Problem Solving in Math (Math 43900) Fall 2013

Week six (October 1) solutions

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A non-Putnam warm-up exercise

Using the trick of repeatedly differentiating the identity

$$\frac{1}{1-x} = 1 + x + x^2 + \dots,$$

find a nice expression for the coefficients of the power series (about 0) of $1/(1-x)^k$. Use this to derive, via generating functions, the identity

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

and if you are feeling masochistic, go on to find a nice closed-form for

$$\sum_{i=0}^{n} i^3$$

using the same idea.

Solution: Differentiating the left-hand side k times, we get

$$\frac{k!}{(1-x)^{k+1}},$$

and differentiating the right-hand side k times, we get an power series where the coefficient of x^{n-k} is $n(n-1)\dots(n-(k-1))$, so the coefficient of x^n is $(n+k)(n+k-1)\dots(n+1)$. Dividing through by $k!$, the coefficient of x^n in $1/(1-x)^{k+1}$ is

$$\frac{(n+k)(n+k-1)\dots(n+1)}{k!} = \binom{n+k}{k}.$$

Let $a_n = \sum_{i=0}^{n} i^2$; a_n satisfies the recurrence $a_0 = 0$ and $a_n = a_{n-1} + n^2$ for $n > 0$. Letting

$A(x) = a_0 + a_1 x + a_2 x^2 \ldots$ be the generating function of the a_n's, we get

$$\begin{aligned}
A(x) &= 0 + (a_0 + 1^2)x + (a_1 + 2^2)x^2 + \ldots \\
&= xA(x) + (1^2 x + 2^2 x^2 + 3^2 x^3 + \ldots) \\
&= xA(x) + [1.0 + 1]x + [2.1 + 2]x^2 + [3.2 + 3]x^3 + \ldots \\
&= xA(x) + (1.0x + 2.1x^2 + 3.2x^3 + \ldots) + (1x + 2x^2 + 3x^3 + \ldots) \\
&= xA(x) + x^2(2.1 + 3.2x + \ldots) + x(1 + 2x + 3x^2 + \ldots) \\
&= xA(x) + x^2 \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) + x\frac{d}{dx}\left(\frac{1}{1-x}\right) \\
&= xA(x) + \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} \\
&= xA(x) + \frac{x^2 + x}{(1-x)^3}
\end{aligned}$$

so

$$A(x) = \frac{x^2 + x}{(1-x)^4} = \frac{x^2}{(1-x)^4} + \frac{x}{(1-x)^4}.$$

This means that a_n consists of two parts — the coefficient of x^{n-2} in $1/(1-x)^4$ and the coefficient of x^{n-1} in $1/(1-x)^4$. By what we established earlier, this is

$$\binom{n+1}{3} + \binom{n+2}{3} = \frac{n(n+1)(2n-1)}{6}.$$

If you were feeling masochistic, you might have used the same method to discover

$$\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

The problems

For these, time got away from me and I was unable to write up full solutions. Instead I've given the source of the problem (all but one are Putnam problems), so you can find the solution either (for pre-2000) in the appropriate book that's on reserve in the math library, or (for post-2000) by following the links on the course website.

1. Define a selfish set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.

   Source: Putnam 1996, B1. Notes: the anser is the nth Fibonacci number!

2. Define a sequence $(p_n)_{n \geq 1}$ recursively by $p_1 = 3$, $p_2 = 7$ and, for $n \geq 3$,

   $$p_n = 4 + p_{n-1} + 2p_{n-2} + \ldots + 2p_1$$

   (so, for example, $p_3 = 4 + p_2 + 2p_1 = 17$ and $p_4 = 4 + p_3 + 2p_2 + 2p_1 = 41$). Find a closed-form expression for p_n for general n.

Solution: The first few values are $p_1 = 3$, $p_2 = 7$, $p_3 = 17$, $p_4 = 41$, $p_3 = 17$, $p_4 = 41$. A pattern seems to be emerging: $p_n = 2p_{n-1} + p_{n-2}$, with $p_1 = 3$, $p_2 = 7$. We verify this by induction on n. It's certainly true for $n = 3$. For $n > 3$,

$$\begin{aligned} p_n &= 4 + p_{n-1} + 2p_{n-2} + 2p_{n-3} + \ldots + 2p_3 + 2p_2 + 2p_1 \\ &= (p_{n-1} + p_{n-2}) + 4 + p_{n-2} + 2p_{n-3} + \ldots + 2p_3 + 2p_2 + 2p_1 \\ &= (p_{n-1} + p_{n-2}) + p_{n-2} \quad \text{(induction)} \\ &= 2p_{n-1} + p_{n-2}, \end{aligned}$$

as required. With this new recurrence, it is easy to apply the method of generating functions, as described in the introduction, to get

$$p_n = \frac{(1+\sqrt{2})^{n+1}}{2} + \frac{(1-\sqrt{2})^{n+1}}{2}.$$

Source: This problem arose in my research. A *graph* is a collection of points, some pairs of which are joined by edges. A *Widom-Rowlinson* coloring of a graph is a coloring of the points using 3 colors, red, white and blue, in such a way that no point colored red is joined by an edge that is colored blue. I was looking at how many Widom-Rowlinson colorings there are of the graph P_n that consists of n points, numbered 1 up to n, with edges from 1 to 2, from 2 to 3, etc., up to from $n-1$ to n. It turns out that there are p_n such colorings, where p_n satisfies the first recurrence. In trying to find a closed form for p_n, I realized that p_n satisfies the Fibonacci-like recurrence described in the solution above, and so was able to solve for p_n explicitly using generating functions.

3. Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

   Source: Putnam 1993, A2.

4. Define a sequence by $a_k = k$ for $k = 1, 2, \ldots, 2006$ and

   $$a_{k+1} = a_k + a_{k-2005}$$

   for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

   Source: Putnam 2006, A3. Notes: the key points are 1) any recursive sequence of this kind can be extended both forward and backwards, and the resulting doubly infinite sequence, reduced to any modulus, is periodic.

5. The last question was clearly written with the years 2005 and 2006 in mind. Does the conclusion remain true for an arbitrary year? That is, fix $m \geq 1$. Define a sequence by $a_k = k$ for $k = 1, 2, \ldots, m+1$ and

   $$a_{k+1} = a_k + a_{k-m}$$

   for $k \geq m+1$. For which m is it true that the sequence has m consecutive terms each divisible by $m+1$?

   Solution: The solution to the previous problem goes through fine with general positive m.

   Source: An idle thought.

6. Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1 + x + x^2)^m$. Prove that for all integers $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor 2k/3 \rfloor} (-1)^i a_{k-i,i} \leq 1.$$

(Here $\lfloor a \rfloor$ denote the round-down of a to the nearest integer at or below a; so for example $\lfloor 3.4 \rfloor = 3$, $\lfloor 2.999 \rfloor = 2$ and $\lfloor 5 \rfloor = 5$.)

Source: Putnam 1997, B4.

7. Define $(a_n)_{n \geq 0}$ by

$$\frac{1}{1 - 2x - x^2} = \sum_{n \geq 0} a_n x^n.$$

Show that for each $n \geq 0$, there is an $m = m(n)$ such that $a_m = a_n^2 + a_{n+1}^2$.

Source: Putnam 1999, A3.