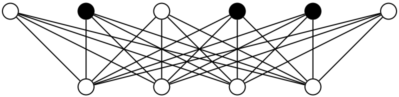

Extremal problems for independent sets

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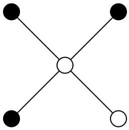
An extremal question for independent sets

Independent set Set of pairwise non-adjacent vertices



- $i(G)$: Number of independent sets in G
- $i_t(G)$: Number of independent sets of size t

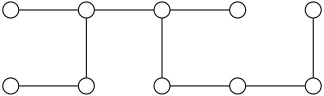
Question

Fix a family $\mathcal{G}$ of graphs.

- *What is the maximum of $i(G)$ as G ranges over $\mathcal{G}$?*
- *What about the maximum of $i_t(G)$ for each t?*

Trees

$\mathcal{T}(n)$: trees on n vertices



Theorem (Prodinger, Tichy, 1982)

For $T \in \mathcal{T}(n)$,

- *$i(G)$ maximized by the star $K_{1,n-1}$*

Theorem (Wingard, 1995)

For $T \in \mathcal{T}(n)$, and all t,

- *$i_t(G)$ maximized by $K_{1,n-1}$*

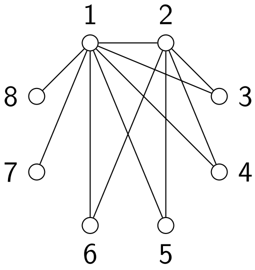
Graphs with given order and size

$\mathcal{H}(n, m)$: graphs on n vertices with m edges

Theorem (Cutler, Radcliffe, 2011)

For $G \in \mathcal{H}(n, m)$,

- *$i(G)$ maximized by the lex graph $L(n, m)$*
- *for all t, $i_t(G)$ maximized by $L(n, m)$*



The lex graph $L(8, 11)$

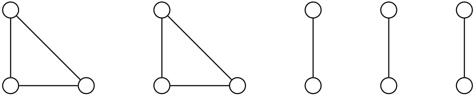
Graphs with given independence number

$\mathcal{I}(n,\alpha)$: graphs on n vertices with $\alpha(G)=\alpha$

Theorem (Zykov, 1952)

For $G \in \mathcal{I}(n,\alpha)$,

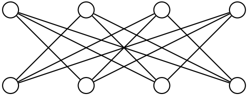
- *$i(G)$ maximized by union of α almost-equal-sized cliques*
- *for all t, $i_t(G)$ maximized by same graph*



The case $n=12$, $\alpha=5$

Regular graphs

$\mathcal{R}(n,d)$: d-regular graphs on n vertices



Theorem (Kahn, 2001; Zhao, 2011)

For $G \in \mathcal{R}(n,d)$,

- $i(G)$ *maximized by* $\frac{n}{2d}K_{d,d}$, *union of* $n/2d$ *copies of* $K_{d,d}$

Conjecture (Kahn, 2001)

For $G \in \mathcal{R}(n,d)$, and all t,

- $i_t(G)$ *maximized by* $\frac{n}{2d}K_{d,d}$

Asymptotic evidence given by Carroll, G., Tetali, and by Zhao

Graphs with given minimum degree

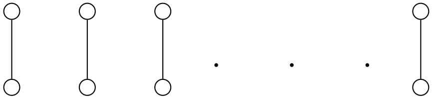
$\mathcal{G}(n, \delta)$: graphs on n vertices with minimum degree δ

Speculation

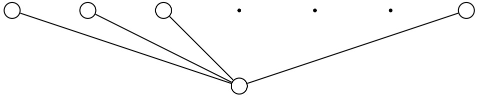
Removing edges increases independent set count, so maybe

- $i(G)$ *maximized by* $\frac{n}{2\delta} K_{\delta,\delta}$

Not true, even for $\delta = 1$



$i(nK_2) = 3^{n/2}$



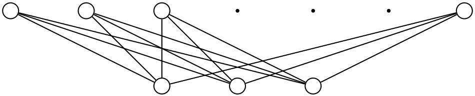
$i(K_{1,n-1}) = 2^{n-1} - 1$

An unbalanced maximizer

Theorem (G., 2011)

For $n \geq 8\delta^2$ and $G \in \mathcal{G}(n, \delta)$,

- *$i(G)$ uniquely maximized by $K_{\delta, n-\delta}$.*



$K_{3,n-3}$

Conjecture (G., 2011)

For $G \in \mathcal{G}(n, \delta)$,

- *for $n \geq 2\delta$, $i(G)$ maximized by $K_{\delta, n-\delta}$*
- *for smaller n, $i(G)$ maximized by $K_{n-\delta, n-\delta, \ldots, n-\delta, x}$ $(x \leq n-\delta)$*

Fixed size in $\mathcal{G}(n, \delta)$

$i_2(G) =$ number of non-edges, so $K_{\delta,n-\delta}$ definitely *not* the maximizer

Conjecture (G., 2011)

For $n \geq 2\delta$, $t \geq 3$ *and* $G \in \mathcal{G}(n, \delta)$,

- $i_t(G)$ *maximized by* $K_{\delta,n-\delta}$

Partial results

- *Bipartite* G *(Alexander, Cutler, Mink, 2012)*
- $\delta \leq 3$ *(Engbers, G., 2012)*
- $t \geq 2\delta + 1$ *for larger* δ *(Engbers, G., 2012)*
- $t \geq 3$, $n \geq C\delta^3$ *(McDiarmid, Law, 2012)*

Leaving, for each $\delta \geq 4$ and non-bipartite G, the box

$$t \in \{3, \ldots, 2\delta\}, \; n \in \{2\delta, \ldots, C\delta^3\}$$

Proof for $t \geq 2\delta + 1$ (I)

Observation

- *Suffices to consider* $t = 2\delta + 1$

Proof Suppose for some $t > \delta$,

$$i_t(G) \leq i_t(K_{\delta,n-\delta}) = \binom{n-\delta}{t}$$

Then

$$\#(\textit{ordered}\text{ independent } t\text{-sets}) \leq (n-\delta)^{\underline{t}}$$

Once t vertices chosen, at least $\delta + t$ ruled out, so

$$\#(\text{ordered ind. } (t+1)\text{-sets}) \leq (n-\delta)^{\underline{t}}(n-(\delta+t)) = (n-\delta)^{\underline{t+1}}$$

and

$$i_{t+1}(G) \leq \binom{n-\delta}{t+1} = i_{t+1}(K_{\delta,n-\delta})$$

Proof for $t \geq 2\delta + 1$ (II)

Proof strategy

- *Prove $t = 2\delta + 1$ case by induction on n*

Base case $n = 3\delta + 1$ is trivial

Induction, case 1 There is $x \in V(G)$ with $\delta(G - x) = \delta$

$$
\begin{aligned}
i_t(G) &= i_t(G - x) + i_{t-1}(G - x - N(x)) \\
&\leq \binom{(n-1)-\delta}{t} \text{ (induction)} + \binom{n-(\delta+1)}{t-1} \text{ (trivial)} \\
&\leq \binom{n-\delta}{t} \text{ (Pascal)}
\end{aligned}
$$

Proof for $t \geq 2\delta + 1$ (III)

Induction, case 2 There is *no* $x \in V(G)$ with $\delta(G - x) = \delta$

Ordered ind. t-sets starting with vertex of degree $> \delta$:

$$N_{>\delta} \leq k(n - (\delta + 2))(n - (\delta + 3)) \ldots (n - (\delta + t))$$

where $k =$ number of vertices of degree $> \delta$

Ordered ind. t-sets starting with vertex of degree $= \delta$:

$$N_{=\delta} \leq (n - k)(n - (\delta + 1))(n - (\delta + 2)) \ldots (n - 2\delta) \\ (n - \widehat{(2\delta + 2)})((n - (2\delta + 2))) \ldots (n - (\delta + t))$$

Why the missing term?

- Worst case: each new vertex shares δ neighbors of first choice
- This *can't* happen $\delta + 1$ times (or we're in case 1)
- $(\delta + 1)$st choice (at worst) removes a new vertex

Proof for $t \geq 2\delta + 1$ (IV)

Have

$$N_{>\delta} \leq k(n-(\delta+2))(n-(\delta+3))\dots(n-(\delta+t))$$

and

$$N_{=\delta} \leq (n-k)(n-(\delta+1))(n-(\delta+2))\dots(n-2\delta)$$
$$\widehat{(n-(2\delta+2))}((n-(2\delta+2)))\dots(n-(\delta+t))$$

Worst case $k=n$, giving bound

$$i_t(G) \leq \frac{n(n-(\delta+2))(n-(\delta+3))\dots(n-(\delta+t))}{t!}$$
$$< \binom{n-\delta}{t}$$

Last inequality uses $t = 2\delta + 1$

Final comments

- Maybe improve result by considering first, second, third ... choices more carefully, and optimizing a linear program
- $\delta = 2, 3$ requires messy case analysis, structural results for δ-critical graphs, with $\delta = 4$ hopeless by our methods

Open questions

- $i_t(G)$ *for all* t *and* n*-vertex,* d*-regular* G
- $i(G)$ *for* $n \leq 8\delta^2$ *for* n*-vertex* G*, min. degree* δ
- $i_t(G)$ *for* $t \in [3, 2\delta]$ *and* $n \in [2\delta, C\delta^3]$ *for* n*-vertex* G*, min. degree* δ
- ...

THANK YOU!

Slides at http://nd.edu/~dgalvin1