Problem Solving in Math (Math 43900) Fall 2013

Week nine (October 29) solutions

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Easy warm-up problems

1. Give a combinatorial proof of the upper summation identity ($\sum_{m=k}^{n}\binom{m}{k} = \binom{n+1}{k+1}$).

   Solution: RHS is number of subsets of $\{1,\ldots,n+1\}$ of size $k+1$, counted directly. LHS counts same, by first specifying largest element in subset (if largest element is $k+1$, remaining k must be chosen from $\{1,\ldots,k\}$, $\binom{k}{k}$ ways; if largest element is $k+2$, remaining k must be chosen from $\{1,\ldots,k+1\}$, $\binom{k+1}{k}$ ways; etc.).

2. Give a combinatorial proof of the parallel summation identity ($\sum_{k=0}^{n}\binom{m+k}{k} = \binom{n+m+1}{n}$.).

   Solution: RHS is number of subsets of $\{1,\ldots,n+m+1\}$ of size n, counted directly. LHS counts same, by first specifying the smallest element *not* in subset (if smallest missed element is 1, all n elements must be chosen from $\{2,\ldots,n+m+1\}$, $\binom{m+n}{n}$ ways, the $k=n$ term; if smallest missed element is 2, then 1 is in subset and remaining $n-1$ elements must be chosen from $\{3,\ldots,n+m+1\}$, $\binom{m+n-1}{n-1}$ ways, the $k=n-1$ term; etc., down to: if smallest missed element is $n+1$, then $\{1,\ldots,n\}$ is in subset and remaining 0 elements must be chosen from $\{n+2,\ldots,k+1\}$, $\binom{m+0}{0}$ ways, the $k=0$ term).

3. Give a combinatorial proof of the square summation identity ($\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$).

   Solution: RHS is number of subsets of $\{\pm 1,\ldots,\pm n\}$ of size n, counted directly. LHS counts same, by first specifying k, the number of positive elements chosen, then selecting k positive elements ($\binom{n}{k}$ ways), then selecting the k negative elements that are *not* chosen (so the $n-k$ that are, for n in total) ($\binom{n}{k}$ ways).

4. Give a combinatorial proof of the Vandermonde identity ($\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k} = \binom{n+m}{r}$).

   Solution: Let $A = \{x_1,\ldots,x_m\}$ and $B = \{y_1,\ldots,y_n\}$ be disjoint sets. RHS is number of subsets of $A\cup B$ of size r, counted directly. LHS counts same, by first specifying k, the number of elements chosen from A, then selecting r elements from A ($\binom{m}{k}$ ways), then selecting the remaining $r-k$ elements from B ($\binom{n}{r-k}$ ways).

5. Evaluate

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}$$

   for $n \geq 1$.

Solution: Applying the binomial theorem with $x = 1, y = 1$ get

$$0 = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k}(-1)^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

so the sum is 0.

Harder warm-up problems

1. The *kth falling power* of x is $x^{\underline{k}} = x(x-1)(x-2)\dots(x-(k-1))$. Prove that for all real x, y, and all $n \geq 1$,

$$(x+y)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\underline{n-k}} y^{\underline{k}}.$$

Solution: Here's a combinatorial proof. Let x and y be positive integers. The number of words in alphabet $\{1, \dots, x\} \cup \{x_1, \dots, x+y\}$ of length n with no two repeating letters, counted by selecting letter-by-letter, is $(x+y)^{\underline{n}}$. If instead we count by first selecting k, the number of letters from $\{x+1, \dots, x+y\}$ used, then locate the k positions in which those letters appear, then selecting the $n-k$ letters from $\{1, \dots, x\}$ letter-by-letter in the order that they appear in the word, and finally selecting the k letters from $\{x+1, \dots, x+y\}$ letter-by-letter in the order that they appear in the word, we get a count of $\sum_{k=0}^{n} \binom{n}{k} x^{\underline{n-k}} y^{\underline{k}}$. So the identity is true for positive integers x, y.

The LHS and RHS are polynomials in x and y of degree n, so the difference is a polynomial in x and y of degree at most n, which we want to show is identically 0. Write the difference as $P(x, y) = p_0(x) + p_1(x)y + \dots + p_n(x)y^n$ where each $p_i(x)$ is a polynomial in x of degree at most n. Setting $x = 1$ we get a polynomial $P(1, y)$ in y of degree at most n. This is 0 for all integers $y > 0$ (by our combinatorial argument), so by the polynomial principle it is identically 0. So each $p_i(x)$ evaluates to 0 at $x = 1$. But the same argument shows that each $p_i(x)$ evaluates to 0 at any positive integer x. So again by the polynomial principle, each $p_i(x)$ is identically 0 and so $P(x, y)$ is. This proves the identity for all real x, y.

2. The *kth rising power* of x is $x^{\overline{k}} = x(x+1)(x+2)\dots(x+(k-1))$. Prove that for all real x, y, and all $n \geq 1$,

$$(x+y)^{\overline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\overline{n-k}} y^{\overline{k}}.$$

Solution: Set $x' = -x$ and $y' = -y$; we have

$$(x+y)^{\overline{n}} = (-x'-y')^{\overline{n}} = (-1)^n (x'+y')^{\underline{n}}$$

and

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} x^{\overline{n-k}} y^{\overline{k}} &= \sum_{k=0}^{n} \binom{n}{k} (-x')^{\overline{n-k}} (-y')^{\overline{k}} \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (x')^{\underline{n-k}} (-1)^k (y')^{\underline{k}} \\
&= (-1)^n \sum_{k=0}^{n} \binom{n}{k} (x')^{\underline{n-k}} (y')^{\underline{k}},
\end{aligned}$$

so the identity follows from the falling power binomial theorem (the previous question).

3. Evaluate
$$\sum_{k=0}^{2n}(-1)^k k^n \binom{2n}{k}$$
for $n \geq 1$.

Solution: It's not easy to deal with this sum in isolation. But, we can generalize: define, for each $n, r \geq 1$, $a_{r,n} = \sum_{k=0}^{2n}(-1)^k k^r \binom{2n}{k}$. We claim that $a_{r,n} = 0$ (and so in particular $a_{n,n}$, our sum of interest, is 0.

We prove the claim for each n by induction on r. It will be helpful to define
$$f_n(x) = \sum_{k=0}^{2n} x^k \binom{2n}{k} = (1+x)^{2n}.$$
Differentiating,
$$f_n'(x) = \sum_{k=0}^{2n} kx^{k-1} \binom{2n}{k} = 2n(1+x)^{2n-1}.$$
Evaluating at $x = -1$ we get
$$\sum_{k=0}^{2n}(-1)^{k-1}k^1 \binom{2n}{k} = 0,$$
and multiplying by -1 gives $a_{1,n} = 0$. This is the base case of the induction.

For the induction step, assume that $a_{j,n} = 0$ for all $1 \leq j < r$ (with $r \geq 2$). The rth derivative of $f_n(x)$ is
$$f_n^{(r)}(x) = \sum_{k=0}^{2n} k(k-1)\dots(k-(r-1))x^{k-r} \binom{2n}{k} = 2n(2n-1)\dots(2n-(r-1))(1+x)^{2n-r}.$$
Now $k(k-1)\dots(k-(r-1))$ is a polynomial in k, of degree r, whose leading coefficient is 1, and for which all other terms are polynomials in r; in other words,
$$k(k-1)\dots(k-(r-1)) = k^r + c_1(r)k^{r-1} + \dots + c_r(r),$$
and so
$$f_n^{(r)}(x) = \sum_{k=0}^{2n} x^{k-r}k^r \binom{2n}{k} + \sum_{j=1}^{r} c_j(r) \sum_{k=0}^{2n} x^{k-r}k^{r-j} \binom{2n}{k}.$$
Evaluating at $x = -1$, and recalling that $f_n^{(r)}(x) = 2n(2n-1)\dots(2n-(r-1))(1+x)^{2n-r}$, we get
$$\sum_{k=0}^{2n}(-1)^{k-r}k^r \binom{2n}{k} + \sum_{j=1}^{r} c_j(r) \sum_{k=0}^{2n}(-1)^{k-r}k^{r-j} \binom{2n}{k} = 0.$$
The sum corresponding to $j = r$ is
$$c_j(r) \sum_{k=0}^{2n}(-1)^{k-r} \binom{2n}{k} = (-1)^{-r}c_j(r) \sum_{k=0}^{2n}(-1)^k \binom{2n}{k} = (-1)^{-r}c_j(r)(1-1)^{2n} = 0.$$

The sum corresponding to each j, $1 \le j < r$ is $c_j(r)(-1)^{-r}a_{r,n}$, so is 0 by induction. So we conclude

$$\sum_{k=0}^{2n}(-1)^{k-r}k^r\binom{2n}{k} = 0,$$

which give $a_{r,n} = 0$ on multiplying by $(-1)^r$.

4. Evaluate

$$\sum_{k=0}^{n} F_{k+1}\binom{n}{k}$$

for $n \ge 0$, where $F_1, F_2, F_3, F_4, F_5, \ldots$ are the Fibonacci numbers $1, 1, 2, 3, 5, \ldots,$.

Solution: When $n = 0$ we get a sum of 1; when $n = 1$ we get a sum of 2; when $n = 2$ we get a sum of 5; when $n = 3$ we get a sum of 13; when $n = 4$ we get a sum of 34; this suggests strongly

$$\sum_{k=0}^{n} F_{k+1}\binom{n}{k} = F_{2n+1}.$$

One way to prove this is to iterative apply the Fibonacci recurrence to F_{2n+1}: on the zeroth iteration,

$$F_{2n+1} = \binom{0}{0}F_{2n+1}.$$

The first iteration leads to

$$F_{2n+1} = F_{2n} + F_{2n-1} = \binom{1}{0}F_{2n} + \binom{1}{0}F_{2n-1}.$$

The second leads to

$$\begin{aligned} F_{2n+1} &= F_{2n} + F_{2n-1} \\ &= (F_{2n-1} + F_{2n-2}) + (F_{2n-2} + F_{2n-3}) \\ &= \binom{2}{0}F_{2n-1} + \binom{2}{1}F_{2n-2} + \binom{2}{2}F_{2n-3}. \end{aligned}$$

This suggest that we prove to more general statement, that for each $0 \le s \le n$,

$$F_{2n+1} = \sum_{j=0}^{s}\binom{s}{j}F_{2n+1-s-j}. \qquad (\star)$$

The case $s = n$ yields

$$F_{2n+1} = \sum_{j=0}^{n}\binom{n}{j}F_{n+1-j},$$

which is the same as what we have to prove (by the symmetry relation $\binom{n}{j} = \binom{n}{n-j}$).

We can prove $(\star)$ by induction on s (for each fixed n), with the case $s = 0$ trivial. For larger s, we begin with the $s-1$ case of the induction hypothesis, then use the Fibonacci recurrence to break each Fibonacci number into the sum of two earlier ones, then use Pascals identity to gather together terms involving the same Fibonacci number. (Details omitted.)

5. (a) Let a_n be the number of 0-1 strings of length n that do not have two consecutive 1's. Find a recurrence relation for a_n (starting with initial conditions $a_0 = 1$, $a_1 = 2$).

   Solution: By considering whether the last term is a 0 or a 1, get the Fibonacci recurrence: $a_n = a_{n-1} + a_{n-2}$.

   (b) Let $a_{n,k}$ be the number of 0-1 strings of length n that have exactly k $1'$ and that do not have two consecutive 1's. Express $a_{n,k}$ as a (single) binomial coefficient.

   Solution: Add a 0 to the beginning and end of such a string. By reading off a_1, the number of 0's before the first 1, then a_2, the number of 0's between the first 1 and the second, and so on up to a_{k+1}, the number of 0's after the last 1, we get a composition $(a_1, \ldots, a_{k+1})$ of $n+2-k$ into $k+1$ parts; and each such composition can be encoded (uniquely) by such a string. So $a_{n,k}$ is the number of compositions of $n+2-k$ into $k+1$ parts, and so equals $\binom{n+1-k}{k}$.

   (c) Use the results of the previous two parts to give a combinatorial proof (showing that both sides count the same thing) of the identity

$$F_n = \sum_{k \geq 0} \binom{n-k-1}{k}$$

   where F_n is the nth Fibonacci number (as defined in the last question).

   Solution: From the recurrence in the first part, we get $a_n = F_{n+2}$, so F_n counts the number of 0-1 strings of length $n-2$ with no two consecutive 1's. We can count such strings by first deciding on k, the number of 1's, and by the second part, the number of such strings is $\binom{n-1-k}{k}$. Summing over k we get the result.

Problems

1. Show that for every $n, m \geq 0$,

$$\int_0^1 x^n (1-x)^m \, dx = \frac{1}{(n+m+1)\binom{n+m}{n}}.$$

Solution: The result is trivial when one or both of $n, m = 0$, so we may assume $n + m \geq 2$. Using integration by parts, we get

$$\int_0^1 x^n (1-x)^m \, dx = \frac{m}{n+1} \int_0^1 x^{n+1} (1-x)^{m-1} \, dx.$$

This suggests that for each $s \geq 2$ we use induction on m to prove the result for all pairs (n, m) with $n + m = s$. The case $m = 0$ has been observed already. For $m > 0$ we have

$$\begin{aligned}
\int_0^1 x^n (1-x)^m \, dx &= \frac{m}{n+1} \int_0^1 x^{n+1} (1-x)^{m-1} \, dx \\
&= \frac{m}{n+1} \frac{1}{((n+1)+(m-1)+1)\binom{(n+1)+(m-1)}{n+1}} \\
&= \frac{m}{(n+1)(n+m+1)\binom{n+m}{n+1}}.
\end{aligned}$$

The result follows if we can show

$$(n+1)\binom{n+m}{n+1} = m\binom{n+m}{n}.$$

This is easily verified, either algebraically, or via the "committee-chair" identity: to choose a committee-with-chair of size $n+1$ from $n+m$ people, we either choose the $n+1$ people for the committee and elect one of them chair ($(n+1)\binom{n+m}{n+1}$ ways) or select the n non-chair members from the $n+m$ people, and choose the chair from among those not yet chosen ($m\binom{n+m}{n}$ ways).

2. Show that the coefficient of x^k in $(1+x+x^2+x^3)^n$ is

$$\sum_{j=0}^{k}\binom{n}{j}\binom{n}{k-2j}.$$

Solution: (This was Putnam problem 1992 B2.) We have

$$\begin{aligned}(1+x+x^2+x^3)^n &= (1+x)^n(1+x^2)^n \\ &= \left(\sum_{i\geq 0}\binom{n}{i}x^i\right)\left(\sum_{i'\geq 0}\binom{n}{2i'}x^{2i'}\right).\end{aligned}$$

We get an x^k term in the product by pairing each $\binom{n}{i}x^i$ from the first sum with $\binom{n}{k-2i}x^{k-2i}$ from the second; so the coefficient of x^k in the product is

$$\sum_{i=0}^{k}\binom{n}{i}\binom{n}{k-2i},$$

as claimed.

3. Let r, s and t be integers with $r, s \geq 0$ and $r+s \leq t$. Prove that

$$\sum_{i=0}^{s}\frac{\binom{s}{i}}{\binom{t}{r+i}} = \frac{t+1}{(t+1-s)\binom{t-s}{r}}.$$

Solution: (This was Putnam problem 1987 B2.) Write the sum on the left-hand side as $F(r,s,t)$; we prove by induction on s that

$$F(r,s,t) = \frac{t+1}{(t+1-s)\binom{t-s}{r}}$$

whenever r, s and t are integers with $r, s \geq 0$ and $r+s \leq t$. The base case $s=0$ is trivial.

For the inductive step, write

$$
\begin{aligned}
F(r,s,t) &= \frac{\binom{s}{0}}{\binom{t}{r}} + \frac{\binom{s}{1}}{\binom{t}{r+1}} + \ldots + \frac{\binom{s}{s-1}}{\binom{t}{r+s-1}} + \frac{\binom{s}{s}}{\binom{t}{r+s}} \\
&= \frac{\binom{s-1}{0}}{\binom{t}{r}} + \frac{\binom{s-1}{0}+\binom{s-1}{1}}{\binom{t}{r+1}} + \ldots + \frac{\binom{s-1}{s-2}+\binom{s-1}{s-1}}{\binom{t}{r+s-1}} + \frac{\binom{s-1}{s-1}}{\binom{t}{r+s}} \\
&= F(r,s-1,t) + F(r+1,s-1,t) \\
&= \frac{t+1}{(t+2-s-1)\binom{t-s+1}{r}} + \frac{t+1}{(t+2-s)\binom{t-s+1}{r+1}} \\
&= \frac{t+1}{(t+1-s)\binom{t-s}{r}}
\end{aligned}
$$

(with the last line following after a little algebra). This completes the inductive step. (See *The William Lowell Putnam Mathematical Competition 1985-2000, Problems, Solutions, and Commentary* for a nice discussion of an alternate, combinatorial approach.

4. Prove that the expression
$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$
is an integer for all pairs of integers $n \geq m \geq 1$.

   Solution: (This was Putnam problem 2000 B2.) We know that $\gcd(m,n) = am + bn$ for some integers a, b; but then
$$\frac{\gcd(m,n)}{n}\binom{n}{m} = a\left(\frac{m}{n}\binom{n}{m}\right) + b\left(\frac{n}{n}\binom{n}{m}\right).$$
   Since
$$\binom{n}{m} = \frac{n}{m}\binom{n-1}{m-1}$$
   (the committee-chair identity again, or easy algebra), we get
$$\frac{\gcd(m,n)}{n}\binom{n}{m} = a\binom{n-1}{m-1} + b\binom{n}{m},$$
   and so (since a, b, $\binom{n-1}{m-1}$ and $\binom{n}{m}$ are all integers) we get the desired result.

5. For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \ldots + |x_n| \leq m$. Show that $f(m,n) = f(n,m)$. (In other words, the number of points in the ℓ_1 ball of radius m in $\mathbb{R}^n$ is the same as the number of points in the ℓ_1 ball of radius n in $\mathbb{R}^m$.)

   Solution: (This was Putnam problem 2005 B5.) We'll produce an explicit formula for $f(m,n)$ that is symmetric in m and n. Each tuple counted by $f(m,n)$ has k entries that are non-zero, for some $k \geq 0$; once k has been decided on, there are $\binom{n}{k}$ ways to decide on the locations of the k non-zeros. Each of these k entries has a sign ($\pm$); there are 2^k ways to decide on the signs. To complete the specification of the tuple, we have find k numbers $a_1, \ldots, a_k$, all strictly positive, that add to at most m. Adding a phantom $(k+1)$st number, this is the same as the number of $(k+1)$-tuples $(a_1, \ldots, a_{k+1})$, all strictly positive integers, that add to *exactly* $m+1$; i.e., the number of compositions of $m+1$ into $k+1$ parts, so $\binom{(m+1)-1}{(k+1)-1} = \binom{m}{k}$. So

$$f(m,n) = \sum_{k \geq 0} \binom{n}{k}\binom{n}{k} 2^k.$$

Since this is symmetric in m and n, $f(m,n) = f(n,m)$.

Note: $f(m,n)$ is also the number of ways of moving from $(0,0)$ to (m,n) on the two-dimensional lattice by taking steps of the following kind: one unit up parallel to y-axis, one unit to the right parallel to the x-axis, or one diagonal unit (Euclidean distance $\sqrt{2}$) up and to the right, parallel to the line $x = y$; the numbers $f(m,n)$ are called the *Delannoy numbers*.