Problem Solving in Math (Math 43900) Fall 2013

Week nine (October 29) problems — Binomial coefficients

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Binomial coefficients crop up quite a lot in Putnam problems. This handout presents some ways of thinking about them.

Introduction

The *binomial coefficient* $\binom{n}{k}$, with $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, can be defined many ways; possibly the most helpful definition from the point of view of problem-solving is the following combinatorial one:

$$\binom{n}{k} \text{ is the number of subsets of size } k \text{ of a set of size } n.$$

In particular, this definition immediately tells us that for all $n \geq 0$ we have $\binom{n}{k} = 0$ if $k > n$ or if $k < 0$, and that $\binom{n}{0} = \binom{n}{n} = 1$ (and so in particular $\binom{0}{0} = 1$).

The binomial coefficients can also be defined by a recurrence relation: for $n \geq 1$, and all $k \in \mathbb{Z}$, we have the recurrence

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \qquad (\textit{Pascal's identity})$$

with initial conditions $\binom{0}{k} = 0$ if $k \neq 0$, and $\binom{0}{0} = 1$. To see that this recurrence does indeed generate the binomial coefficients, think about the combinatorial interpretation: the subsets of $\{1, \ldots, n\}$ of size k ($\binom{n}{k}$ of them) partition into those that don't include element n ($\binom{n-1}{k}$ of them) and those that do include element n ($\binom{n-1}{k-1}$ of them). The recurrence allows us to quickly compute small binomial coefficients via *Pascal's triangle*: the zeroth row of the triangle has length one, and consists just of the number 1. Below that, the first row has two 1's, one below and to the left of the 1 in the zeroth row, and one below and to the right of the 1 in the zeroth row. The second row has three entries, a 1 below and to the left of the leftmost 1 in the first row, a 1 below and to the right of the rightmost 1 in the first row, and in the center a 2. Each subsequent row contains one more entry than the previous row, starting with a 1 below and to the left of the leftmost 1 in the previous row, ending with a 1 below and to the right of the rightmost 1 in the previous row, and with all other entries being the sum of the two entries in the previous row above to the left and to the right of the entry being considered (see the picture below).

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

etc.

Pascal's triangle in numbers

The kth entry in row k (counting from 0 rather than 1 both down and across) is then $\binom{n}{k}$ (this is just a restatement of Pascal's identity) (see the picture below).

$$
\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}
$$

etc.

Pascal's triangle symbolically

Finally, there is a algebraic expression for $\binom{n}{k}$, that makes sense for all $n, k \geq 0$, using the factorial function (defined combinatorially as the number of ways of arranging n distinct objects in order, and algebraically by $n! = n(n-1)(n-2)\ldots(3)(2)(1)$ for $n \geq 1$, with $0! = 1$):

$$
\binom{n}{k} = \frac{n(n-1)\ldots(n-(k-1))}{k!} = \frac{n!}{k!(n-k)!}.
$$

To see this, note that $n(n-1)\ldots(n-(k-1))$ is fairly evidently the number of *ordered lists* of k distinct elements from $\{1, \ldots, n\}$ (often referred to in textbooks as "permutations of n items taken k at a time" — ugh). When the ordered lists are turned into (unordered) subsets, each subset appears $k!$ times (once for each of the $k!$ ways of putting k distinct objects into an ordered list), so we need to divide the ordered count by $k!$ to get the unordered count.

When dealing with binomial coefficients, it is very helpful to bear all three definitions in mind, but in particular the first two.

Identities

The binomial coefficients satisfy a staggering number of identities. The simplest of these are easily understood using either the combinatorial or algebraic definitions; for the more involved ones, that include sums, the algebraic definition is usually next to useless, and often the easiest way to prove the identity is combinatorially, by showing that both sides of the identity count the same thing in different ways (illustration below), though it is often possible also to prove these identities by induction, using the recurrence relation. Another approach that is helpful is that of generating functions.

Here are some of the basic binomial coefficient identities:

1. (Symmetry)
$$\binom{n}{k} = \binom{n}{n-k}$$
(Proof: trivial from the algebraic definition; combinatorially, left-hand side counts selection of subsets of size k from a set of size n, by naming the selected elements; right-hand side also counts selection of subsets of size k from a set of size n, this time by naming the *un*selected elements).

2. (Lower summation)
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$
(Proof: close to impossible using the algebraic definition; combinatorially, very straightforward: left-hand side counts the number of subsets of a set of size n, by first deciding the size of the subset, and then choosing the subset itself; right-hand side also counts the number of subsets of a set of size n, by going through the n elements one-by-one and deciding whether they are in the subset or not).

3. (Upper summation)
$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}.$$

4. (Parallel summation)
$$\sum_{k=0}^{n} \binom{m+k}{k} = \binom{n+m+1}{n}.$$

5. (Square summation)
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

6. (Vandermonde identity, or Vandermonde convolution)

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{n+m}{r}.$$

The binomial theorem

This is the most important identity involving binomial coefficients: for all real x and y, and $n \geq 0$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

This can be proved by induction using Pascal's identity, but the proof is quite awkward. Here's a nice combinatorial proof. First, note that the identity is trivial if either $x = 0$ or $y = 0$, so we may assume $x, y \neq 0$. Dividing through by x^n, the identity is the same as

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k.$$

We will prove this combinatorially when z is a positive integer. The left-hand side counts the number of words of length n from alphabet $\{0, 1, 2, \ldots, z\}$, by deciding on the letters one after the other. The right-hand side also counts the number of words of length n from alphabet $\{0, 1, 2, \ldots, z\}$, as follows: first decide how many of the letters of the word are from $\{1, \ldots, z\}$ (this is the k of the summation). Next, decide the location of these k letters (this is the $\binom{n}{k}$). Finally, decide what specific letters go into those spots, one after another (this is the z^k) (note that the remaining $n-k$ letters must all be 0's).

This only shows the identity for positive integer z. But now we use the fact that both the right-hand and left-hand sides are polynomials of degree n, so if they agree at $n+1$ different values of z, they must agree at all values of z (otherwise, their difference is a not-identically-zero polynomial of degree at most n with $n+1$ distinct roots, an impossibility). And indeed, the two sides agree not just at $n+1$ different values of z, but at infinitely many (all positive integers z). So from the combinatorial argument that shows that the two sides are equal for positive integers z, we infer that they are equal for *all* real z. This argument is often called the *polynomial principle*.

Compositions and weak compositions

A *composition* of a positive integer n into k parts is a vector $(x_1, x_2, \ldots, x_k)$, with each entry a strictly positive integer, and with $\sum_{i=1}^{k} x_i = n$. Foe example, $(2, 1, 1, 3)$ is a composition of 7, as is $(1, 3, 1, 2)$; and, because a composition is a *vector* (ordered list), these two are considered different compositions.

A *weak composition* of a positive integer n into k parts is a vector $(x_1, x_2, \ldots, x_k)$, with each entry a non-negative (possibly 0) integer, and with $\sum_{i=1}^{k} x_i = n$. Foe example, $(2, 0, 1, 3)$ is a weak composition of 6, but not a composition.

How many weak compositions of n are there, into k parts? Put down $n+k-1$ stars in a row. Choose $k-1$ of them to turn into bars. The resulting arrangement of stars-and-bars encodes a weak composition of n into k parts — the number of stars before the first bar is x_1, the number of stars between the first and second bar is x_2, and so on, up to the number of stars after the last bar, which is x_k (notice that only $k-1$ bars are needed to determine k intervals of stars).

Conversely, every weak composition of n into k parts is encoded by one such selection of $k-1$ bars from the initial list of $n+k-1$ stars. For example, the configuration $\star\star||\star|\star\star\star$ encodes the weak composition $(2,0,1,3)$ of 6 into 4 parts. So, the number of weak compositions of n into k parts is a binomial coefficient, $\binom{n+k-1}{k-1}$.

How many compositions of n are there, into k parts? Each such composition $(x_1, x_2, \ldots, x_k)$ gives rise to a weak composition $(x_1-1, x_2-1, \ldots, x_k-1)$ of $n-k$ into k parts, and all weak composition of $n-k$ into k parts are achieved by this process, So, the number of compositions of n into k parts is the same as the number of weak compositions of $n-k$ into k parts, which is $\binom{(n-k)+k-1}{k-1} = \binom{n-1}{k-1}$.

For example: I like plain cake, chocolate cake, blueberry cake and pumpkin cake donuts from Dunkin' Donuts. In how many different ways can I buy a dozen donuts that I like? I must buy x_1 plain, x_2 chocolate, x_3 blueberry and x_4 pumpkin, with $x_1+x_2+x_3+x_4=12$, and with each x_i a non-negative integer (possibly 0). So the number of different purchases I can make is the number of weak compositions of 12 into 4 parts, so $\binom{15}{4} = 1365$.

Easy warm-up problems

1. Give a combinatorial proof of the upper summation identity.

2. Give a combinatorial proof of the parallel summation identity.

3. Give a combinatorial proof of the square summation identity.

4. Give a combinatorial proof of the Vandermonde identity.

5. Evaluate
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}$$
for $n \geq 1$.

Harder warm-up problems

1. The *kth falling power* of x is $x^{\underline{k}} = x(x-1)(x-2)\ldots(x-(k-1))$. Prove that for all real x, y, and all $n \geq 1$,
$$(x+y)^{\underline{n}} = \sum_{k=0}^{n}\binom{n}{k}x^{\underline{n-k}}y^{\underline{k}}.$$

2. The *kth rising power* of x is $x^{\overline{k}} = x(x+1)(x+2)\ldots(x+(k-1))$. Prove that for all real x, y, and all $n \geq 1$,
$$(x+y)^{\overline{n}} = \sum_{k=0}^{n}\binom{n}{k}x^{\overline{n-k}}y^{\overline{k}}.$$

3. Evaluate
$$\sum_{k=0}^{2n}(-1)^k k^n\binom{2n}{k}$$
for $n \geq 1$.

4. Evaluate

$$\sum_{k=0}^{n} F_{k+1}\binom{n}{k}$$

for $n \geq 0$, where $F_1, F_2, F_3, F_4, F_5, \ldots$ are the Fibonacci numbers $1, 1, 2, 3, 5, \ldots,$.

5. (a) Let a_n be the number of 0-1 strings of length n that do not have two consecutive 1's. Find a recurrence relation for a_n (starting with initial conditions $a_0 = 1$, $a_1 = 2$).

   (b) Let $a_{n,k}$ be the number of 0-1 strings of length n that have exactly k $1'$ and that do not have two consecutive 1's. Express $a_{n,k}$ as a (single) binomial coefficient.

   (c) Use the results of the previous two parts to give a combinatorial proof (showing that both sides count the same thing) of the identity

$$F_n = \sum_{k\geq 0}\binom{n-k-1}{k}$$

   where F_n is the nth Fibonacci number (as defined in the last question).

Problems

1. Show that for every $n, m \geq 0$,

$$\int_0^1 x^n(1-x)^m = \frac{1}{(n+m+1)\binom{n+m}{n}}.$$

2. Show that the coefficient of x^k in $(1+x+x^2+x^3)^n$ is

$$\sum_{j=0}^{k}\binom{n}{j}\binom{n}{k-2j}.$$

3. Let r, s and t be integers with $r, s \geq 0$ and $r+s \leq t$. Prove that

$$\sum_{i=0}^{s}\frac{\binom{s}{i}}{\binom{t}{r+i}} = \frac{t+1}{(t+1-s)\binom{t-s}{r}}.$$

4. Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

   is an integer for all pairs of integers $n \geq m \geq 1$.

5. For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, x_22, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \ldots + |x_n| \leq m$. Show that $f(m,n) = f(n,m)$. (In other words, the number of points in the ℓ_1 ball of radius m in $\mathbb{R}^n$ is the same as the number of points in the ℓ_1 ball of radius n in $\mathbb{R}^m$.)