# Problem Solving in Math (Math 43900) Fall 2013

Week five (September 24) problems — another grab-bag

Instructor: David Galvin

I sent out a call for people's favourite problems. This week's set is a grab-bag from among those that I received (if yours wasn't included, don't worry, more later!)

## The problems

1. Let $x$ be a real number such that $x + 1/x$ is an integer. Prove that

$$x^n + \frac{1}{x^n}$$

is also an integer for any positive number $n$.

2. The following scenario is explained to 100 perfect logicians: in the middle of the night they will be put into a dark room, and a certain number of them will be selected to have a blue dot put on their foreheads. Those with the dots don't realize that the dots are being put on, and can't see their own foreheads. The person running the game provides a guarantee that at least one person will have such a blue dot. At dawn on day 1, the lights are turned on, and everyone is allowed to look at the foreheads of everyone else (just not their own!). When everyone has thoroughly examined everyone else, the lights go out. All those who now know for certain that they have a blue dot on their forehead are instructed to slip quietly out of the room; their game is over. At dawn on day 2, the lights are turned on again, and the observation period recommences; then the lights go out and all those who now know for certain that they have a blue dot on their forehead leave. The process is repeated indefinitely.

   Suppose initially that all 100 logicians have a blue dot put on their forehead. What happens?

3. Given five points on the surface of a sphere, show that there exists a closed hemisphere which contains four of those points.

4. Five men crash-land their airplane on a deserted island in the South Pacific. On their first day they gather as many coconuts as they can find into one big pile. They decide that, since it is getting dark, they will wait until the next day to divide the coconuts.

   That night each man took a turn watching for rescue searchers while the others slept. The first watcher got bored so he decided to divide the coconuts into five equal piles. When he did this, he found he had one remaining coconut. He gave this coconut to a monkey, took one of the piles, and hid it for himself. Then he jumbled up the four other piles into one big pile again.

   The second watcher did the same thing, as did all the others: they each divided the pile of coconuts they found at the start of their watch into five equal piles and each found they had one extra coconut left over, which they gave to the monkey. They each took one of the five piles and hid those coconuts. They each came back and jumbled up the remaining four piles into one big pile.

In the morning, none of them admitted to what they had done, so they divided the (rather smaller!) pile of coconuts into five equal piles. When they did this, they found that they had left over, which they gave to the lucky monkey.

How many coconuts were there in the original pile?

5. Using the digits 1 up to 9, two numbers must be made. The product of these two numbers should be as large as possible. All digits must be used exactly once. Which are the requested two numbers?

6. Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$, be sequences of positive reals with $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \ldots$. Assume that the sequence $(b_j)_{j=1}^{\infty}$ is bounded. Prove that the sum

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \ldots a_n}$$

converges, and evaluate $S$.