Problem Solving in Math (Math 43900) Fall 2013

Week seven (October 8) solutions

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Some warm-up problems

1. $n! < \left(\frac{n+1}{2}\right)^n$ for $n = 2, 3, 4, \ldots$.

   Solution: Use the geometric mean - arithmetic mean inequality, with $(a_1, \ldots, a_n) = (1, \ldots, n)$.

2. $\sqrt{3(a+b+c)} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}$ for positive a, b, c.

   Solution: Use the power means inequality, with $(a_1, a_2, a_3) = (a, b, c)$ and $r = 1/2, s = 1$.

3. Minimize $x_1 + \ldots + x_n$ subject to $x_i \geq 0$ and $x_1 \ldots x_n = 1$.

   Solution: Guess: the minimum is n, achieved when all $x_1 = 1$. Then use geometric mean - arithmetic mean inequality to show

$$\left(\frac{x_1 + \ldots + x_n}{n}\right) \geq \sqrt[n]{x_1 \ldots x_n} = 1$$

   for positive x_i satisfying $x_1 \ldots x_n = 1$.

4. Minimize

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}$$

   subject to $x, y, z \geq 0$ and $xyz = 1$.

   Solution: Apply Cauchy-Schwartz with the vectors $\left(\sqrt{y+z}, \sqrt{z+x}, \sqrt{x+y}\right)$ and $\left(\frac{x}{\sqrt{y+z}}, \frac{y}{\sqrt{z+x}}, \frac{z}{\sqrt{x+y}}\right)$ to get

$$(x+y+z)^2 \leq \left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right) 2\left(x+y+z\right),$$

   leading to

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{x+y+z}{2}.$$

   By the AM-GM inequality,

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz} = 1,$$

   so

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}.$$

   This lower bound can be achieved by taking $x = y = z = 1$, so the minimum is $3/2$.

5. If triangle has side lengths a, b, c and opposite angles (measured in radians) A, B, C, then
$$\frac{aA + bB + cC}{a + b + c} \geq \frac{\pi}{3}.$$

   Solution: Assume, without loss of generality, that $a \leq b \leq c$. Then also $A \leq B \leq C$, so by Chebychev,
$$\frac{aA + bB + cC}{3} \geq \left(\frac{a+b+c}{3}\right)\left(\frac{A+B+C}{3}\right) = \left(\frac{a+b+c}{3}\right)\frac{\pi}{3},$$
   from which the result follows.

6. Identify which is bigger:
$$1999!^{(2000)} \quad \text{or} \quad 2000!^{(1999)}.$$
   (Here $n!^{(k)}$ indicates iterating the factorial function k times, so for example $4!^{(2)} = 24!$.)

   Solution: For $n \geq 1$, $n!$ is increasing in n ($1 \leq n < m$ implies $n! < m!$). So, starting from the easy
$$1999! > 2000,$$
   apply the factorial function 1999 more times to get
$$1999!^{(2000)} > 2000!^{(1999)}.$$

7. Identify which is bigger:
$$1999^{1999} \quad \text{or} \quad 2000^{1998}.$$

   Solution: Consider $f(x) = (1999 - x)\ln(1999 + x)$. We have $e^{f(0)} = 1999^{1999}$ and $e^{f(1)} = 2000^{1998}$, so we want to see what f does on the interval $[0, 1]$: increase or decrease? The derivative is
$$f'(x) = -\ln(1999 + x) + \frac{1999 - x}{1999 + x},$$
   which is negative on $[0, 1]$ (since, for example,
$$\frac{1999 - x}{1999 + x} \leq 1 = \ln e < \ln(1999 + x)$$
   on that interval). So
$$2000^{1998} < 1999^{1999}.$$

8. Minimize
$$\frac{\sin^3 x}{\cos x} + \frac{\cos^3 x}{\sin x}$$
   on the interval $0 < x < \pi/2$.

   Solution: We can use the rearrangement inequality on the pairs $\left(\sin^3 x, \cos^3 x\right)$ (which satisfies $\sin^3 x \leq \cos^3 x$ on $[0, \pi/4]$, and $\sin^3 x \geq \cos^3 x$ on $[\pi/4, \pi/2]$), and $(1/\cos x, 1/\sin x)$ (which also satisfies $1/\cos x \leq 1/\sin x$ on $[0, \pi/4]$, and $1/\cos x \geq 1/\sin x$ on $[\pi/4, \pi/2]$), to get
$$\frac{\sin^3 x}{\cos x} + \frac{\cos^3 x}{\sin x} \geq \frac{\sin^3 x}{\sin x} + \frac{\cos^3 x}{\cos x} = \sin^2 x + \cos^2 x = 1$$
   on the whole interval. Since 1 can be achieved (at $x = \pi/4$) the minimum is 1.

Source: These problems were all taken from a Northwestern Putnam prep problem set.

The Putnam problems

1. Show that for non-negative reals $a_1, \dots, a_n$ and $b_1, \dots, b_n$,

$$(a_1 \dots a_n)^{1/n} + (b_1 \dots b_n)^{1/n} \le ((a_1 + b_1) \dots (a_n + b_n))^{1/n}.$$

Solution: If any a_i is 0, the result is trivial, so we may assume all $a_i > 0$. Dividing through by $(a_1 \dots a_n)^{1/n}$, the inequality becomes

$$1 + (c_1 \dots c_n)^{1/n} \le ((1 + c_1) \dots (1 + c_n))^{1/n}$$

for $c_i \ge 0$. Raising both sides to the power n, this is the same as

$$\sum_{k=0}^{n} \binom{n}{k} (c_1 \dots c_n)^{k/n} \le \sum_{k=0}^{n} e_k$$

where e_k is the sum of the products of the c_i's, taken k at a time. So it is enough to show that for each k,

$$\binom{n}{k} (c_1 \dots c_n)^{k/n} \le \sum_{A \subseteq \{1,\dots,n\},\ |A|=k} \prod_{i \in A} c_i.$$

We apply the AM-GM inequality to the numbers $\prod_{i \in A} c_i$ as A ranges over all subsets of size k of $\{1, \dots, n\}$. Note that each a_i appears exactly $\binom{n-1}{k-1}$ times in all these numbers. So we we get

$$(c_1 \dots c_n)^{\binom{n-1}{k-1}/\binom{n}{k}} \le \frac{\sum_{A \subseteq \{1,\dots,n\},\ |A|=k} \prod_{i \in A} c_i}{\binom{n}{k}}.$$

Since $\binom{n-1}{k-1}/\binom{n}{k} = k/n$, this is the same as

$$(c_1 \dots c_n)^{k/n} \le \frac{\sum_{A \subseteq \{1,\dots,n\},\ |A|=k} \prod_{i \in A} c_i}{\binom{n}{k}},$$

which is exactly what we wanted to show.

Source: Putnam 2003 A2

2. Minimize

$$(u - v)^2 + \left(\sqrt{2 - u^2} - \frac{9}{v} \right)^2$$

in the range $0 < u < \sqrt{2}$, $v > 0$.

Solution: The expression to be minimized is the (square of the) distance between a point of the form $(u, \sqrt{2 - u^2})$ on $0 < u < \sqrt{2}$, and a point of the form $(v, 9/v)$ on $v > 0$; in other words, we are looking for the (square of the) distance between the circle $x^2+y^2 = 2$ in the first quadrant and the hyperbola $xy = 9$ in the same quadrant. By symmetry, it strongly seems that the two closed points are $(3, 3)$ on the hyperbola and $(1, 1)$ on the circle (squared distance 8). To prove that this is the minimum, note that the tangent lines to the two curves at those two points are parallel, that the distance between them at these points is the perpendicular distance between the two tangent lines, and that the hyperbola (in the first quadrant) lies completely above its tangent line, while the circle (in the first quadrant) lies completely below its tangent line; so the distance between ant other two points is at least the distance between the two tangent lines.

Source: Putnam 1984 B2

3. For positive integers m, n, show

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

Solution: Rearranging, this is the same as

$$\binom{m+n}{m}\left(\frac{m}{m+n}\right)^m\left(\frac{n}{m+n}\right)^n < 1.$$

This suggests looking at the binomial expansion

$$\left(\frac{m}{m+n}+\frac{n}{m+n}\right)^{m+n}.$$

The whole binomial expansion sums to 1; one term of the expansion is

$$\binom{m+n}{m}\left(\frac{m}{m+n}\right)^m\left(\frac{n}{m+n}\right)^n.$$

Since all terms are strictly positive, we get the required inequality.

Source: Putnam 2004 B2

4. Maximize $x^3 - 3x$ subject to $x^4 + 36 \leq 13x^2$.

Solution: The inequality $x^4 + 36 \leq 13x^2$ factors as $(x+3)(x+2)(x-2)(x-3) \leq 0$, so $-3 \leq x \leq -2$ or $2 \leq x \leq 3$. The derivative of $x^3 - 3x$ is $3x^2 - 3 = 3(x+1)(x-1)$, and so we see that $x^3 - 3x$ is increasing on both $[-3, -2]$ and $[2, 3]$. The maximum is therefore the maximum of $f(-2)$ and $f(3)$, which is $f(3) = 18$.

Source: Putnam 1986 A1

5. Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} \leq 1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

Solution: We estimate the integral of $\ln x$, which is convex and hence easy to estimate. Take the integral from 1 to $2n-1$. This is less than $2(\ln 3 + \ln 5 + \ldots + \ln(2n-1))$. But the antiderivative of $\ln x$ is $x\ln x - x$, so the integral evaluates to $(2n-1)\ln(2n-1) - 2n + 2$. Hence $(2n-1)\ln(2n-1) - (2n-1) < (2n-1)\ln(2n-1) - 2n + 2 < 2(\ln 3 + \ln 5 + \ldots + \ln(2n-1))$. Exponentiating gives the right-hand inequality.

Similarly, the integral from e to $2n+1$ is greater than $2(\ln 3 + \ln 5 + \ldots + \ln(2n-1))$, and an explicit evaluation of the antiderivative here leads to the right-hand side of the inequality. The choice of lower bound e for the integral here is just the righ thing to make the computations work out nicely.

Source: Putnam 1996 B2

6. Let f be a real function with a continuous third derivative such that $f(x), f'(x), f''(x)$ and $f'''(x)$ are positive for all x. Suppose that $f'''(x) \leq f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.

   Solution: See Kedlaya, Poonen & Vakil, *The William Lowell Putnam Mathematical Competition 1985—2000; Problems, Solutions and Commentary*, page 272. This was one of the hardest Putnam Competition problems ever — of the top 205 performers in the 1999 Putnam, only one contestant received a score of more than 0 for this problem, and that score was 2!

   Source: Putnam 1999 B4

7. Maximize
$$\int_0^y \sqrt{x^4 + (y-y^2)^2}\ dx$$
   on the interval $[0,1]$.

   Solution: Let $f(y) = \int_0^y \sqrt{x^4 + (y-y^2)^2}\ dx$. At $y=0$, $f(y)=0$, and at $y=1$, $f(y) = \int_0^1 x^2\ dx = 1/3$. Also, f is non-negative on $[0,1]$. It looks like it will be impossible to evaluate $f(y)$ at any value other than $y = 0, 1$; this strongly leads to the suspicion that the maximum is at $y=1$. To prove this, we need to show that $f'(y) \geq 0$ on the interval $[0,1]$. A process by which this can be done is outlined in Kedlaya, Poonen & Vakil, *The William Lowell Putnam Mathematical Competition 1985—2000; Problems, Solutions and Commentary*, page 138.

   Here's a (really) slick solution from the same source: Since x^2 and $y-y^2$ are positive in the range $0 \leq x \leq 1$, $0 \leq y \leq 1$, we have
$$x^4 + (y-y^2)^2 \leq x^4 + 2x^2(y-y^2) + (y-y^2)^2 = (x^2 + (y-y^2))^2,$$
   and so
$$\int_0^y \sqrt{x^4 + (y-y^2)^2}\ dx \leq \int_0^y \left(x^2 + (y-y^2)\right)\ dx = y^2 - \frac{2}{3}y^3.$$
   The derivative of $y^2 + (2/3)y^3$ is $2y - 2y^2 = 2y(1-y)$ which is non-negative on $[0,1]$, so the maximum is at $y=1$, where it is $1/3$. So
$$\int_0^y \sqrt{x^4 + (y-y^2)^2}\ dx \leq 1/3;$$
   and $1/3$ is achievable, at $y=1$.

   Source: Putnam 1991 A5