Problem Solving in Math (Math 43900) Fall 2013

Week three (September 10) solutions

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1. (a) Show that among any $n+1$ numbers selected from $\{1,\ldots,2n\}$, there must be two (distinct) such that one divides the other.

   Solution: Set up n pigeon holes, numbered $1,3,5,\ldots,2n-1$ (all the odd numbers between 1 and $2n$). Each positive integer $k \leq 2n$ has a unique representation of the form $2^a j$, where $j \in \{1,3,5,\ldots,2n-1\}$ and $a \geq 0$ (just keep dividing by 2 until the number becomes odd; the number of times you divide by 2 is a, the resulting odd number is j). Put number k into hole j. By PHP, at least one j is such that there are two numbers in hole j; the one with the smaller exponent of 2 divides the other.

   Source: Folklore.

   (b) Show that among any $n+1$ numbers selected from $\{1,\ldots,2n\}$, there must be two that share no common factors.

   Solution: Two of the chosen numbers must be consecutive, and so share no common factors! (If $a|m$ and $a|(m+1)$ then $a|((m+1)-m)$ so $a|1$). To formally see that two of the chosen numbers must be consecutive, you could use the PHP. The n holes are labelled $(1,2)$, $(3,4)$, $\ldots$, $(2n-1,2n)$, and a chosen number k goes into hole $(j,j+1)$ exactly if $k \in \{j,j+1\}$.

   Source: This was a favourite puzzle of Paul Erdös. It is sometimes called "Pósa's soup problem"; the reason is given here: `http://www.math.uwaterloo.ca/navigation/ideas/articles/honsberger/index.shtml`.

2. (a) Show that in any group of six people there must EITHER be some three who all know each other OR some three who all don't know each other. (Assuming that "knowing" is a reflexive relation: I know you iff you know me.)

   Solution: Pick one person, A say. Of the other 5 people, A must EITHER know at least 3, OR not know at least three (this is basically PHP: if neither happens, only at most four of the other 5 are accounted for). Suppose A knows three people, say B,C,D. IF any pair among B,C,D know each other, then together with A they form a triple, all three of whom know each other. IF NOT, then B,C,D form a triple, all three of whom don't know each other. The same argument works if there are three people that A doesn't know.

   (b) Is there some number n such that in any group of n people there must EITHER be some four who all know each other OR some four who all don't know each other? What about if "four" is replaced by "five", "six", etc.?

Solution/Source: For every $k \geq 1$, there is a finite $n(k)$ such that in any group of $n(k)$ people there must EITHER be some k who all know each other OR some k who all don't know each other. This is the famous Ramsey's Theorem (see `http://en.wikipedia.org/wiki/Ramsey's_theorem`). For $k = 3$, six works, but five doesn't. For $k = 4$, 18 works but not 17 (this is hard!). For general k, 4^k works but $\sqrt{2}^k$ doesn't, and closing this gap has been a major open problem in discrete mathematics for the past 80 years.

3. Prove that any 55 element subset of $\{1, 2, \ldots, 100\}$ contains elements that differ by 10, 12 and 13, but need not contain elements differing by 11.

   Solution: For 10: Use pigeonholes $(1, 11), (2, 12), \ldots, (10, 20)$, $(21, 31), (22, 32), \ldots, (30, 40)$, $\ldots$, $(81, 91), (82, 92) \ldots, (90, 100)$. 50 in all. Put k in hole $(j, j + 10)$ if $j \in \{j, j + 10\}$. By PHP, one hole has at least 2 numbers in it; these differ by 10.

   For 12 and 13, the same argument work. For 12, there are 52 holes (four labeled with just the single numbers $(97), (98), (99), (100)$); For 13, there are 51 holes (three labeled with just the single numbers $(88), (89), (90)$).

   What happens for 11? The same strategy leads to 55 holes (including 10 holes labeled with just the single numbers $(90), \ldots, (99)$), so this strategy won't work. But it does allow us to prove that NO strategy will work: just (carefully) pick a 55 element subset that includes 1 number from each of the 55 holes, to get a 55 element subset containing no elements differing by 11.

   Source: I took this from Larson's book *Problem solving through problems.*

4. Let a_j, b_j c_j be integers for $1 \leq j \leq N$, with, for each j, at least one of a_j, b_j c_j odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$.

   Solution: All that matters about a_j, b_j c_j is the parity (odd/even) of each. There are 7 possibilities, which we can encode by triples $(1, 1, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 0, 0), (0, 1, 0), (0, 0, 1)$ (with, for example, $(1, 0, 1)$ encoding that a_j is odd, b_j is even, c_j is odd).

   It is useless to try and find r, s, t all even: then $ra_j + sb_j + tc_j$ will always be even. So, considering parity, there are again only 7 possibilities for r, s, t, encoded by triples $(1, 1, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 0, 0), (0, 1, 0), (0, 0, 1)$.

   It is easy to check that if we take any of the possible triples for a_j, b_j c_j, then as we run over each of the 7 possible triples for r, s, t, exactly 4 of the 7 numbers $ra_j + sb_j + tc_j$ is odd.

   So, given any set of N triples a_j, b_j c_j, exactly $4N$ of the $7N$ sums $ra_j + sb_j + tc_j$ are odd as r, s, t runs over its seven possibilities. We can place these $4N$ odd sums among 7 bins, according to which choice for the triple r, s, t we made. At least one of these bins must contain at least $4N/7$ triples; any choice of r, s, t that has that bin's triple associated with it will do.

   Source: Putnam 2000 B1.

5. Let A and B be 2 by 2 matrices with integer entries such that A, $A + B$, $A + 2B$, $A + 3B$ and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that $A + 5B$ is invertible and that its inverse has integer entries.

Solution: When does an integer 2 by 2 matrix have integer inverse? If the entries are c_{ij}, then what's needed is for the determinant $D = c_{11}c_{22} - c_{12}c_{21}$ to divide each of c_{11}, c_{12}, c_{21} and c_{22}. In other words, $c_{11} = aD$, $c_{12} = bD$, $c_{21} = cD$ and $c_{22} = dD$ for integers a, b, c and d. Hence $D = c_{11}c_{22} - c_{12}c_{21} = (ad - bc)D^2$, so $1 = (ad - bc)D$. Since $ad - bc$ is an integer, and so is D, this can only happen if $D = \pm 1$.

Let the entries of A be a_{ij}, and those of B be b_{ij}. We have the following relation (just by doing some algebra):

$$\det(A + nB) = a + nh + n^2 k,$$

where $a = \det(A)$, $h = a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{22}$ and $k = \det(B)$. This is a polynomial in n, of degree 2.

We know that each of $A + nB$, $n = 0, 1, 2, 3, 4$, have integer inverses, so determinants ± 1. So at least three of them must have the same determinant. That means that there are three different inputs n for which the quadratic polynomial $a + nh + n^2 k$ has the same output. For a quadratic polynomial to have this property, it must be a constant, so $h = 0$ and $k = 0$. That means that for *all* n,

$$\det(A + nB) = a = \det\mathrm{A} = \pm 1,$$

and so the inverse of $A + nB$ exists and has integer entries for *all* n (and in particular for $n = 5$).

Source: Putnam 1994 A4.

6. A *partition* of a set X is a collection of disjoint non-empty subsets (the parts) whose union is X. For a partition π of X, and an element $x \in X$, let $\pi(x)$ denote the number of elements in the part containing x.

   Let π, π' be any two partitions of $\{1, 2, \ldots, 9\}$. Show that there are two distinct numbers x and y in $\{1, 2, \ldots, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$.

   Solution: We want to show that at least 4 elements x must have the same value of $\pi(x)$. Why is this helpful? Because these 4 elements can't possible have distinct values of $\pi'(x)$; the smallest possible 4 distinct values are 1, 2, 3 and 4, accounting for 10 elements of a 9-element set. So there are two among the 4, say x, y, with $\pi'(x) = \pi'(y)$, and of course $\pi(x) = \pi(y)$, so we are done.

   Suppose π has a part of size 4 or more. Then definitely at least 4 elements x must have the same value of $\pi(x)$. So assume that every part in π has size 3 or less. The possibilities for the part sizes are:

$$\begin{array}{rcl} 3,3,3 & \to & \text{has 9 elements with same value of } \pi(x) \\ 3,3,2,1 & \to & \text{has 6 elements with same value of } \pi(x) \\ 3,3,1,1,1 & \to & \text{has 6 elements with same value of } \pi(x) \\ 3,2,2,2 & \to & \text{has 6 elements with same value of } \pi(x) \\ 3,2,2,1,1 & \to & \text{has 4 elements with same value of } \pi(x) \\ 3,2,1,1,1,1 & \to & \text{has 4 elements with same value of } \pi(x) \\ 3,1,1,1,1,1,1 & \to & \text{has 6 elements with same value of } \pi(x) \\ 2,2,2,2,1 & \to & \text{has 8 elements with same value of } \pi(x) \\ 2,2,2,1,1,1 & \to & \text{has 6 elements with same value of } \pi(x) \\ 2,2,1,1,1,1,1 & \to & \text{has 5 elements with same value of } \pi(x) \\ 2,1,1,1,1,1,1,1 & \to & \text{has 7 elements with same value of } \pi(x) \\ 1,1,1,1,1,1,1,1,1 & \to & \text{has 9 elements with same value of } \pi(x). \end{array}$$

(Of course this could have been done more efficiently!) In all cases, there are at least 4 elements x with the same value of $\pi(x)$.

Source: Putnam 1995 B1.

7. The Fibonacci numbers are defined by the recurrence $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Show that the Fibonacci sequence is periodic modulo any positive integer. (I.e, show that for each $k \geq 0$, the sequence whose nth term is the remainder of f_n on division by k is a periodic sequence).

   Solution: Consider the sequence obtained from the Fibonacci sequence by taking the remainder of each term on division by k (so the result is a sequence, all terms in $\{0, \ldots, k-1\}$). Suppose that there are two consecutive terms in this sequence, say the mth and $(m+1)$st, taking values a, b, and two *other* consecutive terms , say the nth and $n+1$st, taking the same values a, b (with $m < n$). Then the $(m+2)$nd and $(n+2)$nd terms of the reduced sequence agree.

   [WHY? Because the $(m+2)$nd term is the remainder of F_{m+2} on division by k, which is the remainder of $F_m + F_{m+1}$ on division by k, which is the remainder of F_m on division by k PLUS the remainder of F_{m+1} on division by k, which is the remainder of a on division by k PLUS the remainder of b on division by k, which is the remainder of F_n on division by k PLUS the remainder of F_{n+1} on division by k, which is the remainder of $F_n + F_{n+1}$ on division by k, which is the remainder of F_{n+2} on division by k.]

   The same argument shows that the reduced sequence is periodic beyond the mth terms, with period (at most) $n - m$.

   So all we need to do to find periodicity is to find two consecutive terms in the sequence, that agree with two *other* consecutive terms. There are only k^2 possibilities for a pair of consecutive values in the sequence, and infinitely many consecutive values, so by PHP there has to be a coincidence of the required kind.

   Source: I found this on Northwestern's Putnam prep site.

8. Prove that from a set of ten distinct two-digit numbers, it is possible to select two non-empty disjoint subsets whose members have the same sum.

   Solution: The maximum subset-sum is $99+98+\ldots+90 = 945$. The minimum subset-sum is 10. So there at most 936 possible sums, as we run over non-empty subsets of a 10-element set of two digit numbers. But there are $2^{10} - 1 = 1023$ non-empty subsets of the 10-element set. BY PHP, there are two different non-empty subsets, A and B, with equal sums. PROBLEM: A, B may not be disjoint. SOLUTION: We certainly can't have $A \subset B$ or $B \subset A$ (otherwise one sum would definitely be larger). So $A \setminus (A \cap B)$ and $B \setminus (A \cap B)$ are both non-empty. They are clearly disjoint, and have the same sums, since we obtained them by removing the sam set from both A and B.

   Source: This was problem 1 on the 1972 International Mathematical Olympiad.