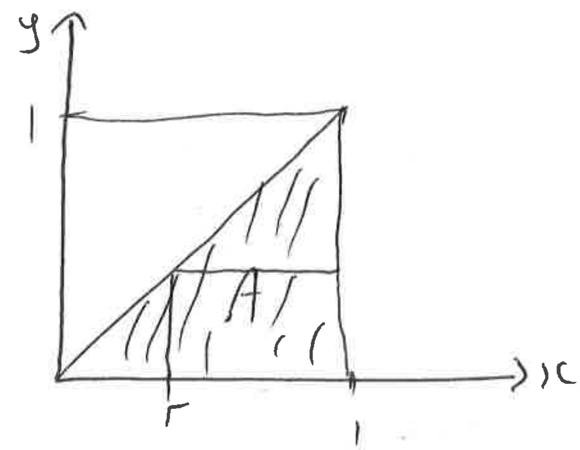
Math 60850, Spring 2016, Homework 2

1) Rosenthal 2.7.21



To show that A is measurable (in 2 dimensional Lebesgue measure), we show that it can be written as the countable union of measurable rectangles:

$$A = \left(\bigcup_{\substack{r \in [0,1] \\ r \text{ rational}}} (r,1] \times [0,r) \right) \cup [0,1] \times \{0\} \cup \{1\} \times [0,1)$$

To find $\lambda(A)$, we use a sandwiching argument. Fix $n \geq 1$. We have

$$A \subseteq \bigcup_{i=1}^{n} \left[\tfrac{i-1}{n}, \tfrac{i}{n}\right) \times \left[0, \tfrac{i}{n}\right]$$, as the union is disjoint.

So $\lambda(A) \leq \sum_{i=1}^{n} \underset{\uparrow}{\frac{i}{n^2}} = \frac{1}{2}\left[1+\frac{1}{n}\right]$

$\lambda\left(\left[\frac{i-1}{n}, \frac{i}{n}\right) \times \left[0, \frac{i}{n}\right]\right)$; we know this because the rectangle is in $\mathcal{S}$, the semialgebra that generates

We also have:

$$A \supseteq \bigcup_{i=1}^{n} \left(\frac{i-1}{n}, \frac{i}{n}\right) \times \left(0, \frac{i-1}{n}\right), \text{ so}$$

$$\lambda(A) \geq \sum_{i=1}^{n} \frac{i-1}{n^2} = \frac{1}{2}\left[1-\frac{1}{n}\right]$$

Conclusion: $\left|\lambda(A) - \frac{1}{2}\right| \leq \frac{1}{n} \quad \forall n \in \mathbb{N}$,

$$\lambda(A) = \frac{1}{2}$$

$$(\Omega, \mathcal{F}, P) = \text{Uniform } [0,1]$$
$$X(w) = w$$
$$Y(w) = w^2 + \frac{3}{16}$$

2) a) $\{X > 1/3\} = \{w \mid X(w) > \frac{1}{3}\}$

$= \{w \mid w > \frac{1}{3}\}$

$= (\frac{1}{3}, 1]$

$\in \mathcal{F}$ because it is an interval (in $\mathcal{F}$)

$$P(\{X > \tfrac{1}{3}\}) = 1 - \frac{1}{3} = \frac{2}{3}$$

b) $\{Y < 1\} = \{w \mid w^2 + \frac{3}{16} < 1\}$

$= \{w \mid w < \frac{13}{16}\}$

$= [0, \sqrt{\frac{13}{16}})$, $\in \mathcal{F}$ as it is an interval

$$P(Y < 1) = \sqrt{\frac{13}{16}}$$

c) $\{X < Y\} = \{w \in [0,1] \mid w < w^2 + \frac{3}{16}\}$

$= (0, \frac{1}{4}) \cup (\frac{3}{4}, 1]$, $\in \mathcal{F}$ as it is ~~an interval~~ a union of intervals.

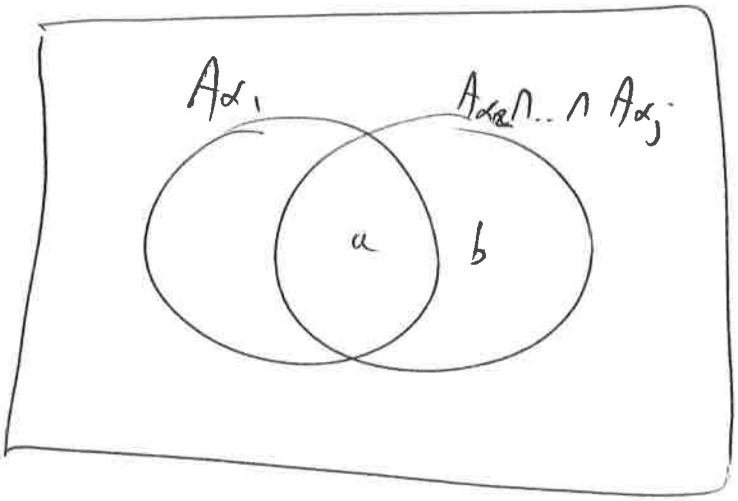
$$P(X < Y) = \frac{1}{2} \left(= \frac{1}{4} + \frac{1}{4}\right)$$

3) Rosenthal 3.2.2

a) Know: $P(A_{\alpha_1} \cap \ldots \cap A_{\alpha_j}) = P(A_{\alpha_1}) \ldots P(A_{\alpha_j})$

Have: $P(A_{\alpha_1}^c \underbrace{\cap}_{b} \ldots \cap A_{\alpha_j}) =$

$$\underbrace{P(A_{\alpha_2} \cap \ldots \cap A_{\alpha_j})}_{a+b} - \underbrace{P(A_{\alpha_1} \cap \ldots \cap A_{\alpha_j})}_{-a}$$



$$= P(A_{\alpha_2}) \ldots P(A_{\alpha_j}) - P(A_{\alpha_1}) \ldots P(A_{\alpha_j}) \quad \left[\begin{matrix}\text{independence} \\ \text{def}\end{matrix}\right]$$

$$= P(A_{\alpha_2}) \ldots P(A_{\alpha_j}) \left[1 - P(A_{\alpha_1})\right]$$

$$= P(A_{\alpha_1}^c) \ldots P(A_{\alpha_j})$$

$\square$

b) Repeatedly apply the argument of part a), replacing one A_{α_i} at a time with $A_{\alpha_i}^c$

c) Follows immediately from b) and definition of independence.

4) Rosenthal 3.6.3 b)

Let Ω consist of 8 equally likely outcomes

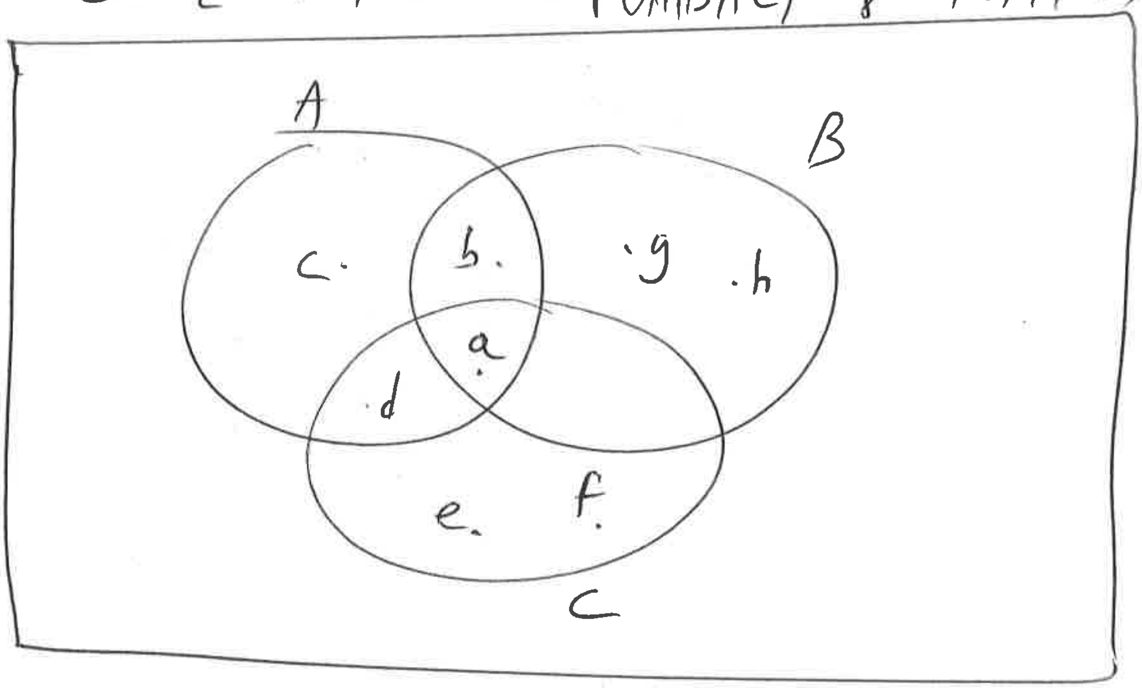
a, b, c, d, e, f, g, h

Let $A = \{a, b, c, d\}$ $\quad P(A) = P(B) = P(C) = \frac{1}{2}$

$B = \{a, b, g, h\}$ $\quad P(A \cap B) = \frac{1}{4} = P(A)P(B)$

$C = \{a, d, e, f\}$ $\quad P(A \cap C) = \frac{1}{4} = P(A)P(C)$

$P(A \cap B \cap C) = \frac{1}{8} = P(A)P(B)P(C)$



<u>But</u> $P(B \cap C) = \frac{1}{8} \neq P(B)P(C)$.

5) Rosenthal 3.6.5

Fix $\omega \in \Omega$

For $i = 1, \ldots, n$, set $B_i = \begin{cases} A_i & \text{if } \omega \in A_i \\ A_i^c & \text{if } \omega \notin A_i \end{cases}$

Then $\omega \in \bigcap_{i=1}^{n} B_i$, so

$$P(\omega) \leq P\left(\bigcap_{i=1}^{n} B_i\right) \underset{\substack{\uparrow \\ \text{independence}}}{=} \prod_{i=1}^{n} P(B_i) = \left(\tfrac{1}{2}\right)^n$$

Since this is true for all n, $P(\omega) = 0$.

But then since Ω is ctble,

$$P(\Omega) = 0$$, Contradiction.

6) Rosenthal 3.6.7

Set $A_n = \begin{cases} \{1,2\} & \text{if } n \equiv 0 \pmod 6 \\ \{1\} & n \equiv 1 \pmod 6 \\ \{1,3\} & \text{if } n \equiv 2 \pmod 6 \\ \{2\} & n \equiv 3 \pmod 6 \\ \{2,3\} & \text{if } n \equiv 4 \pmod 6 \\ \{3\} & n \equiv 5 \pmod 6 \end{cases}$

Then: $P(\liminf A_n) = 0$ [there is no sample point that is in all but finitely many A_n's]

$$\left.\begin{array}{l} \liminf P(A_n) = \frac{1}{3} \\ \limsup P(A_n) = \frac{2}{3} \end{array}\right\} (P(A_n)) = (\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}, \ldots)$$

$P(\limsup A_n) = 1$ [every $w \in \{1,2,3\}$ occurs in ∞ly many A_n]