Problem Solving in Math (Math 43900) Fall 2013

Week six (October 1) problems — recurrences

Instructor: David Galvin

Definition of a recurrence relation

We met recurrences in the induction hand-out.

Sometimes we are either given a sequence of numbers via a recurrence relation, or we can argue that there is such relation that governs the growth of a sequence. A sequence $(b_n)_{n\geq a}$ is defined via a *recurrence relation* if some initial values, $b_a, b_{a+1}, \ldots, b_k$ say, are given, and then a rule is given that allows, for each $n > k$, b_n to be computed as long as we know the values $b_a, b_{a+1}, \ldots, b_{n-1}$.

Sequences defined by a recurrence relation, and proofs by induction, go hand-in-glove. Here's an illustrative example.

Example: Let a_n be the number of different ways of covering a 1 by n strip with 1 by 1 and 1 by 3 tiles. Prove that $a_n < (1.5)^n$.

Solution: We start by figuring out how to calculate a_n via a recurrence. Some initial values of a_n are easy to compute: for example, $a_1 = 1$, $a_2 = 1$ and $a_3 = 2$. For $n \geq 4$, we can tile the 1 by n strip EITHER by first tiling the initial 1 by 1 strip with a 1 by 1 tile, and then finishing by tiling the remaining 1 by $n-1$ strip in any of the a_{n-1} admissible ways; OR by first tiling the initial 1 by 3 strip with a 1 by 3 tile, and then finishing by tiling the remaining 1 by $n-3$ strip in any of the a_{n-3} admissible ways. It follows that for $n \geq 4$ we have $a_n = a_{n-1} + a_{n-3}$. So a_n (for $n \geq 1$) is determined by the recurrence

$$a_n = \begin{cases} 1 & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ 2 & \text{if } n = 3, \text{ and} \\ a_{n-1} + a_{n-3} & \text{if } n \geq 4. \end{cases}$$

Notice that this gives us enough information to calculate a_n for all $n \geq 1$: for example, $a_4 = a_3 + a_1 = 3$, $a_5 = a_4 + a_2 = 4$, and $a_6 = a_5 + a_3 = 6$.

Now we prove, by strong induction, that $a_n < 1.5^n$. That $a_1 = 1 < 1.5^1$, $a_2 = 1 < (1.5)^2$ and $a_3 = 2 < (1.5)^3$ is obvious. For $n \geq 4$, we have

$$\begin{aligned} a_n &= a_{n-1} + a_{n-3} \\ &< (1.5)^{n-1} + (1.5)^{n-3} \\ &= (1.5)^n \left(\frac{2}{3} + \left(\frac{2}{3}\right)^3\right) \\ &= (1.5)^n \left(\frac{26}{27}\right) \\ &< (1.5)^n, \end{aligned}$$

(the second line using the inductive hypothesis) and we are done by induction.

Notice that we really needed strong induction here, and we really needed all three of the base cases $n = 1, 2, 3$ (think about what would happen if we tried to use regular induction, or what would happen if we only verified $n = 1$ as a base case).

Solving via generating functions

Given a sequence $(a_n)_{n \geq 0}$, we can form its *generating function*, the function

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Often we can use a recurrence relation to produce a functional equation that $F(x)$ satisfies, then solve that equation to find a compact (non-infinite-summation) expression for $F(x)$, then finally use knowledge of calculus power-series to extract an exact expression for a_n. There are so many different varieties of this method, that I won't describe it in general, just give an example. The *Perrin sequence* is defined by $p_0 = 3$, $P_1 = 0$, $p_2 = 2$, and

$$p_n = p_{n-2} + p_{n-3} \quad \text{for } n > 2.$$

(A quite interesting sequence: see `http://en.wikipedia.org/wiki/Perrin_number#Primes_and_divisibility`.) The generating function of the sequence is

$$P(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + p_4 x^4 + \ldots.$$

Plugging in the given values for p_0, p_1 and p_2, and the recurrence's right-hand side for all others, we get

$$\begin{aligned} P(x) &= 3 + 2x^2 + (p_0 + p_1)x^3 + (p_1 + p_2)x^4 + \ldots \\ &= 3 + 2x^2 + x^3(p_0 + p_1 x + \ldots) + x^2(p_1 x + p_2 x^2 + \ldots) \\ &= 3 + 2x^2 + x^3 P(x) + x^2(P(x) - p_0) \\ &= 3 - x^2 + (x^3 + x^2)P(x). \end{aligned}$$

We can now solve for $P(x)$ as a rational function in x, and expand using partial fractions:

$$\begin{aligned} P(x) &= \frac{3 - x^2}{1 - x^2 - x^3} \\ &= \frac{A}{1 - \alpha_1 x} + \frac{B}{1 - \alpha_2 x} + \frac{C}{1 - \alpha_3 x} \end{aligned}$$

where A, B and C are some constants and $(1-\alpha_1 x)(1-\alpha_2 x)(1-\alpha_3 x) = 1-x^2-x^3$, or equivalently $(x-\alpha_1)(x-\alpha_2)(x-\alpha_3) = x^3-x-1$. In other words, α_1, α_2 and α_3 are the solutions to $x^3-x-1 = 0$ (it happens that one of them, say α_1, is real, and is roughly 1.32 [it's called the *plastic number*] and the other two are a complex conjugate pair with absolute value smaller than α_1).

Using

$$\frac{1}{1 - kx} = 1 + kx + k^2 x^2 + \ldots,$$

we now get that

$$P(x) = (A+B+C) + (A\alpha_1 + B\alpha_2 + C\alpha_3)x + (A\alpha_1^2 + B\alpha_2^2 + C\alpha_3^2)x^2 + (A\alpha_1^3 + B\alpha_2^3 + C\alpha_3^3)x^3 + \ldots,$$

and so we can read off a formula for p_n (by uniqueness of power-series representations):

$$p_n = A\alpha_1^n + B\alpha_2^n + C\alpha_3^n.$$

But what are A, B and C? One way to figure them out is to use the initial conditions, to get a set of simultaneous equations:

$$\begin{array}{ccccccc} A & + & B & + & C & = & 3 \\ A\alpha_1 & + & B\alpha_2 & + & C\alpha_3 & = & 0 \\ A\alpha_1^2 & + & B\alpha_2^2 & + & C\alpha_3^2 & = & 2. \end{array}$$

It turns out that the unique solution to this system is $A = B = C = 1$. This solution satisfies the first equation above, evidently; it satisfies the second since

$$(x-\alpha_1)(x-\alpha_2)(x-\alpha_3) = x^3-x-1 = x^3-(\alpha_1+\alpha_2+\alpha_3)x^2+(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)x-(\alpha_1\alpha_2\alpha_3) \quad (1)$$

implies $\alpha_1 + \alpha_2 + \alpha_3 = 0$; and it satisfies the last since

$$(\alpha_1 + \alpha_2 + \alpha_3)^2 = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + 2(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3),$$

and from (1) this reduces to $0 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 - 2$ so $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 2$.

So we have an exact formula for p_n:

$$p_n = \alpha_1^n + \alpha_2^n + \alpha_3^n$$

where α_1, α_2 and α_3 are the roots of $x^3 - x - 1$.

Notice that without even doing the explicit computation of A, B and C, we have learned something from the generating function approach about p_n, namely the following: since $\alpha_1 \approx 1.32$ is real and (α_2, α_3) is a complex conjugate pair with absolute value smaller than α_1, we have that for large n,

$$p_n \approx A\alpha_1^n \approx A(1.32)^n.$$

In other words, with very little work, we have isolated the rough growth rate of p_n.

If this business of generating functions interests you, you can find out much more in Herb Wilf's beautiful book *generatingfunctionology* (just google it; it's freely available online).

Solving via characteristic function

Mimicing what we did with the Perrin sequence, we can easily prove the following theorem: let (a_n) be a sequence defined recursively, via the defining relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_k a_{n-k} \quad \text{for } n \geq k,$$

(for some constants c_i) together with initial values $a_0, a_1, a_2, \ldots, a_{k-1}$. Form the polynomial

$$C(x) = x^k - c_1 x^{k-1} - c_2 x^{k-2} - \ldots - c_{k-1}x - c_k$$

(called the *characteristic polynomial* of the recurrence). If $C(x) = (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_k)$ factors into *distinct* linear terms, then there are constants $A_1, A_2, \ldots, A_k$ such that for all $n \geq 0$,

$$a_n = A_1\alpha_1^n + \ldots + A_k\alpha_k^n$$

with the A_i's explicitly findable by solving the k by k system of linear equations

$$\begin{aligned} A_1 + \ldots + A_k &= a_0 \\ A_1\alpha_1 + \ldots + A_k\alpha_k &= a_1 \\ A_1\alpha_1^2 + \ldots + A_k\alpha_k^2 &= a_2 \\ &\ldots \\ A_1\alpha_1^{k-1} + \ldots + A_k\alpha_k^{k-1} &= a_{k-1}. \end{aligned}$$

Even without solving this system, if $C(x)$ has a unique root (say α_1) of greatest absolute value, then we know the asymptotic growth rate

$$p_n \sim A_1\alpha_1^n$$

as $n \to \infty$.

Similar statements can be made when $C(x)$ has repeated roots, with the form of the final answer changing depending on what is the right expression to use in the partial fractions expansion step of the generating function method. I won't make a general statement, because it would be way too cumbersome (but ask me if you want to see more!); instead here's an example:

Suppose $a_n = 4a_{n-1} - 4a_{n-2}$ for all $n \geq 2$, with $a_0 = 0$ and $a_1 = 1$. The generating function method gives that the generating function $A(x)$ satisfies

$$A(x) = \frac{x}{1 - 4x + 4x^2}.$$

The correct partial fractions expansion now is

$$A(x) = \frac{A}{1-2x} + \frac{B}{(1-2x)^2}.$$

The coefficient of x^n in $A/(1-2x)$ is $A(2^n)$. For $B/(1-2x)^2$, we use:

$$\frac{1}{1-kx} = 1 + kx + k^2x^2 + \ldots,$$

so differentiating

$$\frac{k}{(1-kx)^2} = k + 2k^2x + 3k^3x^2 + \ldots + (n+1)k^{n+1}x^n + \ldots,$$

so

$$\frac{1}{(1-kx)^2} = 1 + 2kx + 3k^2x^2 + \ldots + (n+1)k^nx^n + \ldots,$$

so the coefficient of x^n in $B/(1-2x)^2$ is $B(n+1)(2^n)$. [This trick of figuring out new power series from old by differentiation is quite useful!] This gives

$$a_n = A2^n + (n+1)B2^n.$$

Using $a_0 = 0$ we get $A + B = 0$, and using $a_1 = 1$ we get $2A + 4B = 1$, so $A = -1/2$, $B = 1/2$ and

$$a_n = n2^{n-1}$$

(a fact that if we had guessed correctly, we could have easily proven by induction).

Solving via matrices

The Perrin recurrence can be encoded in matrix form: start with $p_{n+2} = p_n + p_{n-1}$, and add the trivial identities $p_{n+1} = p_{n+1}$ and $p_n = p_n$ to get

$$\begin{pmatrix} p_n \\ p_{n+1} \\ p_{n+2} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{pmatrix} p_{n-1} \\ p_n \\ p_{n+1} \end{pmatrix} \tag{2}$$

valid for $n \geq 1$. iteratively applying, we get that for all $n \geq 1$,

$$\begin{pmatrix} p_n \\ p_{n+1} \\ p_{n+2} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^n \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}. \tag{3}$$

Write this as $v_n = A^n v$. If we can diagonalize A (that is, find invertible S with $SAS^{-1} = D$, with D a diagonal matrix), then we can write $A = S^{-1}DS$, so $A^n = S^{-1}D^nS$, from which we can easily find v_n and so p_n explicitly. If you know enough linear algebra, you'll quickly see that this approach requires finding eigenvalues, which are roots of a certain cubic, and the computations quickly reduce to exactly the same ones as those of the previous two methods described. I mention this method just to bring up the matrix point of view of recurrences, which can sometimes be quite helpful.

Solving general recurrences

I've only talked about recurrences with constant coefficients, but of course recurrences can be far more general. While the generating function method is very good to bear in mind for more general problems, there's really no general approach that's sure to work; solving recurrences generally involves ad-ho tool like playing with lots of small examples, and spotting, conjecturing and proving patterns (often by induction).

A non-Putnam warm-up exercise

Using the trick of repeatedly differentiating the identity

$$\frac{1}{1-x} = 1 + x + x^2 + \dots,$$

find a nice expression for the coefficients of the power series (about 0) of $1/(1-x)^k$. Use this to derive, via generating functions, the identity

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

and if you are feeling masochistic, go on to find a nice closed-form for

$$\sum_{i=0}^{n} i^3$$

using the same idea.

The problems

1. Define a selfish set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.

2. Define a sequence $(p_n)_{n\geq 1}$ recursively by $p_1 = 3$, $p_2 = 7$ and, for $n \geq 3$,

$$p_n = 4 + p_{n-1} + 2p_{n-2} + \ldots + 2p_1$$

(so, for example, $p_3 = 4+p_2+2p_1 = 17$ and $p_4 = 4+p_3+2p_2+2p_1 = 41$). Find a closed-form expression for p_n for general n.

3. Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

4. Define a sequence by $a_k = k$ for $k = 1, 2, \ldots, 2006$ and

$$a_{k+1} = a_k + a_{k-2005}$$

for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

5. The last question was clearly written with the years 2005 and 2006 in mind. Does the conclusion remain true for an arbitrary year? That is, fix $m \geq 1$. Define a sequence by $a_k = k$ for $k = 1, 2, \ldots, m + 1$ and

$$a_{k+1} = a_k + a_{k-m}$$

for $k \geq m+1$. For which m is it true that the sequence has m consecutive terms each divisible by $m + 1$?

6. Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1 + x + x^2)^m$. Prove that for all integers $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor 2k/3 \rfloor} (-1)^i a_{k-i,i} \leq 1.$$

(Here $\lfloor a \rfloor$ denote the round-down of a to the nearest integer at or below a; so for example $\lfloor 3.4 \rfloor = 3$, $\lfloor 2.999 \rfloor = 2$ and $\lfloor 5 \rfloor = 5$.)

7. Define $(a_n)_{n\geq 0}$ by

$$\frac{1}{1 - 2x - x^2} = \sum_{n\geq 0} a_n x^n.$$

Show that for each $n \geq 0$, there is an $m = m(n)$ such that $a_m = a_n^2 + a_{n+1}^2$.