Problem Solving in Math (Math 43900) Fall 2013

Week seven (October 8) problems — inequalities

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A list of some of the most important general inequalities to know

Many Putnam problem involve showing that a particular inequality between two expressions holds always, or holds under certain circumstances. There are a huge variety of general inequalities between sets of numbers satisfying certain conditions, that are quite reasonable for you to quote as "well-known". I've listed some of them here, mostly without proofs. If you are interested in knowing more about inequalities, consider looking at the book *Inequalities* by Hardy, Littlewood and Pólya (QA 303 .H223i at the math library).

Squares are positive

Surprisingly many inequalities reduce to the obvious fact that $x^2 \geq 0$ for all real x, with equality iff $x = 0$. I'll highlight one example in what follows.

The triangle inequality

For reals x and y, $|x+y| \leq |x|+|y|$ (called the triangle inequality because it says that the distance travelled along the line in going from x to $-y$ — $|x+y|$ — does not decrease if we demand that we go through the intermediate point 0)

Arithmetic mean — Geometric mean — Harmonic mean inequality

For positive $a_1, \ldots, a_n$

$$\frac{n}{\frac{1}{a_1} + \ldots + \frac{1}{a_1}} \leq \sqrt[n]{a_1 \ldots a_n} \leq \frac{a_1 + \ldots + a_n}{n}$$

with equalities in both inequalities iff all a_i are equal. The three expressions above are the *harmonic mean*, the *geometric mean* and the *arithmetic mean* of the a_i.

For $n = 2$, here's a proof of the second inequality: $\sqrt{a_1 a_2} \leq (a_1 + a_2)/2$ iff $4a_1 a_2 \leq (a_1 + a_2)^2$ iff $a_1^2 - 2a_1 a_2 + a_2^2 \geq 0$ iff $(a_1 - a_2)^2 \geq 0$, which is true by the "squares are positive" inequality; there's equality all along iff $a_1 = a_2$.

For $n = 2$ the first inequality is equivalent to $\sqrt{a_1 a_2} \leq (a_1 + a_2)/2$.

Power means inequality

For a non-zero real r and positive $a_1, \ldots, a_n$ define

$$M^r(a_1, \ldots, a_n) = \left(\frac{a_1^r + \ldots + a_n^r}{n} \right)^{1/r},$$

and set $M^0(a_1, \ldots, a_n) = \sqrt[n]{a_1 \ldots a_n}$. For real numbers $r < s$,

$$M^r(a_1, \ldots, a_n) \leq M^s(a_1, \ldots, a_n)$$

with equality iff all a_i are equal.

Notice that $M^{-1}(a_1, \ldots, a_n)$ is the harmonic mean of the a_i's, and $M^1(a_1, \ldots, a_n)$ is their geometric mean, so this inequality generalizes the Arithmetic mean — Geometric mean — Harmonic mean inequality.

There is a weighted power means inequality: let $w_1, \ldots, w_n$ be positive reals that add to 1, and define

$$M_w^r(a_1, \ldots, a_n) = (w_1 a_1^r + \ldots + w_n a_n^r)^{1/r}$$

for non-zero real r, with $M_w^0(a_1, \ldots, a_n) = a_1^{w_1} \ldots a_n^{w_n}$. For real numbers $r < s$,

$$M_w^r(a_1, \ldots, a_n) \leq M_w^s(a_1, \ldots, a_n).$$

(This reduces to the power means inequality when all $w_i = 1/n$.)

Cauchy-Schwarz-Bunyakovsky inequality

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be real numbers. We have

$$(x_1 y_1 + \ldots + x_n y_n)^2 \leq \left(x_1^2 + \ldots + x_n^2\right)\left(y_1^2 + \ldots + y_n^2\right).$$

Equality holds if one of the sequences $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ is identically zero. If both are not identically zero, then there is equality iff there is some real number λ such that $x_i = \lambda y_i$ for each i.

This is really a very general inequality: if you are familiar with inner products from linear algebra, the CauchySchwarz-Bunyakovsky inequality really says that if $\mathbf{x}$, $\mathbf{y}$ are vectors in an inner product space (over either the reals or the complex numbers) then

$$|\langle \mathbf{x}, \mathbf{y} \rangle|^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle.$$

Equivalently

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq ||\mathbf{x}||\ ||\mathbf{y}||.$$

There is equality iff $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

Hölder's inequality

Fix $p > 1$ and define q by $1/p + 1/q = 1$. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be real numbers. We have

$$|x_1 y_1 + \ldots + x_n y_n| \leq (|x_1|^p + \ldots + |x_n|^p)^{1/p} \left(|y_1|^q + \ldots + |y_n|^2\right)^{1/q}.$$

Notice that Hölder becomes CauchySchwarz-Bunyakovsky in the case $p = 2$.

Chebyshev's sum inequality

If $a_1 \geq \ldots \geq a_n$ and $b_1 \geq \ldots \geq b_n$ are sequences of reals, then

$$\frac{a_1 b_1 + \ldots + a_n b_n}{n} \geq \left(\frac{a_1 + \ldots + a_n}{n}\right)\left(\frac{b_1 + \ldots + b_n}{n}\right).$$

The same holds if $a_1 \leq \ldots \leq a_n$ and $b_1 \leq \ldots \leq b_n$; if either $a_1 \geq \ldots \geq a_n$ and $b_1 \leq \ldots \leq b_n$ or $a_1 \leq \ldots \leq a_n$ and $b_1 \geq \ldots \geq b_n$, then

$$\frac{a_1 b_1 + \ldots + a_n b_n}{n} \leq \left(\frac{a_1 + \ldots + a_n}{n}\right)\left(\frac{b_1 + \ldots + b_n}{n}\right).$$

The rearrangement inequality

If $a_1 \leq \ldots \leq a_n$ and $b_1 \leq \ldots \leq b_n$ are sequences of reals, and $a_{\pi(1)}, \ldots, a_{\pi(n)}$is a permutation (rearrangement) of $a_1 \leq \ldots \leq a_n$, then

$$a_n b_1 + \ldots + a_1 b_n \leq a_{\pi(1)} b_1 + \ldots + a_{\pi(n)} b_n \leq a_1 b_1 + \ldots + a_n b_n.$$

If $a_1 < \ldots < a_n$ and $b_1 < \ldots < b_n$, then there is equality in the first inequality iff π is the reverse permutation $\pi(i) = n + 1 - i$, and there is equality in the second inequality iff π is the identity permutation $\pi(i) = i$.

Jensen's inequality

A real function $f(x)$ is *convex* on the interval $[c, d]$ if for all $c \leq a < b \leq d$, the line segment joining $(a, f(a))$ to $(b, f(b))$ lies entirely above the graph $y = f(x)$ on the interval (a, b), or equivalently, if for all $0 \leq t \leq 1$ we have

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b).$$

If $f(x)$ is convex on the interval $[c, d]$, and $c \leq a_1 \leq \ldots \leq a_n \leq d$, then

$$f\left(\frac{a_1 + \ldots + a_n}{n}\right) \leq \frac{f(a_1) + \ldots + f(a_n)}{n}$$

(note that when $n = 2$, this is just the definition of convexity).

We say that $f(x)$ is *concave* on $[c, d]$ if for all $c \leq a < b \leq d$, and for all $0 \leq t \leq 1$, we have

$$f((1-t)a + tb) \geq (1-t)f(a) + tf(b).$$

If $f(x)$ is concave on the interval $[c, d]$, and $c \leq a_1 \leq \ldots \leq a_n \leq d$, then

$$f\left(\frac{a_1 + \ldots + a_n}{n}\right) \geq \frac{f(a_1) + \ldots + f(a_n)}{n}.$$

As an example, consider the convex function $f(x) = x^2$; for this function Jensen says that

$$\left(\frac{a_1 + \ldots + a_n}{n}\right)^2 \leq \frac{a_1^2 + \ldots + a_n^2}{n},$$

which is equivalent to the powers means inequality $M^1(a_1, \ldots, a_n) \leq M^2(a_1, \ldots, a_n)$; and when $f(x) = -\ln x$ we get

$$\sqrt[n]{a_1 \ldots a_n} \leq \frac{a_1 + \ldots + a_n}{n},$$

the AM-GM inequality.

Two miscellaneous comments

1) Maximization/minimization problems are often problems about inequalities in disguise. For example, to find the minimum of $f(a, b)$ as (a, b) ranges over a set R, it is enough to first guess that the minimum is m, then find an $(a, b) \in R$ with $f(a, b) = m$, and then use inequalities to show that $f(a, b) \geq m$ for all $(a, b) \in R$.

2) If an expression is presented as a sum of n squares, it is sometimes helpful to think of it as the (square of the) distance between two points in n dimensional space, and then think of the problem geometrically.

Some warm-up problems

You should find that these are all fairly easy to prove by direct applications of an appropriate inequality.

1. $n! < \left(\frac{n+1}{2}\right)^n$ for $n = 2, 3, 4, \ldots$.
2. $\sqrt{3(a+b+c)} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}$ for positive a, b, c.
3. Minimize $x_1 + \ldots + x_n$ subject to $x_i \geq 0$ and $x_1 \ldots x_n = 1$.
4. Minimize
$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}$$
subject to $x, y, z \geq 0$ and $xyz = 1$.
5. If triangle has side lengths a, b, c and opposite angles (measured in radians) A, B, C, then
$$\frac{aA + bB + cC}{a+b+c} \geq \frac{\pi}{3}.$$
6. Identify which is bigger:
$$1999!^{(2000)} \quad \text{or} \quad 2000!^{(1999)}.$$
(Here $n!^{(k)}$ indicates iterating the factorial function k times, so for example $4!^{(2)} = 24!$.)
7. Identify which is bigger:
$$1999^{1999} \quad \text{or} \quad 2000^{1998}.$$
8. Minimize
$$\frac{\sin^3 x}{\cos x} + \frac{\cos^3 x}{\sin x}$$
on the interval $0 < x < \pi/2$.

The problems

These are old Putnam problems of varying hardness.

1. Show that for non-negative reals $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$,
$$(a_1 \ldots a_n)^{1/n} + (b_1 \ldots b_n)^{1/n} \leq ((a_1 + b_1) \ldots (a_n + b_n))^{1/n}.$$
2. Minimize
$$(u-v)^2 + \left(\sqrt{2-u^2} - \frac{9}{v}\right)^2$$
in the range $0 < u < \sqrt{2}$, $v > 0$.
3. For positive integers m, n, show
$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

4. Maximize $x^3 - 3x$ subject to $x^4 + 36 \leq 13x^2$.

5. Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} \leq 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

6. Let f be a real function with a continuous third derivative such that $f(x), f'(x), f''(x)$ and $f'''(x)$ are positive for all x. Suppose that $f'''(x) \leq f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.

7. Maximize

$$\int_0^y \sqrt{x^4 + (y - y^2)^2}\, dx$$

on the interval $[0, 1]$.