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Entropy and Graph Homomorphisms

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(joint work with Prasad Tetali [3])

Let G be an n-regular, N-vertex bipartite graph on vertex set $V(G)$, and let H be a fixed graph on vertex set $V(H)$ (perhaps with loops). Set

$$Hom(G,H) = \{f : V(G) \to V(H) \ : \ u \sim v \Rightarrow f(u) \sim f(v)\}.$$

That is, $Hom(G,H)$ is the set of graph homomorphisms from G to H.

When $H = H_{ind}$ consists of one looped and one unlooped vertex connected by an edge, an element of $Hom(G,H_{ind})$ can be thought of as a specification of an independent set (a set of vertices spanning no edges) in G. Our point of departure is the following result of Kahn [4], bounding the size of $\mathcal{I}(G)$, the set of independent sets of G.

Theorem 1 *For any n-regular, N-vertex bipartite graph G,*

$$|\mathcal{I}(G)| \leq (2^{n+1}-1)^{N/2n}.$$

Note that $|Hom(K_{n,n}, H_{ind})| = 2^{n+1}-1$ (where $K_{n,n}$ is the complete bipartite graph with n vertices on each side), so we may paraphrase Theorem 1 by saying that $|Hom(G,H_{ind})|$ is maximum when G is a disjoint union of $K_{n,n}$'s. Our main result is a generalization of this statement (and our proof is a generalization of Kahn's).

Proposition 2 *For any n-regular, N-vertex bipartite G, and any H,*

$$|Hom(G,H)| \leq |Hom(K_{n,n},H)|^{N/2n}.$$

We also consider a weighted version of Proposition 2. Following [1], we put a measure on $Hom(G,H)$ as follows. To each $i \in V(H)$ assign a positive "activity" λ_i, and write Λ for the set of activities. Give each $f \in Hom(G,H)$ weight $w^\Lambda(f) = \prod_{v\in V(G)} \lambda_{f(v)}$. The constant that turns this assignment of weights on $Hom(G,H)$ into a probability distribution is

$$Z^\Lambda(G,H) = \sum_{f\in Hom(G,H)} w^\Lambda(f).$$

When all activities are 1, we have $Z^\Lambda(G,H) = |Hom(G,H)|$, and so the following is a generalization of Proposition 2.

Proposition 3 *For any n-regular, N-vertex bipartite G, any H, and any system Λ of positive activities on $V(H)$,*

$$Z^\Lambda(G,H) \le \left(Z^\Lambda(K_{n,n},H)\right)^{N/2n}.$$

We may put this result in the framework of a well-known mathematical model of physical systems with "hard constraints" (see [1]). We think of the vertices of G as particles and the edges as bonds between pairs of particles, and we think of the vertices of H as possible "spins" that particles may take. Pairs of bonded vertices of G may have spins i and j only when i and j are adjacent in H. Thus the legal spin configurations on the vertices of G are precisely the homomorphisms from G to H. We think of the activities on the vertices of H as a measure of the likelihood of seeing the different spins; the probability of a particular spin configuration is proportional to the product over the vertices of G of the activities of the spins. Proposition 3 concerns the "partition function" of this model — the normalizing constant that turns the above-described system of weights on the set of legal configurations into a probability measure.

Our proofs are based on entropy considerations, and in particular on a lemma of Shearer (see [2, p. 33]) bounding the entropy of a random vector.

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Random Planar Graphs

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(joint work with Colin McDiarmid [3])

Given $0 < p < 1$ and a positive integer n, let $G_{n,p}$ denote the random graph with nodes $v_1, \ldots, v_n$ in which the $\binom{n}{2}$ possible edges appear independently with probability p. We denote by $R_{n,p}$ the random graph $G_{n,p}$ conditioned on it being planar. (We may think of repeatedly sampling a graph $G_{n,p}$ until we find one that is planar.) Also, let us denote $R_{n,\frac{1}{2}}$ by R_n. Thus R_n is uniformly distributed over all labelled planar graphs on n nodes.

Rather little is known about random planar graphs, even about the number of edges in such graphs, which is our focus here. Let us denote the number of edges in a (simple) graph G by $m(G)$. Thus we are interested in the random variable $m(R_n)$ and more generally in $m(R_{n,p})$. Of course $m(G) \leq 3n - 6$ for any planar graph G on n nodes. The expected value $\mathbf{E}[m(R_n)]$ is at least $(3n-6)/2$ – see [2]. It is shown in [1] that $m(R_n) \leq 2.54n$ asymptotically almost surely (aas), that is with probability tending to 1 as $n \to \infty$. This result slightly improves the upper bound of 2.56 in [6]. We will show here in particular that $m(R_n) \geq \frac{13}{7}n + o(n)$ aas, thereby improving on the result from [2] mentioned above.

We now introduce two functions $f(\alpha)$ and $g(p)$ which are needed to state our two main results – see also Figure 1.

Given $1 < \alpha \leq 3$, let $k = k(\alpha) = \lfloor \frac{2\alpha}{\alpha - 1} \rfloor$, and let

$$f(\alpha) = \frac{1}{4}\left(k^2 + k + 6 - (k^2 - 3k + 6)\alpha\right).$$

It is not hard to verify that $f(\alpha)$ is continuous and decreasing on $1 < \alpha \leq 3$, and satisfies $f(\alpha) \to \infty$ as $\alpha \to 1$ and $f(3) = 0$, see also the end of Section 4. (The function f is also piecewise-linear and convex.) For $0 < p < 1$ we may define $g(p)$ to be the unique value $\rho \in (1,3)$ such that $f(\rho)/\rho = (1-p)/p$. The function g is continuous and increasing on $0 < p < 1$, and satisfies $g(p) \to 1$ as $p \to 0$, $g(\frac{1}{2}) = \frac{13}{7}$ and $g(p) \to 3$ as $p \to 1$. We are now able to state our theorem concerning the number of edges of random planar graphs.

Theorem 1 *Let $0 < p < 1$. Then as $n \to \infty$,*

$$\mathbf{E}[m(R_{n,p})] \geq g(p)n + o(n);$$