Problem Solving in Math (Math 43900) Fall 2013

Week three (September 10) problems — pigeonhole principle

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The pigeonhole principle

"If $n + 1$ pigeons settle themselves into a roost that has only n pigeonholes, then there must be at least one pigeonhole that has at least two pigeons."

This very simple principle, sometimes called the *box principle*, and sometimes *Dirichlet's box principle*, can be very powerful.

The proof is trivial: number the pigeonholes 1 through n, and consider the case where a_i pigeons land in hole i. If each $a_i \leq 1$, then $\sum_{i=1}^n a_i \leq n$, contradicting the fact that (since there are $n + 1$ pigeons in all) $\sum_{i=1}^n a_i = n + 1$.

Since it's a simple principle, to get some power out of it it has to be applied cleverly (in the examples, there will be at least one such clever application). Applying the principle requires identifying what the pigeons should be, and what the pigeonholes should be; sometimes this is far from obvious.

The pigeonhole principle has many obvious generalizations. I'll just state one of them: "if more than mn pigeons settle themselves into a roost that has no more than n pigeonholes, then there must be at least one pigeonhole that has at least $m + 1$ pigeons".

Example: 10 points are placed randomly in a 1 by 1 square. Show that there must be some pair of points that are within distance $\sqrt{2}/3$ of each other.

Solution: Divide the square into 9 smaller squares, each of dimension 1/3 by 1/3. These are the pigeonholes. The ten randomly chosen points are the pigeons. By the pigeonhole principle, at least one of the 1/3 by 1/3 squares must have at least two of the ten points in it. The maximum distance between two points in a 1/3 by 1/3 square is the distance between two opposite corners. By Pythagoras this is $\sqrt{(1/3)^2 + (1/3)^2} = \sqrt{2}/3$, and we are done.

Example: Show that there are two people in New York City who have the exactly same number of hairs on their head.

Solution: Trivial, because *surely* there are at least two baldies in NYC! But even if we weren't sure of that: a quick websearch shows that a typical human head has around 150,000 hairs, and it is then certainly reasonable to assume that no one has more than 5,000,000 hairs on their head. Set up 5,000,001 pigeonholes, numbered 0 through 5,000,000, and place a resident of NYC (a "pigeon") into bin i if (s)he has i hairs on her head. Another websearch shows that the population of NYC is around 8,300,000, so there are more pigeons than pigeonholes, and some pigeonhole must have multiple pigeons in it.

Example: Show that every sequence of $nm + 1$ real numbers must contain EITHER a decreasing subsequence of length $n+1$ OR an increasing subsequence of length $m+1$. (In a sequence $a_1, a_2, \ldots,$

an *increasing subsequence* is a subsequence $a_{i_1}, a_{i_2}, \ldots$ [with $i_1 < i_2 < \ldots$] satisfying $a_{i_1} \leq a_{i_2} \leq \ldots$, and a *decreasing subsequence* is defined analogously).

Solution: Let the sequence be $a_1, \ldots, a_{nm+1}$. For each k, $1 \leq k \leq nm+1$, let $f(k)$ be the length of the longest decreasing subsequence that starts with a_k, and let $g(k)$ be the length of the longest increasing subsequence that starts with a_k. Notice that $f(k), g(k) \geq 1$ always.

If there is a k with either $f(k) \geq n+1$ or $g(k) \geq m+1$, we are done. If not, then for every k we have $1 \leq f(k) \leq n$ and $1 \leq g(k) \leq m$. Set up nm pigeonholes, with each pigeonhole labeled by a different pair (i,j), $1 \leq i \leq n$, $1 \leq j \leq m$ (there are exactly nm such pairs). For each k, $1 \leq k \leq nm+1$, put a_k in pigeonhole (i,j) iff $f(k) = i$ and $g(k) = j$. There are $nm+1$ pigeonholes, so one pigeonhole, say hole (r,s), has at least two pigeons in it.

In other words, there are two terms of the sequence, say a_p and a_q (where without loss of generality $p < q$), with $f(p) = f(q) = r$ and $g(p) = g(q) = s$.

Suppose $a_p \geq a_q$. Then we can find a decreasing subsequence of length $r+1$ starting from a_p, by starting a_p, a_q, and then proceeding with any decreasing subsequence of length r that starts with a_q (one such exists, since $f(q) = r$). But that says that $f(p) \geq r+1$, contradicting $f(p) = r$.

On the other hand, suppose $a_p \leq a_q$. Then we can find an increasing subsequence of length $s+1$ starting from a_p, by starting a_p, a_q, and then proceeding with any increasing subsequence of length s that starts with a_q (one such exists, since $g(q) = s$). But that says that $g(p) \geq s+1$, contradicting $g(p) = s$.

So, whether $a_p \geq a_q$ or $a_p \leq a_q$, we get a contradiction, and we CANNOT ever be in the case where there is NO k with either $f(k) \geq n+1$ or $g(k) \geq m+1$. This completes the proof.

Remark: This beautiful result was discoverd by P. Erdös and G. Szekeres in 1935; the incredibly clever application of pigeonholes was given by A. Seidenberg in 1959.

The problems

1. (a) Show that among any $n+1$ numbers selected from $\{1, \ldots, 2n\}$, there must be two (distinct) such that one divides the other.

   (b) Show that among any $n+1$ numbers selected from $\{1, \ldots, 2n\}$, there must be two that share no common factors.

2. (a) Show that in any group of six people there must EITHER be some three who all know each other OR some three who all don't know each other. (Assuming that "knowing" is a reflexive relation: I know you iff you know me.)

   (b) Is there some number n such that in any group of n people there must EITHER be some four who all know each other OR some four who all don't know each other? What about if "four" is replaced by "five", "six", etc.?

3. Prove that any 55 element subset of $\{1, 2, \ldots, 100\}$ contains elements that differ by 10, 12 and 13, but need not contain elements differing by 11.

4. Let a_j, b_j c_j be integers for $1 \leq j \leq N$, with, for each j, at least one of a_j, b_j c_j odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$.

5. Let A and B be 2 by 2 matrices with integer entries such that A, $A+B$, $A+2B$, $A+3B$ and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

6. A *partition* of a set X is a collection of disjoint non-empty subsets (the parts) whose union is X. For a partition π of X, and an element $x \in X$, let $\pi(x)$ denote the number of elements in the part containing x.

   Let π, π' be any two partitions of $\{1, 2, \ldots, 9\}$. Show that there are two distinct numbers x and y in $\{1, 2, \ldots, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$.

7. The Fibonacci numbers are defined by the recurrence $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Show that the Fibonacci sequence is periodic modulo any positive integer. (I.e, show that for each $k \geq 0$, the sequence whose nth term is the remainder of f_n on division by k is a periodic sequence).

8. Prove that from a set of ten distinct two-digit numbers, it is possible to select two non-empty disjoint subsets whose members have the same sum.