Sampling independent sets in the discrete torus

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Abstract

The even discrete torus is the graph $T_{L,d}$ on vertex set $\{0, \ldots, L-1\}^d$ (with L even) in which two vertices are adjacent if they differ on exactly one coordinate and differ by $1 \pmod L$ on that coordinate. The *hard-core measure with activity* λ on $T_{L,d}$ is the probability distribution π_λ on the independent sets (sets of vertices spanning no edges) of $T_{L,d}$ in which an independent set I is chosen with probability proportional to $\lambda^{|I|}$. This distribution occurs naturally in problems from statistical physics and the study of communication networks.

We study Glauber dynamics, a single-site update Markov chain on the set of independent sets of $T_{L,d}$ whose stationary distribution is π_λ. We show that for $\lambda = \omega(d^{-1/4} \log^{3/4} d)$ and d sufficiently large the convergence to stationarity is (essentially) exponentially slow in L^{d-1}. This improves a result of Borgs *et al.*, who had shown slow mixing of Glauber dynamics for λ growing exponentially with d.

Our proof, which extends to ρ-local chains (chains which alter the state of at most a proportion ρ of the vertices in each step) for suitable ρ, closely follows the conductance argument of Borgs *et al.*, adding to it some combinatorial enumeration methods that are modifications of those used by Galvin and Kahn to show that the hard-core model with parameter λ on the integer lattice $\mathbb{Z}^d$ exhibits phase coexistence for $\lambda = \omega(d^{-1/4} \log^{3/4} d)$.

The discrete even torus is a bipartite graph, with partition classes $\mathcal{E}$ (consisting of those vertices the sum of whose coordinates is even) and $\mathcal{O}$. Our result can be expressed combinatorially as the statement that for each sufficiently large λ, there is a $\rho(\lambda) > 0$ such that if I is an independent set chosen according to π_λ, then the probability that $||I \cap \mathcal{E}| - |I \cap \mathcal{O}||$ is at most $\rho(\lambda) L^d$ is exponentially small in L^{d-1}.

Key words and phrases: Glauber dynamics, mixing time, independent sets, hard-core model, conductance, discrete torus, Peierl's argument.

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In particular, we obtain the combinatorial result that for all $\varepsilon > 0$ the probability that a uniformly chosen independent set from $T_{L,d}$ satisfies $||I \cap \mathcal{E}| - |I \cap \mathcal{O}|| \leq (.25 - \varepsilon)L^d$ is exponentially small in L^{d-1}.

1 Introduction and statement of the result

Let $\Sigma = (V, E)$ be a simple, loopless, finite graph on vertex set V and edge set E. (For graph theory basics, see *e.g.* [2], [7].) Write $\mathcal{I}(\Sigma)$ for the set of independent sets (sets of vertices spanning no edges) in V. For $\lambda > 0$ we define the *hard-core probability measure with activity* λ on $\mathcal{I}(\Sigma)$ by

$$\pi_\lambda(\{I\}) = \frac{\lambda^{|I|}}{Z_\lambda(\Sigma)} \quad \text{for } I \in \mathcal{I}(\Sigma)$$

where $Z_\lambda(\Sigma) = \sum_{I \in \mathcal{I}} \lambda^{|I|}$ is the appropriate normalizing constant or *partition function*. Note that π_1 is uniform measure on $\mathcal{I}(\Sigma)$.

The hard-core measure originally arose in statistical physics (see *e.g.* [8, 1]) where it serves as a model of a gas with particles of non-negligible size. The vertices of Σ we think of as sites that may or may not be occupied by particles; the rule of occupation is that adjacent sites may not be simultaneously occupied. In this context the activity λ measures the likelihood of a site being occupied.

The measure also has a natural interpretation in the context of multicast communications networks (see *e.g.* [16]). Here the vertices of Σ are thought of as locations from which calls can be made; when a call is made, the call location is connected to all its neighbours, and throughout its duration, no call may be placed from any of the neighbours. Thus at any given time, the set of locations from which calls are being made is an independent set in Σ. If calls are attempted independently at each vertex as a Poisson process of rate λ and have independent exponential mean 1 lengths, then the process has stationary distribution π_λ.

Unless L and d are small, it is unfeasible to explicitly compute the partition function Z_λ and the distribution π_λ. It is therefore of great interest to understand the effectiveness of algorithms which approximate Z_λ and/or π_λ. In this paper we study *Glauber dynamics*, a Monte Carlo Markov chain (MCMC) which simulates π_λ. MCMC's occur frequently in computer science in algorithms designed to sample from or estimate the size of large combinatorially defined structures; they are also used in statistical physics and the study of networks to help understand the behavior of models of physical systems and networks in equilibrium. Glauber dynamics is the single-site update Markov chain $\mathcal{M}_\lambda = \mathcal{M}_\lambda(\Sigma)$ on state space $\mathcal{I}(\Sigma)$ with transition probabilities $P_\lambda(I, J),\ I, J \in \mathcal{I}(\Sigma)$, given by

$$P_\lambda(I,J) = \begin{cases} 0 & \text{if } |I \triangle J| > 1 \\ \frac{1}{|V|}\frac{\lambda}{1+\lambda} & \text{if } |I \triangle J| = 1,\ I \subseteq J \\ \frac{1}{|V|}\frac{1}{1+\lambda} & \text{if } |I \triangle J| = 1,\ J \subseteq I \\ 1 - \sum_{I \neq J' \in \mathcal{I}(\Sigma)} P_\lambda(I, J') & \text{if } I = J. \end{cases}$$

We may think of $\mathcal{M}_\lambda$ dynamically as follows. From an independent set I, choose a vertex v uniformly from V. Then add v to I with probability proportional to λ, and remove it with probability proportional to 1; that is, set

$$I' = \begin{cases} I \cup \{v\} & \text{with probability } \frac{\lambda}{1+\lambda} \\ I \setminus \{v\} & \text{with probability } \frac{1}{1+\lambda}. \end{cases}$$

Finally, move to I' if I' is an independent set, and stay at I otherwise.

It is readily checked that $\mathcal{M}_\lambda$ is an ergodic Markov chain with (unique) stationary distribution π_λ. A natural and important question to ask about $\mathcal{M}_\lambda$ is how quickly it converges to its stationary distribution. It is traditional to define the *mixing time* $\tau_{\mathcal{M}_\lambda(\Sigma)}$ of $\mathcal{M}_\lambda(\Sigma)$ to be

$$\tau_{\mathcal{M}_\lambda(\Sigma)} = \max_{I \in \mathcal{I}(\Sigma)} \min \left\{ t_0 : \frac{1}{2} \sum_{J \in \mathcal{I}(\Sigma)} |P^t(I,J) - \pi_\lambda(J)| \leq \frac{1}{e} \quad \forall t > t_0 \right\},$$

where $P^t(I, \cdot)$ is the distribution of the chain at time t, given that it started in state I. The mixing time of $\mathcal{M}_\lambda$ captures the speed at which the chain converges to its stationary distribution: for every $\epsilon > 0$, in order to get a sample from $\mathcal{I}(\Sigma)$ which is within ϵ of π_λ (in variation distance), it is necessary and sufficient to run the chain from some arbitrarily chosen distribution for some multiple (depending on ϵ) of the mixing time. For surveys of issues related to the mixing time of a Markov chain, see *e.g.* [19, 20].

Here we study $\tau_{\mathcal{M}_\lambda(T_{L,d})}$, where $T_{L,d}$ is the even discrete torus. This is the graph on vertex set $\{0, \ldots, L-1\}^d$ (with L even) in which two strings are adjacent if they differ on only one coordinate, and differ by $1 \pmod{L}$ on that coordinate. For $L \geq 4$ this is a $2d$-regular bipartite graph with unique bipartition $\mathcal{E} \cup \mathcal{O}$ where $\mathcal{E}$ is the set of even vertices of $T_{L,d}$ (those strings the sum of whose coordinates is even) and $\mathcal{O}$ is the set of odd vertices.

Much work has been done on the question of bounding $\tau_{\mathcal{M}_\lambda}$ above for various classes of graphs. The most general results available to date are due to Luby and Vigoda [18] and Dyer and Greenhill [11], who have shown that for any graph Σ with maximum degree Δ, $\tau_{\mathcal{M}_\lambda(\Sigma)}$ is a polynomial in $|V(\Sigma)|$ whenever $\lambda < 2/(\Delta - 2)$, which implies that $\tau_{\mathcal{M}_\lambda(T_{L,d})}$ is a polynomial in L^d whenever $\lambda < 1/(d-1)$. More recently, Weitz [22] has improved this general bound in the case of graphs with sub-exponential growth, and in particular has shown that $\tau_{\mathcal{M}_\lambda(T_{L,d})}$ is a polynomial in L^d whenever $\lambda \leq (2d-1)^{2d-1}/(2d-2)^{2d} \approx e/2d$.

Recently, attention has been given to the question of regimes of *inefficiency* of Glauber and other dynamics. Dyer, Frieze and Jerrum [10] considered the case $\lambda = 1$ and showed that for each $\Delta \geq 6$ a random (uniform) Δ-regular, n-vertex bipartite Σ almost surely (with probability tending to 1 as n tends to infinity) satisfies $\tau_{\mathcal{M}_1}(\Sigma) \geq 2^{\gamma n}$ for some absolute constant $\gamma > 0$. The first result in this vein that applied specifically to $T_{L,d}$ was due to Borgs *et al.* [5], who used a conductance argument to obtain the following.

Theorem 1.1 *There is $c(d) > 0$ (independent of L) such that for λ sufficiently large and all even $L \geq 4$,*

$$\tau_{\mathcal{M}_\lambda(T_{L,d})} > \exp\left\{ \frac{c(d) L^{d-1}}{\log^2 L} \right\}.$$

An examination of [5] reveals that "sufficiently large" may be quantified as $\lambda > c^d$ for a suitable constant $c > 1$. One motivation for [5] was to show that for values of λ for which the hard-core model on the integer lattice $\mathbb{Z}^d$ exhibits multiple Gibbs phases (to be explained below), the mixing of the Glauber dynamics on $T_{L,d}$ should be slow. Dobrushin [8] showed that as long as λ is sufficiently large, there are indeed multiple Gibbs phases in the hard-core model. Specifically, write $\mathcal{E}$ and $\mathcal{O}$ for the sets of even and odd vertices of $\mathbb{Z}^d$ (defined in the obvious way). Equip $\mathbb{Z}^d$ with the usual nearest neighbour adjacency and set

$$\Lambda_L = [-L, L]^d \quad \text{and} \quad \partial\Lambda_L = [-L, L]^d \setminus [-(L-1), L-1]^d.$$

For $\lambda > 0$, choose $\mathbb{I}$ from $\mathcal{I}(\Lambda_L)$ with $\Pr(\mathbb{I} = I) \propto \lambda^{|I|}$. Dobrushin showed that for λ large

$$\lim_{L\to\infty} \mathbb{P}\left(\vec{0} \in \mathbb{I} \mid \mathbb{I} \supseteq \partial\Lambda_L \cap \mathcal{E}\right) > \lim_{L\to\infty} \mathbb{P}\left(\vec{0} \in \mathbb{I} \mid \mathbb{I} \supseteq \partial\Lambda_L \cap \mathcal{O}\right) \tag{1}$$

where $\vec{0} = (0, \ldots, 0)$. Thus, roughly speaking, the influence of the boundary on behavior at the origin persists as the boundary recedes. Informally, this suggests that for λ large, the typical independent set chosen from $T_{L,d}$ according to the hard-core measure is either predominantly odd or predominantly even, and so there is a highly unlikely bottleneck set of balanced independent sets separating the predominantly odd sets from the predominantly even ones. It is the existence of this bottleneck that should cause the mixing of the Glauber dynamics chain to be slow. No explicit bound is given in [8], but several researchers report that Dobrushin's argument works for $\lambda > c^d$ for a suitable constant $c > 1$. A key tool in the proof of Theorem 1.1 is an appeal to a (suitable generalization) of a lemma of Dobrushin from [9], and our main lemma, Lemma 3.5, is of a similar flavour.

In light of a recent result of Galvin and Kahn [12], it is tempting to believe that slow mixing on $T_{L,d}$ should hold for smaller values of λ; even for values of λ tending to 0 as d grows. The main result of [12] is that the hard-core model on $\mathbb{Z}^d$ exhibits multiple Gibbs phases for $\lambda = \omega(d^{-1/4}\log^{3/4} d)$. Specifically, Galvin and Kahn show that for $\lambda \geq cd^{-1/4}\log^{3/4} d$ for sufficiently large c, (1) holds.

In [13], some progress was made towards establishing slow mixing on $T_{L,d}$ for small λ. Let Q_d be the usual discrete hypercube (the graph on $\{0,1\}^d$ in which two strings are adjacent if they differ on exactly one coordinate). Note that $T_{2,d}$ is isomorphic to Q_d. A corollary of the main result of [13] is that for $\lambda = \omega(d^{-1/4}\log^{3/2} d)$,

$$\tau_{\mathcal{M}_\lambda(Q_d)} \geq \exp\left\{\Omega\left(\frac{2^d}{d^2}\right)\right\}.$$

In the present paper, using different methods, we show that for d sufficiently large Glauber dynamics does indeed mix slowly on $T_{L,d}$ for all even $L \geq 4$ for some small values of λ.

Theorem 1.2 *There are constants $c, d_0 > 0$ for which the following holds. For*

$$\lambda \geq cd^{-1/4}\log^{3/4} d, \tag{2}$$

$d \geq d_0$ and $L \geq 4$ even, the Glauber dynamics chain $\mathcal{M}_\lambda$ on $\mathcal{I}(T_{L,d})$ satisfies

$$\tau_{\mathcal{M}_\lambda(T_{L,d})} \geq \exp\left\{\frac{L^{d-1}}{d^4 \log^2 L}\right\}.$$

Our techniques actually apply to the class of ρ-local chains (considered in [5] and also in [10], where the terminology $\rho|V|$-*cautious* is employed) for suitable ρ. A Markov chain $\mathcal{M}$ on state space $\mathcal{I}$ is ρ-*local* if in each step of the chain the states of at most $\rho|V|$ vertices are changed; that is, if

$$P_{\mathcal{M}}(I_1, I_2) \neq 0 \Rightarrow |I_1 \triangle I_2| \leq \rho|V|.$$

Our main theorem is the following.

Theorem 1.3 *There are constants $c, d_0 > 0$ for which the following holds. For λ satisfying (2), $d \geq d_0$, $L \geq 4$ even and ρ satisfying*

$$\rho + \frac{1}{2d^{1/2}} \leq \frac{\lambda}{1+\lambda} \tag{3}$$

and

$$H\left(\frac{1}{2d^{1/2}}\right) + H\left(\rho + \frac{1}{2d^{1/2}}\right) + \left(\frac{1}{d^{1/2}} + \rho\right)\log_2 \lambda + \frac{10}{d^4 L \log^2 L} \leq \log_2(1+\lambda) \tag{4}$$

(where $H(\alpha) = -\alpha \log_2 \alpha - (1-\alpha)\log_2(1-\alpha)$ is the usual binary entropy function), if $\mathcal{M}$ is an ergodic ρ-local Markov chain on state space $\mathcal{I}(T_{L,d})$ with stationary distribution π_λ then

$$\tau_{\mathcal{M}(T_{L,d})} \geq \exp\left\{\frac{L^{d-1}}{d^4 \log^2 L}\right\}.$$

With $\rho = L^{-d}$, (4) is satisfied for all λ satisfying (2) (for sufficiently large d). An L^{-d}-local chain is a single-site update chain and so Theorem 1.2 is a corollary of Theorem 1.3. Taking $\lambda = 1$ we may satisfy (4) with ρ any constant less than $1/2$ by taking d large enough (as a function of ρ). We therefore obtain a further corollary of Theorem 1.3.

Corollary 1.4 *Fix $\rho < 1/2$. There is a constant $d_0 = d_0(\rho) > 0$ for which the following holds. For $L \geq 4$ even and $d \geq d_0$, if $\mathcal{M}$ is an ergodic ρ-local Markov chain on state space $\mathcal{I}(T_{L,d})$ with uniform stationary distribution then*

$$\tau_{\mathcal{M}(T_{L,d})} \geq \exp\left\{\frac{L^{d-1}}{d^4 \log^2 L}\right\}.$$

We prove Theorem 1.3 via a well-known conductance argument (introduced in [15]). A particularly useful form of the argument was given by Dyer, Frieze and Jerrum [10]. Let $\mathcal{M}$

be an ergodic Markov chain on state space Ω with transition probabilities P and stationary distribution π. Let $A \subseteq \Omega$ and $M \subseteq \Omega \setminus A$ satisfy $\pi(A) \leq 1/2$ and

$$\omega_1 \in A, \omega_2 \in \Omega \setminus (A \cup M) \Rightarrow P(\omega_1, \omega_2) = 0.$$

Then from [10] we have

$$\tau_{\mathcal{M}} \geq \frac{\pi(A)}{8\pi(M)}. \tag{5}$$

The intuition behind (5) is that if we start the chain at some state in A, then in order to mix, it must at some point leave A and so pass through M. The ratio of $\pi(A)$ to $\pi(M)$ is a measure of how long the chain must run before it transitions from A to M. So we may think of M as a bottleneck set through which any run of the chain must pass in order to mix; if the bottleneck has small measure, then the mixing time is high.

Now let us return to the setup of Theorem 1.3. Set

$$\mathcal{I}_{b,\rho} = \mathcal{I}_{b,\rho}(T_{L,d}) = \{I \in \mathcal{I}(T_{L,d}) : ||I \cap \mathcal{E}| - |I \cap \mathcal{O}|| \leq \rho L^d/2\}$$

($\mathcal{I}_{b,\rho}$ is the set of *balanced* independent sets) and

$$\mathcal{I}_{\mathcal{E},\rho} = \mathcal{I}_{\mathcal{E},\rho}(T_{L,d}) = \{I \in \mathcal{I}(T_{L,d}) : |I \cap \mathcal{E}| > |I \cap \mathcal{O}| + \rho L^d/2\}.$$

By symmetry, $\pi_\lambda(\mathcal{I}_{\mathcal{E},\rho}) < 1/2$. Notice that since $\mathcal{M}$ changes the state of at most ρL^d vertices in each step, we have that if $I_1 \in \mathcal{I}_{\mathcal{E},\rho}$ and $I_2 \in \mathcal{I}(T_{L,d}) \setminus (\mathcal{I}_{\mathcal{E},\rho} \cup \mathcal{I}_{b,\rho})$ then $P_{\mathcal{M}}(I_1, I_2) = 0$. From (5) we obtain

$$\tau_{\mathcal{M}} \geq \frac{\pi_\lambda(\mathcal{I}_{\mathcal{E},\rho})}{8\pi_\lambda(\mathcal{I}_{b,\rho})} = \frac{1 - \pi_\lambda(\mathcal{I}_{b,\rho})}{16\pi_\lambda(\mathcal{I}_{b,\rho})}.$$

Theorem 1.3 thus follows from the following theorem, whose proof will be the main business of this paper.

Theorem 1.5 *There are constants $c, d_0 > 0$ for which the following holds. For λ satisfying (2), $d \geq d_0$, $L \geq 4$ even and ρ satisfying (3) and (4),*

$$\pi_\lambda(\mathcal{I}_{b,\rho}) \leq \exp\left\{-\frac{2L^{d-1}}{d^4 \log^2 L}\right\}.$$

Theorem 1.5 is the statement that if an independent set I is chosen from $\mathcal{I}(T_{L,d})$ according to the hard-core distribution π_λ, then, as long as λ is sufficiently large, it is extremely unlikely that I is balanced. In particular, if we take $\lambda = 1$ we obtain the following appealing combinatorial corollary.

Corollary 1.6 *Fix $\varepsilon > 0$. There is a constant $d_0 = d_0(\varepsilon) > 0$ for which the following holds. For $L \geq 4$ even and $d \geq d_0$, if $\mathbb{I}$ is a uniformly chosen independent set from $T_{L,d}$ then*

$$\mathbb{P}\left(||\mathbb{I} \cap \mathcal{E}| - |\mathbb{I} \cap \mathcal{O}|| \leq (.25 - \varepsilon)L^d\right) \leq \exp\left\{-\frac{2L^{d-1}}{d^4 \log^2 L}\right\}.$$

2 Overview of the proof of Theorem 1.5

Consider an independent set $I \in \mathcal{I}(T_{L,d})$. Some regions of $T_{L,d}$ consist predominantly of even vertices from I together with their neighbours (the *even-occupied* regions) and some regions consist predominantly of odd vertices from I with their neighbours. These regions are separated by a collection of connected unoccupied two-layer moats or *cutsets* γ. In Section 3.1 we follow [5] and describe a procedure which selects a collection $\Gamma(I)$ of these γ's with the properties that $i)$ the interiors of those $\gamma \in \Gamma(I)$ are mutually disjoint (the *interior* of γ is the smaller of the two parts into which its deletion breaks a graph) and $ii)$ either the interiors of all $\gamma \in \Gamma(I)$ are predominantly even-occupied or they are all predominantly odd-occupied. We do this in the setting of an arbitrary bipartite graph. We also point out some properties of γ that are specific to the torus, including an isoperimetric inequality that gives a lower bound on $|\gamma|$ (the number of edges in γ) in terms of the number of vertices it encloses.

Our main technical result, Lemma 3.5, is the assertion that for each specification of cutset sizes $c_1, \ldots, c_\ell$ and vertices $v_1, \ldots, v_\ell$, the probability that an independent set I has among its associated cutsets $\Gamma(I)$ a collection $\gamma_1, \ldots, \gamma_\ell$ with $|\gamma_i| = c_i$ and with v_i in the interior of γ_i is exponentially small in the sum of the c_i's. The case $\ell = 1$ is essentially contained in [12], and our generalization draws heavily on that paper. It may be worthwhile to compare our Lemma 3.5 with [5, Lemma 6] in which is obtained an exponential bound on the probability of I having a *particular* collection of cutsets.

We use a Peierl's argument (see *e.g.* [14]) to prove Lemma 3.5. For simplicity, we describe the argument here for $\lambda = 1$. For fixed $c_1, \ldots, c_\ell$, $v_1, \ldots, v_\ell$, let $\mathcal{I}_{spec}$ be the collection of $I \in \mathcal{I}(T_{L,d})$ which have a collection of associated cutsets $\gamma_1, \ldots, \gamma_\ell$ with $|\gamma_i| = c_\ell$ and with v_i in the interior of γ_i. For an $I \in \mathcal{I}_{spec}$, fix one such collection $\gamma_1, \ldots, \gamma_\ell$. By modifying I carefully in the interior of each γ_i (specifically, by shifting I one unit in a carefully chosen direction) we can identify a collection of subsets S_i of the vertices of γ_i with $|S_i| = c_i/2d$ which can be added to the modified I, the resulting set still being independent. (Here we exploit the fact that the cutset can be thought of as two unoccupied layers separating the interior from the exterior). By adding arbitrary subsets of each S_i to the modified I, we get a one-to-many map φ from $\mathcal{I}_{spec}$ to $\mathcal{I}(T_{L,d})$ with $|\varphi(I)|$ exponential in the sum of the c_i's.

If the $\varphi(I)$'s would be disjoint for distinct I's, we would essentially be done, having shown that there are exponentially more (in the sum of the c_i's) independent sets than sets in $\mathcal{I}_{spec}$. To deal with the issue of overlaps between the $\varphi(I)$'s, we define a flow $\nu : \mathcal{I}_{spec} \times \mathcal{I}(T_{L,d}) \to [0, \infty)$ supported on pairs (I, J) with $J \in \varphi(I)$ in such a way that the flow out of every $I \in \mathcal{I}_{spec}$ is 1. Any uniform bound we can obtain on the flow into vertices of $\mathcal{I}(T_{L,d})$ is then easily seen to be a bound on $\pi_1(\mathcal{I}_{spec})$.

We define the flow via a notion of approximation modified from [12]. To each cutset γ we associate a set $A(\gamma)$ which approximates the interior of γ in a precise sense, in such a way that as we run over all possible γ, the total number of approximate sets used is small (and in particular, much smaller than the total number of cutsets). There is a clear trade-off here: the more precise the notion of approximation used, the greater the number of approximate sets needed. Then for each $J \in \mathcal{I}(T_{L,d})$ and each collection of approximations $A_1, \dots, A_\ell$ we consider the set of those $I \in \mathcal{I}_{spec}$ with $J \in \varphi(I)$ and with A_i the approximation to γ_i. We define the flow in such a way that if this set is large, then $\nu(I, J)$ is small for each I in the set. In this way we control the flow into J corresponding to each collection of approximations $A_1, \dots, A_\ell$; and since the total number of approximations is small, we control the total flow into J.

In the language of statistical physics, there is a tradeoff between *entropy* and *energy* that we need to control. Each $I \in \mathcal{I}_{spec}$ has high energy — by the shift operation described above, we can perturb it only slightly and map it to an exponentially large collection of independent sets. But before exploiting this fact to show that $\pi_1(\mathcal{I}_{spec})$ is small, we have to account for a high entropy term — there are exponentially many possible cutsets of size c_i that could be associated with an $I \in \mathcal{I}_{spec}$. There are about $\exp\{\Omega(c_i \log d/d)\}$ cutsets of size c_i (this count comes from [17]), each one giving rise to about $\exp\{\Theta(c_i/d)\}$ independent sets, so the entropy term exceeds the energy term and the Peierl's argument cannot succeed. One way to overcome this problem is to allow λ to grow exponentially with d, increasing the energy term (the independent sets obtained from the shift are larger than the pre-shifted sets, and so have greater weight) while not changing the entropy term. This is the approach taken in [5]. Alternatively we could try to salvage the argument for $\lambda = 1$ by somehow decreasing the entropy term. This is where the idea of approximate cutsets comes in. Instead of specifying a cutset γ_i by its c_i edges, we specify a connected collection of roughly $c_i/d^{3/2}$ vertices nearby (in a sense to be made precise) to the cutset, from which a good approximation to the cutset can be constructed in a specified (algorithmic) way. Our entropy term drops to roughly $\exp\{O(c_i \log d/d^{3/2})\}$, much lower than the energy term; so much lower, in fact, that we can rescue the Peierl's argument for values of λ tending to 0 as d grows. The bound $\exp\{O(c_i \log d/d^{3/2})\}$ on the number of connected subsets of $T_{L,d}$ of size $O(c_i/d^{3/2})$ is based on the fact that a Δ-regular graph has at most $2^{O(n \log \Delta)}$ connected induced subgraphs of size n passing through a fixed vertex.

The precise statement of Lemma 3.5 appears in Section 3.2 and the proof appears in Section 4. It is here that the precise notion of approximation used is given, together with the verification that there is a ν that satisfies our diverse requirements. We defer a more detailed discussion of the proof to that section.

Given Lemma 3.5, the proof of Theorem 1.5 is relatively straightforward. We begin by using a naive count to observe that the total measure of those $I \in \mathcal{I}_{b,\rho}$ with $\min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} \leq L^d/4d^{1/2}$ is exponentially small in L^d. This drives our specification of ρ, which is chosen as large as possible so that the naive count gives an exponentially small bound. This allows us in the sequel to consider only those $I \in \mathcal{I}(T_{L,d})$ with $\min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} > L^d/4d^{1/2}$. The naive count consists of considering those subsets X of $T_{L,d}$ with $\min\{|X \cap \mathcal{E}|, |X \cap \mathcal{O}|\} \leq L^d/4d^{1/2}$ and $\max\{|X \cap \mathcal{E}|, |X \cap \mathcal{O}|\} \leq L^d/4d^{1/2} + \rho L^d/2$, without regard for whether $X \in \mathcal{I}(T_{L,d})$.

It remains to consider the case where balanced I satisfies $\min\{|I \cap \mathcal{E}|, |I \cap \mathcal{O}|\} > L^d/4d^{1/2}$. In this case the isoperimetric inequality in the torus allows us to conclude that

$\Gamma(I)$ contains a small subset of cutsets, all with similar lengths, the sum of whose lengths is essentially L^{d-1}. We then use Lemma 3.5 and a union bound to say that the measure of the large balanced independent sets is at most the product of a term that is exponentially small in L^{d-1} (from Lemma 3.5), a term corresponding to the choice of a fixed vertex in each of the interiors, and a term corresponding to the choice of the collection of lengths. The second term will be negligible because our special collection of contours is small and the third will be negligible because the contours all have similar lengths. The detailed proof appears in Section 3.3.

3 Proof of Theorem 1.5

3.1 Cutsets

We describe a way of associating with each $I \in \mathcal{I}(T_{L,d})$ a collection of minimal edge cutsets, following the approach of [5]. Much of the discussion is valid for any bipartite graph, so we present it in that generality.

Let $\Sigma = (V, E)$ be a connected bipartite graph on at least 3 vertices with partition classes $\mathcal{E}$ and $\mathcal{O}$. For $X \subseteq V$, write $\nabla(X)$ for the set of edges in E which have one end in X and one end outside X; $\overline{X}$ for $V \setminus X$; $\partial_{int} X$ for the set of vertices in X which are adjacent to something outside X; $\partial_{ext} X$ for the set of vertices outside X which are adjacent to something in X; X^+ for $X \cup \partial_{ext} X$; $X^{\mathcal{E}}$ for $X \cap \mathcal{E}$ and $X^{\mathcal{O}}$ for $X \cap \mathcal{O}$. Further, for $x \in V$ set $\partial x = \partial_{ext}\{x\}$. In what follows we abuse notation slightly, identifying sets of vertices of V and the subgraphs they induce.

For each $I \in \mathcal{I}(\Sigma)$, each component R of $(I^{\mathcal{E}})^+$ or $(I^{\mathcal{O}})^+$ and each component C of $\overline{R}$, set $\gamma = \gamma_{RC}(I) = \nabla(C)$ and $W = W_{RC}(I) = \overline{C}$. Evidently C is connected, and W consists of R, which is connected, together with a number of other components of $\overline{R}$, each of which is connected and joined to R, so W is connected also. It follows that γ is a minimal edge-cutset in Σ. Define the *size* of γ to be $|\gamma| = |\nabla C|$ $(= |\nabla(W)|)$. Define $\operatorname{int}\gamma$, the *interior* of γ, to be the smaller of C, W (if $|W| = |C|$, take $\operatorname{int}\gamma = W$) and say that γ is *enveloping* if $\operatorname{int}\gamma = W$ (so that R, the component that gives rise to γ, is contained in the interior of γ). Say that I is *even* (respectively, *odd*) if it satisfies the following condition: for every component R of $(I^{\mathcal{E}})^+$ (respectively, $(I^{\mathcal{O}})^+$) there exists a component C of $\overline{R}$ such that $\gamma_{RC}(I)$ is enveloping. Note that there must be an unique such C for each R since the components of $\overline{R}$ are disjoint and each one that gives rise to an enveloping cutset must have more than $|V|/2$ vertices.

Lemma 3.1 *Each $I \in \mathcal{I}(\Sigma)$ is either odd or even.*

Proof: Suppose that I is not even. Then there is a component R of $(\mathcal{I}^{\mathcal{E}})^+$ such that for all components C of $\overline{R}$, $|C| < |V|/2$. Consider a component R' of $(\mathcal{I}^{\mathcal{O}})^+$. It lies inside some component C of $\overline{R}$, so one of the components of $\overline{R'}$, say C', contains $\overline{C}$. Since $|\overline{C}| \geq |V|/2$ the cutset $\gamma_{R'C'}(I)$ is enveloping. It follows that I is odd. □

Lemma 3.2 *For each even $I \in \mathcal{I}(\Sigma)$ there is an associated collection $\Gamma(I)$ of enveloping cutsets with mutually disjoint interiors such that $I^{\mathcal{E}} \subseteq \cup_{\gamma \in \Gamma(I)} \mathrm{int}\, \gamma$.*

Proof: Let $R_1, \ldots, R_m$ be the components of $(I^{\mathcal{E}})^+$. For each i there is one component, C_i say, of $\overline{R_i}$ such that $\gamma_i = \gamma_{R_i C_i}$ is enveloping. We have $I^{\mathcal{E}} \subseteq \cup_{i=1}^m \mathrm{int}\, \gamma_i$.

We claim that for each $i \neq j$ one of $\mathrm{int}\, \gamma_i \subseteq \mathrm{int}\, \gamma_j$, $\mathrm{int}\, \gamma_i \supseteq \mathrm{int}\, \gamma_j$, $\mathrm{int}\, \gamma_i \cap \mathrm{int}\, \gamma_j = \emptyset$ holds. To see this, we consider cases. If $R_j \subseteq C'$ for some component $C' \neq C_i$ of $\overline{R_i}$ then $\mathrm{int}\, \gamma_j \subseteq C' \subseteq \mathrm{int}\, \gamma_i$ $(= \overline{C_i})$. Otherwise, $R_j \subseteq C_i$. In this case, either $C_j \subseteq C_i$ (so $\mathrm{int}\, \gamma_j \supseteq \mathrm{int}\, \gamma_i$) or $C_j \supseteq \overline{C_i}$ (so $\mathrm{int}\, \gamma_j \cap \mathrm{int}\, \gamma_i = \emptyset$). We may take

$$\Gamma(I) = \{\gamma_i \ : \ \text{for all } j \neq i \text{ either } \mathrm{int}\, \gamma_j \subseteq \mathrm{int}\, \gamma_i \text{ or } \mathrm{int}\, \gamma_i \cap \mathrm{int}\, \gamma_j = \emptyset\}.$$

□

The following lemma identifies some key properties of $\gamma \in \Gamma(I)$ for even I. In the proof of Theorem 1.5 these properties only come into play through Lemma 3.5.

Lemma 3.3 *For each even I and $\gamma \in \Gamma(I)$, we have the following.*

$$\partial_{int} W \subseteq \mathcal{O} \ \textit{and} \ \partial_{ext} W \subseteq \mathcal{E}; \tag{6}$$

$$\partial_{int} W \cap I = \emptyset \ \textit{and} \ \partial_{ext} W \cap I = \emptyset; \tag{7}$$

$$\forall x \in \partial_{int} W, \ \partial x \cap W \cap I \neq \emptyset \tag{8}$$

and

$$W^{\mathcal{O}} = \partial_{ext} W^{\mathcal{E}} \ \textit{and} \ W^{\mathcal{E}} = \left\{ y \in \mathcal{E} : \partial y \subseteq W^{\mathcal{O}} \right\}. \tag{9}$$

Proof: We begin by noting that $\partial_{int} W \subseteq \partial_{int} R$ (specifically, $\partial_{int} W = \partial_{int} R \cap \partial_{ext} C = \partial_{ext} C$) and $\partial_{ext} W = \partial_{int} C$. Since $\partial_{int} R \subseteq \mathcal{O}$ and $\partial_{int} C \subseteq \mathcal{E}$, (6) follows immediately from these observations.

By construction, $R \cap \mathcal{O} \cap I = \emptyset$, so $\partial_{int} W \cap I = \emptyset$. If there is $x \in \partial_{int} C \cap I$ then, since $x \in \mathcal{E}$ and there is $y \in R$ adjacent to x, we would have $x \in R$, a contradiction; so $\partial_{int} C \cap I = \emptyset$, giving (7).

It is clear that for all $x \in \partial_{int} R$ there is $y \in R \cap I$ with x adjacent to y; so (8) follows from $\partial_{int} W \subseteq \partial_{int} R$.

Since $\partial_{int} W \subseteq \mathcal{O}$, we have $W^{\mathcal{O}} \supseteq \partial_{ext} W^{\mathcal{E}}$. If there is $y \in W^{\mathcal{O}}$ with $\partial y \cap W^{\mathcal{E}} = \emptyset$, then the connectivity of W implies that $W = W^{\mathcal{O}}$ (and that $W^{\mathcal{O}}$ consists of a single vertex). But $W^{\mathcal{E}}$ is non-empty; so we get the reverse containment $W^{\mathcal{O}} \subseteq \partial_{ext} W^{\mathcal{E}}$.

The containment $W^{\mathcal{E}} \subseteq \{y \in \mathcal{E} : \partial y \subseteq W^{\mathcal{O}}\}$ follows immediately from $W^{\mathcal{O}} \supseteq \partial_{ext} W^{\mathcal{E}}$. For the reverse containment, consider (for a contradiction) $y \in \mathcal{E}$ with $\partial y \subseteq W^{\mathcal{O}}$ but $y \notin W^{\mathcal{E}}$. We must have $y \in C$; but y is not adjacent to anything else in C, and $|C| > 1$ (indeed, $|C| \geq |V|/2 > 1$ since γ is enveloping), a contradiction since C is connected. So we have $W^{\mathcal{E}} \supseteq \{y \in \mathcal{E} : \partial y \subseteq W^{\mathcal{O}}\}$. □

We now return to $T_{L,d}$. Set $\mathcal{I}_{even} = \{I \in \mathcal{I}(T_{L,d}) : I \text{ even}\}$ and define $\mathcal{I}_{odd}$ analogously. The next lemma establishes some of the geometric properties of $T_{L,d}$ that we will need. Before stating it we need some more notation.

For $k \geq 1$, we say that $S \subseteq V(T_{L,d})$ is *k-clustered* if for every $x, y \in S$ there is a sequence $x = x_0, \ldots, x_m = y$ of vertices of S such that $d(x_{i-1}, x_i) \leq k$ for all $i = 1, \ldots m$, where $d(\cdot, \cdot)$ is the usual graph distance. Note that S can be partitioned uniquely into maximal k-clustered subsets; we refer to these as the *k-components* of S.

For a cutset γ, we define a graph G_γ as follows. The vertex set of G_γ is the set of edges of $T_{L,d}$ that comprise γ. Declare $e, f \in \gamma$ to be adjacent in G_γ if either e and f share exactly one endpoint and if the coordinate on which the endpoints of e differ is different from the coordinate on which the endpoints of f differ (*i.e.*, e and f are not parallel) or if the endpoints of e and f determine a cycle of length four (a square) in $T_{L,d}$. (This is equivalent to the following construction, well known in the statistical physics literature: for $e \in \gamma$, let $e^\star$ be the dual $(d-1)$-dimensional cube which is orthogonal to e and bisects it when $T_{L,d}$ is considered as immersed in the continuum torus. Then declare $e, f \in \gamma$ to be adjacent if $e^\star \cap f^\star$ is a $(d-2)$-dimensional cube.) We say that a cutset γ is *trivial* if G_γ has only one component.

Lemma 3.4 *For each $I \in \mathcal{I}_{even}$ and $\gamma \in \Gamma(I)$,*

$$|\gamma| \geq |W|^{1-1/d}; \tag{10}$$

$$\textit{for large enough } d,\ |\gamma| \geq d^{1.9}; \tag{11}$$

$$\textit{if } \gamma \textit{ is not trivial then each component of } G_\gamma \textit{ has at least } L^{d-1} \textit{ edges} \tag{12}$$

and

$$\textit{either } \partial_{int} W \textit{ is 2-clustered or each of its 2-components has size at least } L^{d-1}/2d. \tag{13}$$

Proof: For (10) and (11) we appeal to an isoperimetric inequality of Bollobás and Leader [4] which states that if $A \subseteq V(T_{L,d})$ with $|A| \leq L^d/2$, then

$$|\partial_{ext} A| \geq \min\left\{2|A|^{1-1/r} r L^{(d/r)-1} : r = 1, \ldots, d\right\}.$$

From this (10) follows easily, as does (11) once we observe that $|W| \geq 2d+1$ (since $W^{\mathcal{E}} \neq \emptyset$) and that $|\gamma| \geq |\partial_{ext} W|$.

From [5, Lemma 3] we have (12). Finally we turn to (13). Let $C_1, \ldots, C_\ell$ be the components of G_γ, and for each i let C_i' be the vertices of $\partial_{int} W$ which are endpoints of edges of C_i. It is readily checked that each C_i' is 2-clustered and that $\partial_{int} W = \cup_i C_i'$. If $\ell = 1$ we therefore have that $\partial_{int} W$ is 2-clustered. If $\ell > 1$, we have (by (12)) that each C_i has at least L^{d-1} edges. Since each vertex in $T_{L,d}$ has degree $2d$, it follows that each C_i' has size at least $L^{d-1}/2d$. Since the C_i''s are 2-clustered, each 2-component of $\partial_{int} W$ has size at least $L^{d-1}/2d$, establishing (13). □

3.2 The main lemma

For $c \in \mathbb{N}$ and $v \in V(T_{L,d})$ set

$$\mathcal{W}(c,v) = \left\{\gamma \,:\, \gamma \in \Gamma(I) \text{ for some } I \in \mathcal{I}_{even}, |\gamma| = c, v \in W^{\mathcal{E}}\right\}$$

and set $\mathcal{W} = \cup_{c,v}\mathcal{W}(c,v)$. A *profile* of a collection $\{\gamma_1, \ldots, \gamma_\ell\} \subseteq \mathcal{W}$ is a vector $\underline{p} = (c_1, v_1, \ldots, c_\ell, v_\ell)$ with $\gamma_i \in \mathcal{W}(c_i, v_i)$ for all i. Given a profile vector $\underline{p}$ set

$$\mathcal{I}(\underline{p}) = \{I \in \mathcal{I}_{even} :\, \Gamma(I) \text{ contains a subset with profile } \underline{p}\}.$$

Our main lemma is the following.

Lemma 3.5 *There are constants $c, c', d_0 > 0$ such that the following holds. For all even $L \geq 4$, $d \geq d_0$, λ satisfying (2) and profile vector $\underline{p}$,*

$$\pi_\lambda(\mathcal{I}(\underline{p})) \leq \exp\left\{-\frac{c'\beta(\lambda)\sum_{i=1}^{\ell} c_i}{d}\right\}, \tag{14}$$

where $\beta(\lambda) = 2\log(1+\lambda) - \log(1+2\lambda)$.

This may be thought of as an extension of the main result of [12], which treats only $\ell = 1$ and in a slightly less general setting. We will derive Theorem 1.5 from Lemma 3.5 in Section 3.3 before proving the lemma in Section 4. From here on we assume that the conditions of Theorem 1.5 and Lemma 3.5 are satisfied (with c and d_0 sufficiently large to support our assertions). All constants implied in O and Ω statements will be absolute. When it makes no difference to do otherwise, we assume that all large numbers are integers. We note for future reference that for λ satisfying (2) we have

$$\frac{\lambda}{1+\lambda} = \omega\left(\frac{1}{d^{1/4}}\right) \quad \text{and} \quad \beta(\lambda) = \omega\left(\frac{1}{d^{1/2}}\right). \tag{15}$$

3.3 The proof of Theorem 1.5

We begin with an easy count that dispenses with small balanced independent sets. Set

$$\mathcal{I}_{small} = \left\{I \in \mathcal{I}_{b,\rho} : \min\{|I^{\mathcal{E}}|, |I^{\mathcal{O}}|\} \leq L^d/4d^{1/2}\right\}.$$

and $\mathcal{I}_{large} = \mathcal{I}_{b,\rho} \setminus \mathcal{I}_{small}$.

Lemma 3.6

$$\pi_\lambda(\mathcal{I}_{small}) \leq \exp\left\{-\frac{3L^{d-1}}{d^4 \log^2 L}\right\}.$$

Proof: We need a well-known result of Chernoff [6] (see also [3], p.11). Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli random variables with $\mathbf{P}(X_1 = 1) = p$. Then for $k \le pn$

$$\mathbf{P}\left(\sum_{i=1}^{n} X_i \le k\right) \le 2^{nH_p\left(\frac{k}{n}\right)}$$

where $H_p(x) = x\log_2(p/x) + (1-x)\log_2((1-p)/(1-x))$. Note that $H_p(x) = H(x) + x\log_2 p + (1-x)\log_2(1-p)$ where $H(x)$ is the usual binary entropy function. Taking $p = \lambda/(1+\lambda)$ we see that for a set X with $|X| = n$ and for $c \le \lambda/(1+\lambda)$,

$$\begin{aligned}\sum_{A\subseteq X,\ |A|\le cn} \frac{\lambda^{|A|}}{(1+\lambda)^n} &\le 2^{nH_{\lambda/(1+\lambda)}(c)} \\ &= 2^{n\left(H(c)+c\log_2\frac{\lambda}{1+\lambda}+(1-c)\log_2\frac{1}{1+\lambda}\right)} \\ &= 2^{n(H(c)+c\log_2\lambda-\log_2(1+\lambda))}\end{aligned}$$

from which it follows that

$$\sum_{A\subseteq X,\ |A|\le cn} \lambda^{|A|} \le 2^{n(H(c)+c\log_2\lambda)}. \tag{16}$$

Now using $(1+\lambda)^{L^d/2}$ as a trivial lower bound on $\sum_{I\in\mathcal{I}(T_{L,d})}\lambda^{|I|}$ and with the subsequent inequalities justified below, we have

$$\begin{aligned}\pi_\lambda(\mathcal{I}_{small}) &\le 2\left(\sum_{A\subseteq\mathcal{E},\ |A|\le L^d/4d^{1/2}} \lambda^{|A|}\right)\left(\sum_{B\subseteq\mathcal{O},\ |B|\le(1/2d^{1/2}+\rho)L^d/2} \lambda^{|B|}\right)(1+\lambda)^{-L^d/2} \\ &\le \frac{2\exp_2\left\{\frac{L^d}{2}\left(H\left(\frac{1}{2d^{1/2}}\right)+H\left(\frac{1}{2d^{1/2}}+\rho\right)+\left(\frac{1}{d^{1/2}}+\rho\right)\log_2\lambda\right)\right\}}{(1+\lambda)^{L^d/2}} \qquad (17) \\ &\le \exp\left\{-\frac{2L^{d-1}}{d^4\log^2 L}\right\}. \qquad (18)\end{aligned}$$

In (17) we use (16) (legitimate since $1/2d^{1/2} \le \lambda/(1+\lambda)$ and $1/2d^{1/2}+\rho \le \lambda/(1+\lambda)$, the former by (15) and the latter by (3)); (18) follows from (4). $\square$

Set $\mathcal{I}_{large,\ even} = \mathcal{I}_{large} \cap \mathcal{I}_{even}$ and define $\mathcal{I}_{large,\ odd}$ analogously. By Lemma 3.1 $\mathcal{I}_{large} = \mathcal{I}_{large,\ even} \cup \mathcal{I}_{large,\ odd}$ and by symmetry $\pi_\lambda(\mathcal{I}_{large,\ even}) = \pi_\lambda(\mathcal{I}_{large,\ odd})$. In the presence of Lemma 3.6, Theorem 1.5 reduces to bounding (say)

$$\pi_\lambda(\mathcal{I}_{large,\ even}) \le \exp\left\{-\frac{3L^{d-1}}{d^4\log^2 L}\right\}. \tag{19}$$

Set $\mathcal{I}^{non-trivial}_{large,\ even} = \{I \in \mathcal{I}_{large,\ even} :$ there is $\gamma \in \Gamma(I)$ with $|\gamma| \ge L^{d-1}\}$ and $\mathcal{I}^{trivial}_{large,\ even} = \mathcal{I}_{large,\ even} \setminus \mathcal{I}^{non-trivial}_{large,\ even}$. With the sum below running over all vectors $\underline{p}$ of the form (c, v)

with $v \in V(T_{L,d})$ and $c \geq L^{d-1}$, and with the inequalities justified below, we have

$$\begin{aligned}
\pi_\lambda(\mathcal{I}^{non-trivial}_{large,\ even}) &\leq \sum_{\underline{p}} \pi_\lambda(\mathcal{I}(\underline{p})) \\
&\leq L^{2d} \exp\left\{-\Omega\left(\frac{L^{d-1}\beta(\lambda)}{d}\right)\right\} \qquad (20) \\
&\leq \exp\left\{-\Omega\left(\frac{L^{d-1}}{d^{3/2}}\right)\right\} \qquad (21)
\end{aligned}$$

We have used Lemma 3.5 in (20) and the factor of L^{2d} is for the choices of c and v. In (21) we have used (15).

For $I \in \mathcal{I}^{trivial}_{large,\ even}$ and $\gamma \in \Gamma(I)$ we have $|\gamma| \geq |\mathrm{int}\,\gamma|^{1-1/d}$ (by (10)) and so

$$\sum_{\gamma \in \Gamma(I)} |\gamma|^{d/(d-1)} \geq \sum_{\gamma \in \Gamma(I)} |\mathrm{int}\,\gamma| \geq |I^{\mathcal{E}}| \geq L^d/4d^{1/2}.$$

The second inequality is from Lemma 3.2 and the third follows since $I \notin \mathcal{I}_{small}$.

Set $\Gamma_i(I) = \{\gamma \in \Gamma(I) : 2^{i-1} \leq |\gamma| < 2^i\}$. Note that $\Gamma_i(I)$ is empty for $2^i < d^{1.9}$ (recall (11)) and for $2^{i-1} > L^{d-1}$ so we may assume that

$$1.9 \log d \leq i \leq (d-1)\log L + 1. \qquad (22)$$

Since $\sum_{m=1}^{\infty} 1/m^2 = \pi^2/6$, there is an i such that

$$\sum_{\gamma \in \Gamma_i(I)} |\gamma|^{\frac{d}{d-1}} \geq \Omega\left(\frac{L^d}{d^{1/2} i^2}\right). \qquad (23)$$

Choose the smallest such i set $\ell = |\Gamma_i(I)|$. We have $\sum_{\gamma \in \Gamma_i(I)} |\gamma| \geq \Omega(\ell 2^i)$ (this follows from the fact that each $\gamma \in \Gamma_i(I)$ satisfies $|\gamma| \geq 2^{i-1}$) and

$$O\left(\frac{dL^d}{2^i}\right) \geq \ell \geq \Omega\left(\frac{L^d}{2^{\frac{id}{d-1}} i^2 d^{1/2}}\right). \qquad (24)$$

The first inequality follows from that fact that $\sum_\gamma |\gamma| \leq dL^d = |E(T_{L,d})|$; the second follows from (23) and the fact that each γ has $|\gamma|^{d/(d-1)} \leq 2^{di/(d-1)}$. We therefore have $I \in \mathcal{I}(\underline{p})$ for some $\underline{p} = (c_1, v_1, \ldots, c_\ell, v_\ell)$ with ℓ satisfying (24), with

$$\sum_{j=1}^{\ell} c_j \geq O(\ell 2^i), \qquad (25)$$

with

$$c_j \leq 2^i \qquad (26)$$

for each j and with i satisfying (22). With the sum below running over all profile vectors $\underline{p}$ satisfying (22), (24), (25) and (26) we have

$$\begin{aligned}\pi_\lambda(\mathcal{I}^{trivial}_{large,\ even}) &\leq \sum_{\underline{p}} \pi_\lambda(\mathcal{I}(\underline{p})) \\ &\leq d \log L \max_{i \text{ satisfying (22)}} 2^{\ell i} \binom{L^d}{\ell} \exp\left\{-\Omega\left(\frac{\ell 2^i \beta(\lambda)}{d}\right)\right\}. \end{aligned} \tag{27}$$

In (27) we have used Lemma 3.5. The factor of $d \log L$ is an upper bound on the number of choices for i; the factor of $2^{\ell i}$ is for the choice of the c_j's; and the factor $\binom{L^d}{\ell}$ is for the choice of the ℓ (distinct) v_j's. By (22), the second inequality in (24) and the second inequality in (15) we have (for d sufficiently large)

$$\begin{aligned}2^{\ell i}\binom{L^d}{\ell} &\leq 2^{\ell i}\left(\frac{L^d}{\ell}\right)^\ell \\ &\leq 2^{\ell i}\left(O\left(2^{\frac{id}{d-1}} i^2 d^{1/2}\right)\right)^\ell \\ &\leq 2^{4\ell i} \\ &= \exp\left\{o\left(\frac{2^i \beta(\lambda)}{d}\right)\right\}.\end{aligned}$$

Inserting into (27) we finally get

$$\begin{aligned}\pi_\lambda(\mathcal{I}^{trivial}_{large,\ even}) &\leq d \log L \max_i \exp\left\{-\Omega\left(\frac{2^i \beta(\lambda) \ell}{d}\right)\right\} \\ &\leq d \log L \max_i \exp\left\{-\Omega\left(\frac{2^i \beta(\lambda) L^d}{d 2^{\frac{id}{d-1}} i^2 d^{1/2}}\right)\right\} &(28) \\ &\leq \exp\left\{-\frac{4L^{d-1}}{d^4 \log^2 L}\right\}. &(29)\end{aligned}$$

In (28) we have taken ℓ as small as possible, and in (29) we have taken i as large as possible and used (15).

Combining (29) and (21) we obtain (19) and so Theorem 1.5.

4 Proof of Lemma 3.5

Our strategy is the following. Let a profile vector $\underline{p} = (c_1, v_1, \ldots, c_\ell, v_\ell)$ be given. Set $\underline{p'} = (c_2, v_2, \ldots, c_\ell, v_\ell)$. We will show

$$\frac{\pi_\lambda(\mathcal{I}(\underline{p}))}{\pi_\lambda(\mathcal{I}(\underline{p'}))} \leq \exp\left\{-\Omega\left(\frac{c_1 \beta(\lambda)}{d}\right)\right\}. \tag{30}$$

Then by a telescoping product

$$\pi_\lambda(\mathcal{I}(\underline{p})) \leq \frac{\pi_\lambda(\mathcal{I}(\underline{p}))}{\pi_\lambda(\mathcal{I}_{even})} \leq \exp\left\{-\Omega\left(\frac{\beta(\lambda)\sum_{i=1}^{\ell} c_i}{d}\right)\right\}$$

as claimed. To obtain (30) we employ a general strategy to bound $\pi_\lambda(\mathcal{S})/\pi_\lambda(\mathcal{T})$ for $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{I}(T_{L,d})$ (note that $\mathcal{I}(\underline{p}) \subseteq \mathcal{I}(\underline{p'})$). We define a one-to-many map φ from $\mathcal{S}$ to $\mathcal{T}$. We then define a flow $\nu : \mathcal{S} \times \mathcal{T} \to [0, \infty)$ supported on pairs (I, J) with $J \in \varphi(I)$ satisfying

$$\forall I \in \mathcal{S}, \quad \sum_{J \in \varphi(I)} \nu(I, J) = 1 \tag{31}$$

and

$$\forall J \in \mathcal{T}, \quad \sum_{I \in \varphi^{-1}(J)} \lambda^{|I|-|J|}\nu(I, J) \leq M. \tag{32}$$

This gives

$$\begin{aligned}
\sum_{I \in \mathcal{S}} \lambda^{|I|} &= \sum_{I \in \mathcal{S}} \lambda^{|I|} \sum_{J \in \varphi(I)} \nu(I, J) \\
&= \sum_{J \in \mathcal{T}} \lambda^{|J|} \sum_{I \in \varphi^{-1}(J)} \lambda^{|I|-|J|}\nu(I, J) \\
&\leq M \sum_{J \in \mathcal{T}} \lambda^{|J|}
\end{aligned}$$

and so $\pi_\lambda(\mathcal{S})/\pi_\lambda(\mathcal{T}) \leq M$. So our task is to define φ and ν for $\mathcal{S} = \mathcal{I}(\underline{p})$ and $\mathcal{T} = \mathcal{I}(\underline{p'})$ for which (32) holds with M given by the right-hand side of (30).

Much of what follows is modified from [12]. The main result of [12] has already been described in Section 1. It will be helpful here to describe the main technical work of that paper. Let Λ_L be the box $[-L, L]^d$ in $\mathbb{Z}^d$ with boundary $\partial^\star\Lambda_L = [-L, L]^d \setminus [-(L-1), L-1]^d$. Write $\mathcal{J}$ for the set of independent sets in Λ_L which extend $\partial^\star\Lambda_L \cap \mathcal{O}$ and, for a fixed vertex $v_0 \in \Lambda_L \cap \mathcal{E}$, write $\mathcal{I}$ for those $I \in \mathcal{J}$ with $v_0 \in I$. The stated aim of [12] is to show, using a similar strategy to that described above, that $\pi_\lambda(\mathcal{I})/\pi_\lambda(\mathcal{J}) \leq (1+\lambda)^{-2(d-o(1))}$. More specifically, for each $I \in \mathcal{I}$ let $\gamma'(I)$ be the cutset associated with that component of $(I^{\mathcal{E}})^+$ that includes v_0. For each w_o, w_e write $\mathcal{I}(w_o, w_e)$ for those $I \in \mathcal{I}$ with $|W^{\mathcal{E}}| = w_e$ and $|W^{\mathcal{O}}| = w_o$, where W is the subset of Λ_L associated with $\gamma'(I)$ as described in Section 3.1. It is shown in [12] (inequalities (62) and (63) of that paper) that for λ satisfying (2) we have

$$\frac{\pi_\lambda(\mathcal{I}(w_o, w_e))}{\pi_\lambda(\mathcal{J})} \leq \begin{cases} \exp\{-\Omega(\lambda^2(w_o - w_e))\} & \text{for } \lambda < 2 \text{ and} \\ \lambda^{-\Omega(w_o - w_e)} & \text{for larger } \lambda \end{cases} \tag{33}$$

from which the stated bound on $\pi_\lambda(\mathcal{I})/\pi_\lambda(\mathcal{J})$ is easily obtained by a summation. The remainder of this paper is devoted to an explanation of how the proof of (33) needs to be augmented and modified to obtain our main lemma, and we do not state the proofs of many of our intermediate lemmas, since they can be found in the generality we need in [12]. The main technical issue we have to deal with in moving from (33) to Lemma 3.5 relates to dealing with γ that are non-trivial (in the sense defined before the proof of Lemma 3.4); this is not an issue in [12] because it is shown there that the cutsets $\gamma'(I)$ described above are always trivial.

One technical issue aside, the specification of φ is relatively straightforward. For each $s \in \{\pm 1, \ldots, \pm d\}$, define σ_s, the *shift in direction* s, by $\sigma_s(x) = x + e_s$, where e_s is the sth standard basis vector if $s > 0$ and $e_s = -e_{-s}$ if $s < 0$. For $X \subseteq V(T_{L,d})$, write $\sigma_s(X)$ for $\{\sigma_s(x) : x \in X\}$. For a cutset $\gamma \in \mathcal{W}$ set $W^s = \{x \in \partial_{int} W : \sigma_s^{-1}(x) \notin W\}$. We will obtain $\varphi(I)$ by shifting I inside W in a certain direction s and adding arbitrary subsets of W^s to the result, where W is associated with a cutset $\gamma \in \Gamma(I) \cap \mathcal{W}(c_1, v_1)$. The success of this process depends on the fact that I is disjoint from the vertex set of γ. We now formalize this.

Lemma 4.1 *Let $I \in \mathcal{I}(\underline{p})$ be given. Let $\gamma \in \Gamma(I)$ be such that $|\gamma| = c_1$ and $v_1 \in W^{\mathcal{E}}$ where $W = \operatorname{int} \gamma$. For any choice of s, it holds that*

$$I_0 := (I \setminus W) \cup \sigma_s(I \cap W) \text{ is in } \mathcal{I}(\underline{p'})$$

and has the same size as I. Moreover, the sets I_0 and W^s are mutually disjoint and

$$I_0 \cup W^s \in \mathcal{I}(\underline{p'}).$$

Proof: That $I_0 \cup W^s$ is an independent set and that I_0 is the same size as I is the content of [12, Proposition 2.12]. Because $\operatorname{int} \gamma$ is disjoint from the interiors of the remaining cutsets and the shift operation that creates $I_0 \cup W^s$ only modifies I inside W it follows that $I_0, I_0 \cup W^s \in \mathcal{I}(\underline{p'})$. □

For $I \in \mathcal{I}(\underline{p})$ we define

$$\varphi(I) = \{I_0 \cup S : S \subseteq W^s\}$$

for a certain s to be chosen presently. In light of Lemma 4.1, $\varphi(I) \subseteq \mathcal{I}(\underline{p'})$ regardless of this choice.

To define ν and s we employ the notion of approximation also used in [12] and introduced by Sapozhenko in [21]. For $\gamma \in \mathcal{W}$ we say that $A \subseteq V(T_{L,d})$ is an *approximation* of γ if

$$A^{\mathcal{E}} \supseteq W^{\mathcal{E}} \quad \text{and} \quad A^{\mathcal{O}} \subseteq W^{\mathcal{O}}, \tag{34}$$

$$d_{A^{\mathcal{O}}}(x) \geq 2d - \sqrt{d} \text{ for all } x \in A^{\mathcal{E}} \tag{35}$$

and

$$d_{\mathcal{E} \setminus A^{\mathcal{E}}}(x) \geq 2d - \sqrt{d} \text{ for all } y \in \mathcal{O} \setminus A^{\mathcal{O}}, \tag{36}$$

where $d_X(x) = |\partial x \cap X|$. Note that since $W_{\mathcal{O}} = \partial W_{\mathcal{E}}$, W is an approximation of γ.

To motivate the definition of approximation, note that by (9) if u is in $W^{\mathcal{E}}$ then all of its neighbors are in $W^{\mathcal{O}}$, and if u' is in $\mathcal{O} \setminus W^{\mathcal{O}}$ then all of its neighbors are in $\mathcal{E} \setminus W^{\mathcal{E}}$. If we think of $A^{\mathcal{E}}$ as approximate-$W^{\mathcal{E}}$ and $A^{\mathcal{O}}$ as approximate-$W^{\mathcal{O}}$, (35) says that if $u \in \mathcal{E}$ is in approximate-$W^{\mathcal{E}}$ then almost all of its neighbors are in approximate-$W^{\mathcal{O}}$ while (36) says that if $u' \in \mathcal{O}$ is not in approximate-$W^{\mathcal{O}}$ then almost all of its neighbors are not in approximate-$W^{\mathcal{E}}$.

Before stating our main approximation lemma, which is a slight modification of [12, Lemma 2.18], it will be convenient to further refine our partition of cutsets. To this end set

$$\mathcal{W}(w_e, w_o, v) = \{\gamma : \gamma \in \Gamma(I) \text{ for some } I \in \mathcal{I}_{even}, |W^{\mathcal{O}}| = w_o, |W^{\mathcal{E}}| = w_e, v \in W^{\mathcal{E}}\}.$$

Note that (by (9))

$$|\gamma| = |\nabla(W)| = 2d(|W^{\mathcal{O}}| - |W^{\mathcal{E}}|)$$

so $\mathcal{W}(w_e, w_o, v) \subseteq \mathcal{W}((w_o - w_e)/2d, v)$.

Lemma 4.2 *For each w_e, w_o and v there is a family $\mathcal{A}(w_e, w_o, v)$ satisfying*

$$|\mathcal{A}(w_e, w_o, v)| \leq \exp\left\{O\left((w_o - w_e)d^{-\frac{1}{2}} \log^{\frac{3}{2}} d\right)\right\}$$

and a map $\Pi : \mathcal{W}(w_e, w_o, v) \to \mathcal{A}(w_e, w_o, v)$ such that for each $\gamma \in \mathcal{W}(w_e, w_o, v)$, $\Pi(\gamma)$ is an approximation for γ.

The proof of this lemma is deferred to Section 4.1. Our bound on the number of approximate cutsets with parameters w_e, w_o and v is much smaller than any bound we are able to obtain on the number of cutsets with the same set of parameters. This is where we make the entropy gain discussed in Section 2.

We are now in a position to define ν and s. Our plan for each fixed $J \in \mathcal{I}(\underline{p'})$ is to fix w_e, w_o and $A \in \mathcal{W}(w_e, w_o)$ and to consider the contribution to the sum in (32) from those $I \in \varphi^{-1}(J)$ with $\Pi(\gamma(I)) = A$. We will try to define ν in such a way that each of these individual contributions to (32) is small; to succeed in this endeavour we must first choose s with care. To this end, given $\gamma \in \mathcal{W}(w_e, w_o, v)$, set

$$Q^{\mathcal{E}} = A^{\mathcal{E}} \cap \partial_{ext}(\mathcal{O} \setminus A^{\mathcal{O}}) \quad \text{and} \quad Q^{\mathcal{O}} = (\mathcal{O} \setminus A^{\mathcal{O}}) \cap \partial_{ext} A^{\mathcal{E}},$$

where $A = \Pi(\gamma)$ in the map guaranteed by Lemma 4.2. To motivate the introduction of $Q^{\mathcal{E}}$ and $Q^{\mathcal{O}}$, note that for $\gamma \in \Pi^{-1}(A)$ we have

$$\begin{aligned} A^{\mathcal{E}} \setminus Q^{\mathcal{E}} &\subseteq W^{\mathcal{E}} \\ \mathcal{E} \setminus A^{\mathcal{E}} &\subseteq \mathcal{E} \setminus W^{\mathcal{E}} \\ A^{\mathcal{O}} &\subseteq W^{\mathcal{O}} \end{aligned}$$

and

$$\mathcal{O} \setminus (A^{\mathcal{O}} \cup Q^{\mathcal{O}}) \subseteq \mathcal{O} \setminus W^{\mathcal{O}}$$

(all using (9) and (34)). It follows that for each $\gamma \in \Pi^{-1}(A)$, $Q^{\mathcal{E}} \cup Q^{\mathcal{O}}$ contains all of the vertices whose location in the partition $T_{L,d} = W \cup \overline{W}$ is as yet unknown.

Lemma 4.3 *For $\gamma \in \mathcal{W}(w_e, w_o, v)$, there is an $s \in \{\pm 1, \ldots, \pm d\}$ such that both of*

$$|W^s| \geq .8(w_o - w_e) \quad \textit{and} \quad |\sigma_s(Q^{\mathcal{E}}) \cap Q^{\mathcal{O}}| \leq \frac{5|W^s|}{\sqrt{d}}$$

hold.

Proof: [12, (49) and (50)]. □

We choose the smallest such s to be the lattice direction associated with γ. Note that s depends on γ but not on I.

Now for each $I \in \mathcal{I}(\underline{p})$ let $\gamma \in \Gamma(I)$ be a particular cutset with $\gamma \in \mathcal{W}(c_1, v_1)$. Let $\varphi(I)$ be as defined before, with s as specified by Lemma 4.3. Define

$$C = W^s \cap A^{\mathcal{O}} \cap \sigma_s(Q^{\mathcal{E}})$$

and

$$D = W^s \setminus C,$$

and for each $J \in \varphi(I)$ set

$$\nu(I, J) = \lambda^{|J \cap W^s|} \left(\frac{\lambda}{(1+\lambda)^2}\right)^{|C \cap J|} \left(\frac{1+2\lambda}{(1+\lambda)^2}\right)^{|C \setminus J|} \left(\frac{1}{1+\lambda}\right)^{|D|}.$$

Note that for $I \in \varphi^{-1}(J)$, $\nu(I, J)$ depends on W but not on I itself.

Noting that $C \cup D$ partitions W we have

$$\begin{aligned}
\sum_{J \in \varphi(I)} \nu(I, J) &= \sum_{A \subseteq C,\ B \subseteq D} \lambda^{|A|+|B|} \left(\frac{\lambda}{(1+\lambda)^2}\right)^{|A|} \left(\frac{1+2\lambda}{(1+\lambda)^2}\right)^{|C|-|A|} \left(\frac{1}{1+\lambda}\right)^{|D|} \\
&= \sum_{B \subseteq D} \frac{\lambda^{|B|}}{(1+\lambda)^{|D|}} \sum_{A \subseteq C} \left(\frac{\lambda^2}{1+2\lambda}\right)^{|A|} \left(\frac{1+2\lambda}{(1+\lambda)^2}\right)^{|C|} \\
&= \frac{(1+\lambda)^{|D|}}{(1+\lambda)^{|D|}} \left(\frac{1+2\lambda+\lambda^2}{1+2\lambda}\right)^{|C|} \left(\frac{1+2\lambda}{(1+\lambda)^2}\right)^{|C|} \\
&= 1,
\end{aligned}$$

so ν satisfies (31). To obtain (14) we must establish (32) with M given by the right-hand side of (30).

Fix w_e, w_o such that $2d(w_o - w_e) = c_1$. Fix $A \in \mathcal{A}(w_e, w_o, v_1)$ and $s \in \{\pm 1, \ldots, \pm d\}$. For I with $\gamma(I) \in \mathcal{W}(w_e, w_o, v_1)$ write $I \sim_s A$ if it holds that $\Pi(\gamma) = A$ and $s(I) = s$. The next lemma, which bounds the contribution to the sum in (32) from those $I \in \varphi^{-1}(J)$ with $I \sim_s A$, is the heart of the whole proof, and perhaps the principal inequality of [12]. We extract it directly from [12]; although the setting here is slightly different, the proof is identical to the equivalent statement in [12].

Lemma 4.4 *For $J \in \mathcal{I}(\underline{p'})$,*

$$\sum \left\{ \lambda^{|I|-|J|} \nu(I,J) : I \sim_s A,\ I \in \varphi^{-1}(J) \right\} \leq \left(\frac{\sqrt{1+2\lambda}}{1+\lambda} \right)^{w_o - w_e}.$$

Proof: [12, Section 2.12]. $\square$

We are now only a short step away from (14). With the steps justified below we have that for each $J \in \mathcal{I}(\underline{p'})$

$$\begin{aligned} \sum_{I \in \varphi^{-1}(J)} \lambda^{|I|-|J|} \nu(I,J) &\leq \sum_{w_e, w_o} \sum_{s,A} \sum \left\{ \lambda^{|I|-|J|} \nu(I,J) : I \sim_s A,\ I \in \varphi^{-1}(J) \right\} \\ &\leq 2dc_1^{\frac{2d}{d-1}} |\mathcal{A}(w_e, w_o, v_1)| \left(\frac{\sqrt{1+2\lambda}}{1+\lambda} \right)^{\frac{c_1}{2d}} &(37) \\ &\leq 2dc_1^{\frac{2d}{d-1}} \exp\left\{ -\Omega\left(\frac{c_1 \beta(\lambda)}{d} \right) \right\} &(38) \\ &\leq \exp\left\{ -\Omega\left(\frac{c_1 \beta(\lambda)}{d} \right) \right\} &(39) \end{aligned}$$

completing the proof of (32). In (37), we note that there are $|\mathcal{A}(w_e, w_o, v_1)|$ choices for the approximation A, $2d$ choices for s and $c_1^{d/(d-1)}$ choices for each of w_e, w_o (this is because $c_1 \geq (w_e + w_o)^{1-1/d}$ by (10)), and we apply Lemma 4.4 to bound the summand. In (38) use Lemma 4.2 and the fact that for any $c > 0$ we can choose $c' > 0$ such that whenever $\lambda > c' d^{-1/4} \log^{3/4} d$ and $d = d(c)$ is sufficiently large we have

$$\exp\left\{ cd^{-\frac{1}{2}} \log^{\frac{3}{2}} d \right\} \frac{\sqrt{1+2\lambda}}{1+\lambda} \leq \exp\left\{ -\frac{\beta(\lambda)}{4} \right\}.$$

Finally in (39) we use $c_1 \geq d^{1.9}$ (by (11)) and the second inequality in (15) to bound $2dc_1^{2d/(d-1)} = \exp\{o(c_1 \beta(\lambda)/d)\}$.

4.1 Proof of Lemma 4.2

We obtain Lemma 4.2 by combining a sequence of lemmas. Lemma 4.5, which we extract directly from [12], establishes the existence for each γ of a very small set of vertices nearby to γ whose neighbourhood can be thought of as a coarse approximation to γ. (We will elaborate on this after the statement of the lemma.) Lemma 4.6 shows that there is a small collection of these coarse approximations such that every $\gamma \in \mathcal{W}(w_e, w_o, v)$ is approximated by one of the collection. Our proof of this lemma for γ trivial is from [12], but we need to add a new ingredient to deal with non-trivial γ. Finally Lemma 4.7, which we extract directly from [12], turns the coarse approximations of Lemma 4.6 into the more refined approximations of Lemma 4.2 without increasing the number of approximations too much.

Given $\gamma \in \mathcal{W}(w_e, w_o, v)$ set

$$\partial'_{int}W = \{x \in \partial_{int}W : d_{W^{\mathcal{E}}}(x) \le d\} \quad \text{and} \quad \partial'_{int}C = \{x \in \partial_{int}C : d_{C^{\mathcal{O}}}(x) \le d\}.$$

(Recall that $d_X(x) = |\partial x \cap X|$.)

Lemma 4.5 *For each $\gamma \in \mathcal{W}(w_e, w_o, v)$ there is a U with the following properties.*

$$U \subseteq N(\partial'_{int}W \cup \partial'_{int}C) \tag{40}$$

$$N(U) \supseteq \partial'_{int}W \cup \partial'_{int}C \tag{41}$$

and

$$|U| \le O\left((w_o - w_e)\sqrt{\frac{\log 2d}{2d}}\right) \tag{42}$$

where $N(X) = \cup_{x \in X}\partial x$.

To motivate Lemma 4.5, let us point out that in [12, (34)] it is observed that for U satisfying (40) and (41) the removal of $N(U)$ from $V(T_{L,d})$ separates W from C. U may therefore be thought of a coarse approximation to γ: removing U and its neighbourhood achieves the same effect as removing γ. However, U is very much smaller than γ (γ has $2d(w_o - w_e)$ edges). By focusing on specifying U instead of γ, we lose some information, but we gain because fewer choices have to be made to specify U. The engine driving the proof of Lemma 3.5 is the fact that the gain far outweighs the loss. Lemma 4.5 is [12, Lemma 2.15] and we omit the proof.

Lemma 4.6 *For each w_e, w_o and v there is a family $\mathcal{U}(w_e, w_o, v)$ satisfying*

$$|\mathcal{U}(w_e, w_o, v)| \le \exp\left\{O\left((w_o - w_e)d^{-\frac{1}{2}}\log^{\frac{3}{2}} d\right)\right\}$$

and a map $\Pi^{\mathcal{U}} : \mathcal{W}(w_e, w_o, v) \to \mathcal{U}(w_e, w_o, v)$ such that for each $\gamma \in \mathcal{W}(w_e, w_o, v)$, $\Pi^{\mathcal{U}}(\gamma)$ satisfies (40), (41) and (42).

Proof: It is observed in [12, paragraph after (35)] that for U satisfying (40) and (41) we have

$$\text{for all } x \in \partial_{int}W,\ d(x, U) \le 2 \tag{43}$$

and

$$\text{for all } y \in U,\ d(y, \partial_{int}W) \le 2. \tag{44}$$

Let U satisfy (43), (44) and (42) for some $\gamma \in \mathcal{W}(w_e, w_o, v)$ and let $W_1, \ldots, W_k$ be the 2-components of $\partial_{int}W$. For each $j = 1, \ldots, k$ let

$$U_j = \{y \in U : d(y, x) \le 2 \ \text{ for some } x \in W_j\}.$$

We claim that each U_j is 6-clustered. To see this, fix $u, v \in W_j$ and take $x_u \in W_j$ at distance at most 2 from u and $x_v \in W_j$ at distance at most 2 from v. Let $x_u = x_0, \ldots, x_\ell = x_v$ be a sequence of vertices in W_j with $d(x_{i-1}, x_i) \leq 2$ for each i. For $i = 1, \ldots, \ell - 1$, take $u_i \in W_j$ with $d(u_i, x_i) \leq 2$. Then the sequence $u = u_0, u_1, \ldots, u_{\ell-1}, u_\ell = v$ has the property that $d(u_{i-1}, u_i) \leq 6$ for each i, establishing the claim.

To bound the number of possibilities for U we first consider the case $2d(w_o - w_e) \leq L^{d-1}$. In this case, by (13), all γ under consideration are trivial (in the sense defined before the statement of Lemma 3.3) and $k = 1$.

We show that there is a small (size $O(w_o d^2)$) set of vertices meeting all possible U's in this case. Fix a linear ordering $\ll$ of $\mathcal{O}$ satisfying

$$d(v, y_1) < d(v, y_2) \quad \Longrightarrow \quad y_1 \ll y_2,$$

and let T be the initial segment of $\ll$ of size w_o. We claim that $T \cap \partial_{int} W \neq \emptyset$. If $T = W^{\mathcal{O}}$, this is clear; if not, consider a shortest $y - v$ path in $T_{L,d}$ for some $y \in T \setminus W^{\mathcal{O}}$. This path intersects $W^{\mathcal{O}}$ (since $\partial v \subseteq W^{\mathcal{O}}$). Let y' be the largest (with respect to $\ll$) vertex of $W^{\mathcal{O}}$ on the path; then $y' \in \partial_{int} W \cap T$, establishing our claim. There are at most w_o possibilities for $y' \in \partial_{int} W \cap T$, so at most $O(w_o d^2)$ possibilities for a vertex x' with $d(x', y') \leq 2$; and by (43) U must contain such an x'.

In this case we may take $\mathcal{U}(w_e, w_o, v)$ to be the collection of all 6-connected subsets of $V(T_{L,d})$ of size at most $O((w_o - w_e)\sqrt{\log 2d/2d})$ containing one of the $O(w_o d^2)$ vertices described in the last paragraph. Using the fact that in any graph with maximum degree Δ the number of connected, induced subgraphs of order n containing a fixed vertex is at most $(e\Delta)^n$ (see, *e.g.*, [12, Lemma 2.1]) we infer that

$$\begin{aligned} |\mathcal{U}(w_e, w_o, v)| &\leq O(w_o d^2)(d^7)^{O\ (w_o - w_e)\sqrt{\frac{\log 2d}{2d}}} && (45) \\ &\leq \exp\left\{O\left((w_o - w_e)d^{-\frac{1}{2}}\log^{\frac{3}{2}} d\right)\right\}, && (46) \end{aligned}$$

as required. The factor of $O(w_o d^2)$ in (45) accounts for the choice of a fixed vertex in U; the exponent $O((w_o - w_e)\sqrt{\log 2d/2d})$ is from (42); and the d^7 accounts for the fact that U is connected in a graph with maximum degree at most $65d^6$. In (46) we use (10) to bound $2d(w_o - w_e) \geq (w_o + w_e)^{1-1/d} \geq w_o^{3/4}$ and so (since $w_o \geq 2d$) $\log(w_o d^2) = o((w_o - w_e)d^{-1/2}\log^{3/2} d)$.

In the case where $2d(w_o - w_e) > L^{d-1}$, by (13) each of the components of γ has at least L^{d-1} edges, so γ has at most $dL^d/L^{d-1} = dL$ components and U at most dL 6-components. In this case we may take $\mathcal{U}(w_e, w_o, v)$ to be the collection of all subsets of $V(T_{L,d})$ of size at most $O((w_o - w_e)\sqrt{\log 2d/2d})$ containing at most dL 6-components. As in the previous case we have

$$\begin{aligned} |\mathcal{U}(w_e, w_o, v)| &\leq (L^d)^{dL}(d^7)^{O\ (w_o - w_e)\sqrt{\frac{\log 2d}{2d}}} \sum_{j=1}^{dL} \binom{O\left(((w_o - w_e)\sqrt{\frac{\log 2d}{2d}}\right) + j - 1}{j - 1} \\ &\leq \exp\left\{O\left((w_o - w_e)d^{-\frac{1}{2}}\log^{\frac{3}{2}} d\right)\right\}, && (47) \end{aligned}$$

as required, the extra factors in the first inequality accounting for the choice of a fixed vertex in each of the at most dL 6-components and of the sizes of each of the 6-components. To obtain (47) we use $w_o \leq L^d$ to bound $(L^d)^{dL} \sum_{j=1}^{dL} \binom{O((w_o-w_e)\sqrt{\log 2d/2d})+j-1}{j-1} \leq 2^{O(d^2 L \log L)}$ and $2d(w_o - w_e) \geq L^{d-1}$ to bound $d^2 L \log L = o((w_o - w_e)d^{-1/2} \log^{3/2} d)$. $\square$

The next lemma turns $\mathcal{U}(w_e, w_o, v)$ into the collection of approximations postulated in Lemma 4.2. It is a straightforward combination of [12, Lemmas 2.16, 2.17, 2.18], and we omit the proof. Combining Lemmas 4.6 and 4.7 we obtain Lemma 4.2.

Lemma 4.7 *For each $U \in \mathcal{U}(w_e, w_o, v)$ there is a family $\mathcal{V}(w_e, w_o, v)$ satisfying*

$$|\mathcal{V}(w_e, w_o, v)| \leq \exp\left\{O\left((w_o - w_e)d^{-\frac{1}{2}} \log^{\frac{3}{2}} d\right)\right\}$$

and a map $\Pi^{\mathcal{V}} : \mathcal{U}(w_e, w_o, v) \to \mathcal{V}(w_e, w_o, v)$ such that for each $\gamma \in \mathcal{W}(w_e, w_o, v)$ and $U \in \mathcal{U}(w_e, w_o, v)$ with $\Pi^{\mathcal{U}}(\gamma) = U$, $\Pi^{\mathcal{V}}(U)$ is an approximation of γ.

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