Problem Solving in Math (Math 43900) Fall 2013

Week eight (October 15) problems — a mock Putnam

Instructor: David Galvin

Think of this as a "mock Putnam". The questions are intended to go from easier to harder, as in the real Putnam papers.

Homework: One important Putnam skill is writing clear, clean, complete and correct solutions. At our next meeting (Tuesday October 29) I want you to hand in a solution to one of these problems (your choice), so I can give you feedback on your presentation. Some things to bear in mind as you present your solution:

1. Write clearly and neatly, using full sentences.
2. Explain each logical step & deduction.
3. If you use a fact that you feel should be common knowledge (and so doesn't need a proof), state clearly what the fact is, and that you are assuming it as well-known.
4. At the end, read back over the question to make sure you are answering exactly what was asked.
5. ~~Don't forget to phrase your answer in the form of a question!~~

The problems

1. Let $f(n)$ denote the nth term in the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots$ (a block of one 1, followed by two 2's, three 3's, etc.). Find $f(2013)$, and (with proof) a simple general formula for $f(n)$ (which may use the floor or ceiling function: $\lfloor x \rfloor$ is the largest integer not bigger than x, and $\lceil x \rceil$ is the smallest integer not smaller than x).

2. Let S be a set, and $\star$ a binary operation on S satisfying $x \star x = x$ for all $x \in S$, and $(x \star y) \star z = (y \star z) \star x$ for all $x, y, z \in S$. Prove that $\star$ is commutative (that is, that $x \star y = y \star x$ for all $x, y \in S$).

3. For which real numbers c is it true that
$$\frac{1}{2}\left(e^x + e^{-x}\right) \le e^{cx^2}$$
for all real numbers x? Carefully justify your answer

4. Find, with proof, a simple formula for the value of the sum
$$\sum_{k=n}^{2n} \binom{k}{n} 2^{2n-k}.$$

5. Let $s(n)$ be the number of ordered pairs (a, b) from $\{1, 2, \ldots, n\}$ such that $a + b$ is a perfect square (so, for example, $s(5) = 6$, the pairs being $(1, 1), (1, 3), (2, 2), (3, 1), (4, 5)$ and $(5, 4)$). Prove that the limit $\lim_{n\to\infty} s(n)n^{-3/2}$ exists, and find it. [Express your final answer in the form $r(\sqrt{s} - t)$, where s and t are integers and r is rational.]

6. Let $x_1, x_2, \ldots,$ and $y_1, y_2, \ldots,$ be sequences of positive real numbers satisfying

$$y_1 \geq y_2 \geq y_3 \geq \ldots$$

and

$$x_1 x_2 \ldots x_k \geq y_1 y_2 \ldots y_k$$

for all $k = 1, 2, 3, \ldots$. Prove that

$$x_1 + x_2 + \ldots + x_k \geq y_1 + y_2 + \ldots + y_k$$

for all $k = 1, 2, 3, \ldots$.