

(b) $\prod_{k=1}^{n} (2k)! \geq ((n+1)!)^n$.

Solution: We prove this for $n \geq 1$ by induction. For $n = 1$, the claimed inequality is $2! \geq 2!^1$, which is true. For $n > 1$ we do

$$\begin{aligned} \prod_{k=1}^{n} (2k)! &= (2n)! \prod_{k=1}^{n-1} (2k)! \\ &\geq (2n)!(n!)^{n-1}, \end{aligned}$$

the inequality by induction. To finish we need to show

$$(2n)!(n!)^{n-1} \geq ((n+1)!)^n.$$

The $(n!)^{n-1}$ term divides out, leaving

$$(2n)! > (n+1)^{n-1}(n+1)!.$$

This is true: $(2n)!$ is the product of $2n$ numbers, specifically $1, 2, \ldots, n+1, n+2, n+3, \ldots, 2n$, while $(n+1)^{n-1}(n+1)!$ is also the product of $2n$ numbers, specifically $1, 2, \ldots, n+1, n+1, n+1, \ldots, n+1$. Listed in the order given, each factor of $(2n)!$ is at least as large as the corresponding factor of $(n+1)^{n-1}(n+1)!$.

Source: A pair of classics; I found both on NYU's Putnam prep site.

3. Define a sequence $(a_n)_{n \geq 1}$ by

$$a_1 = 1, \quad a_{2n} = a_n, \quad \text{and} \quad a_{2n+1} = a_n + 1.$$

Prove that a_n counts the number of 1's in the binary representation of n.

Solution: Let $f(n)$ count the number of 1's in the binary representation of n. We first show that

$$f(1) = 1, \quad f(2n) = f(n), \quad \text{and} \quad f(2n+1) = f(n) + 1.$$

This is easy: Clearly $f(1) = 1$; if the binary representation of n is $a_1a_2\dots a_k$, then the binary representation of $2n$ is $a_1a_2\dots a_k0$, so $f(2n) = f(n)$; if the binary representation of n is $a_1a_2\dots a_k$, then the binary representation of $2n+1$ is $a_1a_2\dots a_k1$, so $f(2n+1) = f(n)+1$.

Since $f(n)$ satisfies the same initial conditions as a_n, and the same recurrence, it seems clear that $f(n) = a_n$. Formally, we prove the statement "$f(n) = a_n$" for all $n \geq 1$ by by strong induction. For $n = 1$ it's clear. For $n > 1$, if $n = 2m$ is even then we have

$$f(n) = f(m) = a_m = a_n,$$

the first equality by what we've proved about f, the second by (strong) induction, and the third by hypothesis on a. Similarly if $n = 2m+1$ is odd then we have

$$f(n) = f(m) + 1 = a_m + 1 = a_n,$$

and we are done.

Remark: The above induction prove works (suitably modified) to establish rigourously the evident but important fact that if two sequences are defined recursively, with the same initial conditions and same recurrence relations, then they are in fact the same sequence.

Source: I found this on Stanford's Putnam prep site, where it is sourced to the book "The Art and Craft of Problem Solving" by P. Zeitz.

4. Prove that for all $n \geq 2$, it is possible to write $n! - 1$ as the sum of $n-1$ numbers, each of which is a divisor of $n!$.

   Solution: This is a problem that seems impossible at first; where could one find such precisely regulated numbers? But once one starts an induction proof, it just falls out. Start with $n = 2$: $2! - 1 = 1$, which can be written as the sum of $2 - 1 = 1$ number, which is a divisor of $2! = 2$.

   For $n > 2$ we have

$$\begin{aligned} n! - 1 &= n[(n-1)! - 1] + (n-1) \\ &= n[k_1 + k_2 + \dots + k_{n-2}] + (n-1), \end{aligned}$$

   where each k_i is a divisor of $(n-1)!$. (Here we are using the inductive hypothesis: re-expressing $n! - 1$ in terms of $(n-1)! - 1$, and using the fact that we are assuming the existence of a good decomposition of $(n-1)! - 1$). Now we have

$$n! - 1 = nk_1 + nk_2 + \dots + nk_{n-2} + (n-1),$$

   so we have written $n! - 1$ as the sum of $n-1$ numbers, each of which divides $n!$ (nk_i divides $n!$ because k_i divides $(n-1)!$, and $n-1$ clearly divides $n!$).

   Done by induction!

   Source: I found this on Northwestern's Putnam prep site.

5. Find (with proof!) all sequences $(a_n)_{n\geq 0}$ of positive real numbers for which

$$\frac{a_1 + 2a_2 + 3a_3 + \dots + ka_k}{a_1 + a_2 + \dots + a_k} = \frac{k+1}{2}$$

for all $k \geq 1$.

Solution: After a little experimentation, you might start believing that the *only* sequences that work are constant sequences (with all terms equal). We'll verify this belief **strong** induction. Specifically, let a fixed sequence $(a_n)_{n\geq 0}$ of positive real numbers by given, for which

$$\frac{a_1 + 2a_2 + 3a_3 + \ldots + ka_k}{a_1 + a_2 + \ldots + a_k} = \frac{k+1}{2}$$

for all $k \geq 1$, and suppose for convenience that $a_1 = a$. We will prove by strong induction on n that the statement $P(n)$ is true for all $n \geq 1$, where $P(n)$ is the statement "$a_n = a$".

$P(1)$ is evident. Suppose we know that $P(1), P(2), \ldots, P(n)$ are all true, for some $n \geq 1$. Let's try to prove $P(n+1)$. One thing we know is that

$$\frac{a_1 + 2a_2 + 3a_3 + \ldots + (n+1)a_{n+1}}{a_1 + a_2 + \ldots + a_{n+1}} = \frac{(n+1)+1}{2} = \frac{n+2}{2}.$$

Another thing we know, by strong induction, is that $a_1 = \ldots = a_n = a$. So what we wrote above reduces to

$$\frac{(1+2+\ldots+n)a + (n+1)a_{n+1}}{na + a_{n+1}} = \frac{n+2}{2}.$$

Now a basic fact (which is an easy example of induction!) is that $1+2+\ldots+n = n(n+1)/2$, so the above becomes

$$\frac{\frac{n(n+1)a}{2} + (n+1)a_{n+1}}{na + a_{n+1}} = \frac{n+2}{2}.$$

Solving for a_{n+1} yields $a_{n+1} = a$, and we are done.

Source: I found this on Northwestern's Putnam prep site.

6. Define a sequence $(a_n)_{n\geq 0}$ by

$$a_n = \begin{cases} 9 & \text{if } n = 0, \text{ and} \\ 3a_{n-1}^4 + 4a_{n-1}^3 & \text{if } n \geq 1. \end{cases}$$

Prove that for all $n \geq 0$, a_n ends with at least 2^n 9's in its decimal representation.

Solution: Induction! $a_0 = 9$ certainly ends with at least $2^0 = 1$ 9's, so the base case is fine. For the induction, suppose that a_n ends with at least 2^n 9's in its decimal representation. This means that $a_n + 1$ ends with 2^n zeros. Write $a_n + 1 = a10^{2^n} := b$, so $a_n = b - 1$ (with a, b integers). Using $a_{n+1} = 3a_n^4 + 4a_n^3$, we get

$$\begin{aligned} a_{n+1} &= 3(b-1)^4 + 4(b-1)^3 \\ &= 3b^4 - 8b^3 + 6b^2 - 1 \\ &= b^2(3b^2 - 8b + 6) - 1 \\ &= \left(10^{2^n}\right)^2 a^2(3b^2 - 8b + 6) - 1 \\ &= 10^{2^{n+1}}[a^2(3b^2 - 8b + 6)] - 1. \end{aligned}$$

Since $a^2(3b^2 - 8b + 6)$ is an integer, we conclude that $a_{n+1} + 1$ ends with at least 2^{n+1} 0's, so a_{n+1} ends with at least 2^{n+1} 9's.

Source: I found this on Northwestern's Putnam prep site.

7. Prove if $a_1, \dots, a_n$ are positive reals, then

$$\frac{a_1 + \ldots + a_n}{n} \geq (a_1 \ldots a_n)^{1/n}.$$

Solution: We'll prove this by an odd kind of induction. First, we prove (by strong induction on the exponent) that it is true when n is a power of 2. It's certainly true when $n = 1$ (exponent 0). For a larger power of 2, say 2^k for $k \geq 1$, we have

$$\begin{aligned}
\frac{a_1 + \ldots + a_{2^k}}{2^k} &= \frac{\frac{a_1+\ldots+a_{2^{k-1}}}{2^{k-1}} + \frac{a_{2^{k-1}+1}+\ldots+a_{2^k}}{2^{k-1}}}{2} \\
&\geq \frac{(a_1 \ldots a_{2^{k-1}})^{1/2^{k-1}} + (a_{2^{k-1}+1} \ldots a_{2^k})^{1/2^{k-1}}}{2} \\
&\geq \left((a_1 \ldots a_{2^{k-1}})^{1/2^{k-1}} (a_{2^{k-1}+1} \ldots a_{2^k})^{1/2^{k-1}}\right)^{1/2} \\
&= (a_1 \ldots a_{2^k})^{1/2^k}
\end{aligned}$$

where the first inequality uses (twice) the induction hypothesis for exponent $k-1$, and the second uses it for exponent 1. But notice that I've cheated! If I look at the prove I've given in the special case exponent 1, I'm assuming exactly that case, not an earlier one! So I need to do $n = 2$ separately. But $n = 2$ is the assertion

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

Multiplying both sides by 2, squaring, and rearranging, this reduces to

$$a^2 - 2ab + b^2 \geq 0,$$

which is true since $a^2 - 2ab + b^2 = (a-b)^2$.

OK, so by induction we've proved the inequality for all powers of 2. What about for non-powers of 2? What we'll show is that if the statement is true for n, it's also true for $n-1$. This allows us to get the truth for any n by first going to a power of 2 above n, and then repeatedly *subtracting*! (This is why it's an odd kind of induction).

So, suppose we know

$$\frac{a_1 + \ldots + a_n}{n} \geq (a_1 \ldots a_n)^{1/n} \tag{1}$$

where $a_1, \dots, a_n$ are arbitrary positive reals. Let $a_1, \dots, a_{n-1}$, positive reals, be given. Set $a_n = \frac{a_1+\ldots+a_{n-1}}{n-1}$, and apply the above inequality (1). The left hand side is

$$\frac{a_1 + \ldots + a_{n-1} + \frac{a_1+\ldots+a_{n-1}}{n-1}}{n} = \frac{a_1 + \ldots + a_{n-1}}{n-1}.$$

The right-hand side is

$$\left(a_1 \ldots a_{n-1} \left(\frac{a_1 + \ldots + a_{n-1}}{n-1}\right)\right)^{1/n} = (a_1 \ldots a_{n-1})^{1/n} \left(\frac{a_1 + \ldots + a_{n-1}}{n-1}\right)^{1/n}.$$

So in this instance (1) becomes

$$\frac{a_1 + \ldots + a_{n-1}}{n-1} \geq (a_1 \ldots a_{n-1})^{1/n} \left(\frac{a_1 + \ldots + a_{n-1}}{n-1}\right)^{1/n}.$$

A little rearrangement shows that this is the same as

$$\frac{a_1 + \ldots + a_{n-1}}{n-1} \geq (a_1 \ldots a_{n-1})^{1/(n-1)},$$

which is exactly the $n-1$ case of the inequality — so we are done.

Source: This is the famous and useful arithmetic mean - geometric mean inequality, sometimes called the AM-GM inequality. The wikipwdia page `http://en.wikipedia.org/wiki/Inequality_of_arithmetic_and_geometric_means` gives many proofs of the stronger statement that the inequality holds, and can only hold with equality when all the a_i are equal.

8. You are given a 64 by 64 chessboard, and 1365 L-shaped tiles (2 by 2 tiles with one square removed). One of the squares of the chessboard is painted purple. Is it possible to tile the chessboard using the given tiles, leaving only the purple square exposed?

   Solution: Yes! We'll prove the more general fact: "Given a 2^n by 2^n chessboard, with one of the squares painted purple, and $(2^{2n}-1)/3$ L-shaped tiles, it is possible to tile the chessboard using the given tiles, leaving only the purple square exposed". We'll proceed by induction on n, with $n = 1$ trivial. For $n > 1$, simply divide the chessboard into 4 2^{n-1} by 2^{n-1} subchessboards in the obvious way. For each of the three subchessboard that do not include the purple square, mark the corner square closest to the middle of the board. By induction, each of the three subchessboard that do not include the purple square can be tiled leaving only the marked corner exposed, and the subchessboard that does include the purple square can be tiled leaving only that square exposed. One tile is now left over, which can be used to cover the three marked squares (they form an L!).

   Remark: A picture helps enormously here!

   Source: This is a classic problem, that I first learned from Andrew Thomason.