Problem Solving in Math (Math 43900) Fall 2013

Week five (September 24) solutions

Instructor: David Galvin

1. Let x be a real number such that $x + 1/x$ is an integer. Prove that

$$x^n + \frac{1}{x^n}$$

is also an integer for any positive number n.

Solution: Note that $x \neq 0$. Try induction on n; base case $n = 1$ is given. For $n > 1$,

$$x^n + \frac{1}{x^n} = \left(x + \frac{1}{x}\right)\left(x^{n-1} + \frac{1}{x^{n-1}}\right) - \left(x^{n-2} + \frac{1}{x^{n-2}}\right).$$

By (strong) induction, all three terms in parentheses on the right-hand side are integers (not quite: when $n = 2$, one of the terms is $x^0 + 1/x^0$, which is an integer but not by induction), so the right-hand side is an integer.

Source: Jay.

2. The following scenario is explained to 100 perfect logicians: in the middle of the night they will be put into a dark room, and a certain number of them will be selected to have a blue dot put on their foreheads. Those with the dots don't realize that the dots are being put on, and can't see their own foreheads. The person running the game provides a guarantee that at least one person will have such a blue dot. At dawn on day 1, the lights are turned on, and everyone is allowed to look at the foreheads of everyone else (just not their own!). When everyone has thoroughly examined everyone else, the lights go out. All those who now know for certain that they have a blue dot on their forehead are instructed to slip quietly out of the room; their game is over. At dawn on day 2, the lights are turned on again, and the observation period recommences; then the lights go out and all those who now know for certain that they have a blue dot on their forehead leave. The process is repeated indefinitely.

Suppose initially that all 100 logicians have a blue dot put on their forehead. What happens?

Solution: I haven't yet found a completely satisfactory solution to this yet ... I'll post one when I do!

Source: Patrick & Seamus

3. Given five points on the surface of a sphere, show that there exists a closed hemisphere which contains four of those points.

Solution: Fix two of the points, and consider the great circle through them. This great circle divides the sphere into two open hemispheres A and B, each of which, together with the great circle, forms a closed hemisphere; call these A', B'. If any two other points lie on the great circle, then both A' and B' have at least 4 of the points. If exactly one other of the points lie on the great circle, then by PHP at least one of A, B must have at least one of the remaining two points; if A, then A' has at least 4 points, and if B, then B' does. If no other point lies on the great circle, then by PHP at least one of A, B must have at least two of the remaining three points; if A, then A' has at least 4 points, and if B, then B' does. This covers all possibilities, so we are done.

Source: John; also Putnam 2002 problem A2

4. Five men crash-land their airplane on a deserted island in the South Pacific. On their first day they gather as many coconuts as they can find into one big pile. They decide that, since it is getting dark, they will wait until the next day to divide the coconuts.

   That night each man took a turn watching for rescue searchers while the others slept. The first watcher got bored so he decided to divide the coconuts into five equal piles. When he did this, he found he had one remaining coconut. He gave this coconut to a monkey, took one of the piles, and hid it for himself. Then he jumbled up the four other piles into one big pile again.

   The second watcher did the same thing, as did all the others: they each divided the pile of coconuts they found at the start of their watch into five equal piles and each found they had one extra coconut left over, which they gave to the monkey. They each took one of the five piles and hid those coconuts. They each came back and jumbled up the remaining four piles into one big pile.

   In the morning, none of them admitted to what they had done, so they divided the (rather smaller!) pile of coconuts into five equal piles. When they did this, they found that they had left over, which they gave to the lucky monkey.

   How many coconuts were there in the original pile?

   Solution: Let x be the number of coconuts. From the first man's action we deduce $x \equiv 1$ (5), or $x = 5k + 1$. After the first man is done, the pile of coconuts has size $4(x-1)/5$. From the second man's action we deduce $4(x-1)/5 \equiv 1$ (5), or $(4x-4)/5 = 5\ell + 1$, so $4x = 25\ell + 9$, or $4x \equiv 9$ (25). To get information about x from this, we must multiply by that number b such that $4b = 1$ (25); $b = 19$ works. So $(19)4x \equiv (19)9$ (25) or $x \equiv 21$ (25).

   After the second man is done, the pile of coconuts has size $4((4(x-1)/5)-1)/5 = (16x-36)/25$. From the third man's action we deduce $(16x-36)/25 \equiv 1$ (5), or $(16x-36)/25 = 5\ell + 1$, so $16x = 125\ell + 61$, or $16x \equiv 61$ (125). To get information about x from this, we must multiply by that number b such that $16b = 1$ (125); $b = 86$ works. So $(86)16x \equiv (86)61$ (125) or $x \equiv 121$ (125).

   A pattern is emerging: maybe it's better to thing of $x \equiv -4$ (5), $x \equiv -4$ (25), $x \equiv -4$ (125), instead of $x \equiv 1$ (5), $x \equiv 21$ (25), $x \equiv 121$ (125). Let's try: After the third man is done, the pile of coconuts has size $(4/5)((16x-36)/25-1) = (64x-244)/125$. From the fourth man's action we deduce $(64x-244)/125 \equiv -4$ (5), or $(64x-244)/125 = 5m-4$, so $64x = 625\ell - 256$. To make this useful, we need to multiply by the inverse of 64 modulo 625; it is 459 (I used a modular arithmetic calculator online). So $(459)64x \equiv x \equiv (459)(-256) \equiv -4$ (625).

Similar, but increasingly more computationally intensive, calculations lead us to $x \equiv -4\ (3125)$ (from fifth man's actions) and $x \equiv -4\ (15625)$ (the action of the morning). Notice that any x satisfying $x \equiv -4\ (15625)$ satisfies all of the other recurrences; so the only restriction on x is $x \equiv -4\ (15625)$. The smallest possible option is $x = 15621$ (a lot of coconuts!)

Here's a "cheap" solution: we can go through the above process without actually calculating the various inverses: First man's action tells us that $x \equiv 1\ (5)$. He leaves $(4/5)(x-1)$ coconuts. Second man's action tells us that $(4/5)(x-1) \equiv 1\ (5)$, or $4x \equiv 9\ (25)$, or $x \equiv (4^{-1})9\ (25)$. (Because $4 and 25$ have no factors in common, we know that there are a and b with $4a+25b = 1$, telling us that 4 does have an inverse modulo 25.) Second man leaves $(4/5)((4/5)(x-1)-1)$ coconuts. His action, using the cheap strategy just employed, tells us that there is some k such that $x \equiv k\ (125)$. Keep repeating, until finally we discover that after the action of the next morning, a necessary condition on x for x to work is that $x \equiv k\ (15625)$ for some (explicitly computable) k. Of course, this is a vacuous statement unless we can actually figure out what k could be.

But we can! Take $k = -4$, and to check that this works consider what happen if there are initially -4 coconuts. The first man divides them into five equal piles of size -1, leaving $+1$ left over, which goes to the monkey. He takes his share of -1, leaving exactly what he started with: -4 coconuts! So every other action also works, and also leaves -4 coconuts.

Of course, -4 is not a physical solution; but it tells us that the necessary condition is $x \equiv -4 \equiv 15621\ (15625)$. We can then easily check that any x of the form $x = 15625k + 15621$ works.

Source: Kevin

5. Using the digits 1 up to 9, two numbers must be made. The product of these two numbers should be as large as possible. All digits must be used exactly once. Which are the requested two numbers?

   Solution: First observation: the digits of the requested two numbers have to form descending sequences (if not, the product can be increased by swapping a later bigger digit with an earlier smaller digit in one of the numbers; I'm thinking of reading the numbers from left-to-right in decimal.

   First task: figure out the best product using a 5 digit number versus a 4 digit number. Suppose two numbers are $9abcd$ and $efgh$. It's a quick check that we should have $e > a$ (if not we can increase the product by instead taking $9efgh$ and $abcd$). Since one of a, e must be 8 (if not one of the numbers wouldn't be in decreasing order), so $e = 8$ and our two numbers are $9abcd$ and $8fgh$.

   Suppose $a = 7$. Then the numbers are $97bcd$ and $8fgh$. Write these as $97000+A$ and $8000+B$ to get the product $(97000)(8000) + 97000B + 8000A + AB$. Claim: it's best to have $B > A$. For suppose $A > B$. Then using $97000+B$ and $8000+A$ we get the product $(97000)(8000) + 97000A + 8000B + AB$, which is bigger than $(97000)(8000) + 97000B + 8000A + AB$ (the difference is $89000(A-B)$). So we should have $f = 6$, making our numbers $97bcd$ and $86gh$.

   If $a \neq 7$, then by the decreasing-sequence property we must have $f = 7$, making our numbers $9abcd$ and $87gh$.

   So we have two possibilities: 1) $97bcd$ and $86gh$ and 2) $9abcd$ and $87gh$. Similar sort of reasoning can narrow things down more from here, or we could just try all possibilities, to see which gives the bigger product.

There's another 5-versus-4 possibility: *abcde* and $9fgh$. One can analyse this possibility similarly, reducing down to a managable number of possibilities to check.

It turns out that the best 5-versus-4 possibility is 9642 and 87531 (and the product of these two numbers is 843,973,902).

One then has to check that no 6-versus-3, 7-versus-2 or 8-versus-1 possibility beats this, using similar thinking.

This is quite a bit of work! Does anyone see a more streamlined approach?

Source: Lindsay

6. Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$, be sequences of positive reals with $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \ldots$. Assume that the sequence $(b_j)_{j=1}^{\infty}$ is bounded. Prove that the sum

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \ldots a_n}$$

converges, and evaluate S.

Solution: The question strongly suggests that the sum is independent of the choice of sequences, so try an example: all $b_i = 1$. This makes $a_n = 2$ for $n \geq 2$, and the sum becomes $1 + (1/2) + (1/2)^2 + \ldots = 3/2$. This suggests $S = 3/2$.

To prove this, lets re-write S in terms of the b's (that's about all we can do!). Using $a_n = (b_n + 2)/b_{n-1}$ (and $b_1 = 1$), the terms in S are

$$1, \frac{1}{b_2+2}, \frac{1}{b_2+2}\frac{b_2}{b_3+2}, \frac{1}{b_2+2}\frac{b_2}{b_3+2}\frac{b_3}{b_4+2}, \ldots.$$

We want the partial sums of this sequence to converge to 3/2, so we look at the difference between 3/2 and the partial sums. The first difference is 1/2. The second is

$$\frac{3}{2} - 1 - \frac{1}{b_2+2} = \frac{b_2}{2(b_2+2)}.$$

The third is

$$\frac{3}{2} - 1 - \frac{1}{b_2+2} - \frac{b_2}{(b_2+2)(b_3+2)} = \frac{b_2}{2(b_2+2)} - \frac{b_2}{(b_2+2)(b_3+2)} = \frac{b_2 b_3}{2(b_2+2)(b_3+2)}.$$

A pattern is emerging: it seems that 3/2 minus the sum of the first k terms is

$$\frac{b_2 \ldots b_k}{2(b_2+2)\ldots(b_k+2)}.$$

We prove this by induction on k, with some base cases already established. For the inductive step, we observe that (by induction) 3/2 minus the sum of the first $k+1$ terms is

$$\frac{b_2 \ldots b_k}{2(b_2+2)\ldots(b_k+2)} - \frac{b_2 \ldots b_k}{(b_2+2)\ldots(b_{k+1}+2)} = \frac{b_2 \ldots b_{k+1}}{2(b_2+2)\ldots(b_{k+1}+2)},$$

so we have what we want by induction.

To finish off, we need to show that

$$\frac{b_2 \dots b_k}{2(b_2+2)\dots(b_k+2)} \to 0$$

as $k \to \infty$. The terms of the sequence are positive. Since $(b_k)_{k\geq 0}$ is bounded, there exist $C > 0$ such that $b_j < C$ for all j. This means $1 + (2/b_j) > 1 + (2/C)$ for all j, so

$$\frac{b_j}{b_j+2} < \frac{1}{1+(2/C)} = \frac{C}{2+C}.$$

It follows that

$$\frac{b_2 \dots b_k}{2(b_2+2)\dots(b_k+2)} < \frac{1}{2}\left(\frac{C}{2+C}\right)^k$$

for all k. The right-hand side tends to 0 as $k \to \infty$, so

$$\frac{b_2 \dots b_k}{2(b_2+2)\dots(b_k+2)} \to 0$$

as $k \to \infty$; so we are done, and have proved that $S = 3/2$.

Source: Jonathan, and also Putnam 2011 problem A2