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On the independence ratio of distance graphs

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ABSTRACT

A distance graph is an undirected graph on the integers where two integers are adjacent if their difference is in a prescribed distance set. The *independence ratio* of a distance graph G is the maximum density of an independent set in G. Lih et al. (1999) showed that the independence ratio is equal to the inverse of the fractional chromatic number, thus relating the concept to the well studied question of finding the chromatic number of distance graphs.

We prove that the independence ratio of a distance graph is achieved by a periodic set, and we present a framework for discharging arguments to demonstrate upper bounds on the independence ratio. With these tools, we determine the exact independence ratio for several infinite families of distance sets of size three and determine asymptotic values for others.

1. Introduction

For a set S of positive integers, the *distance graph* $G(S)$ is the infinite graph with vertex set $\mathbb{Z}$ where two integers i and j are adjacent if and only if $|i - j| \in S$. Intense study of distance graphs began when Eggleton, Erdős, and Skilton [15,16] defined them as a modified version of the Hadwiger–Nelson problem of coloring the unit-distance graph on $\mathbb{R}^2$. The chromatic number of distance graphs has since been widely studied [2,5,8–11,13,14,19,15–18,22,24–27,29,31–37,40–42].

A particularly effective tool for finding lower bounds on the chromatic number is to determine the *fractional chromatic number*, $\chi_f(S) = \chi_f(G(S))$. A *fractional coloring* of a graph G is a function c from the independent sets I of G to nonnegative real numbers such that for every vertex v, the sum $\sum_{I \ni v} c(I) \geq 1$, and the *value* of the coloring is the sum $\sum_I c(I)$ taken over all independent sets I. The fractional chromatic number $\chi_f(G)$ is the minimum value of a fractional coloring, and provides a lower bound on the chromatic number. In fact, $\lceil \chi_f(G) \rceil \leq \chi(G)$ and this inequality is frequently sharp in the case of distance graphs [8–10,29,42].

For an independent set A in $G(S)$ the density $\delta(A)$ is equal to $\limsup_{N\to\infty} \frac{|A\cap[-N,N]|}{2N+1}$. The *independence ratio* $\overline{\alpha}(S)$ is the supremum of $\delta(A)$ over all independent sets A in $G(S)$. Lih, Liu and Zhu showed that determining the fractional chromatic number $\chi_f(S)$ of a distance graph $G(S)$ is equivalent to determining its independence ratio.

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Theorem 1 (*Lih, Liu, and Zhu [31]*). *Let S be a finite set of positive integers. Then* $\chi_f(S) = \overline{\alpha}(S)^{-1}$.

Thus, the previous results computing the fractional chromatic number of distance graphs apply to the independence ratio. We summarize these results in Section 3. The inequality $\chi_f(S) \le \overline{\alpha}(S)^{-1}$ is easy to determine if you have access to a *periodic* independent set in $G(S)$ of density $\overline{\alpha}(S)$. While this was acknowledged by previous work, it is surprising that no one proved that such an independent set exists. Thus, we demonstrate that such periodic extremal sets exist.

Theorem 2. *Let S be a finite set of positive integers and let* $s = \max S$. *There exists a periodic independent set A in* $G(S)$ *with period at most* $s2^s$ *where* $\delta(A) = \overline{\alpha}(S)$.

Our proof of Theorem 2 uses a lemma (Lemma 3) about extremal walks in finite digraphs that we also apply to show there exist periodic extremal sets for dominating sets and identifying codes in finitely-generated distance graphs. This lemma may be also applicable in other situations.

We also develop several fundamental techniques for determining the independence ratio, which we then apply to several infinite families of distance sets of size three. To prove upper bounds on $\overline{\alpha}(S)$, we develop a new discharging method. The resulting *Local Discharging Lemma* (Lemma 21) is then used extensively to give exact values of $\overline{\alpha}(S)$ for several infinite families of distance sets. We witness several common themes among these proofs, and these themes may be evidence that discharging arguments of this type could be used to determine almost all values of $\overline{\alpha}(S)$.

For an integer n, the *circulant graph* $G(n, S)$ is the graph whose vertices are the integers modulo n where two integers i and j are adjacent if and only if $|i-j| \equiv k \pmod{n}$, for some $k \in S$. The distance graph $G(S)$ can be considered to be the limit structure of the sequence of circulant graphs $G(n, S)$. Thus, extremal questions over the circulant graphs lead to extremal questions on the distance graph. For instance, the equality $\overline{\alpha}(S) = \limsup_{n\to\infty} \alpha(G(n, S))/n$ is a consequence of Theorem 1.

Since the complement of a circulant graph is also a circulant graph, and the independence number of a graph is the clique number of its complement, studying the independence ratio of distance graphs is strongly related to determining the independence number and clique number of circulant graphs. In particular, simultaneously bounding the independence number and clique number of circulant graphs has shown lower bounds on Ramsey numbers [30]. So far, these parameters have been studied for circulant graphs $G(n, S)$ when limited to special classes of sets S, whether algebraically defined [1,4,6,12,20,28,39] or with S finite and n varying [3,7,23].

A recent development is the discovery that certain circulant graphs $G(n, S)$ are *uniquely* K_r*-saturated*, including three infinite families [21]. A graph H is uniquely K_r-saturated if H contains no copy of K_r and for every pair $uv \notin E(H)$ there is a unique copy of K_r in $H + uv$. The first step in proving this property is showing that the clique number of $G(n, S)$ is equal to $r - 1$. In the three infinite families, the generating set S uses a growing number of elements, but the complement of the graph uses a finite number of elements. The complement $\overline{G(n, S)}$ is another circulant graph $G(n, S')$ where S' has a finite number of elements. The independence number of $G(n, S')$ is of particular interest. Our discharging method is an adaptation of the discharging method used in [21] to determine the independence number in circulant graphs.

We start in Section 2 by proving Theorem 2, that the independence ratio in a distance graph is achievable by a periodic independent set. We take the opportunity there to show two quick applications of the proof technique to related density problems for other types of subset of $G(S)$. In Section 3 we summarize previous results on the independence ratio. The next section collects some introductory results concerning $\overline{\alpha}(S)$, and then in Section 5 we define our discharging process and its connection to the independence ratio, proving the Local Discharging Lemma. We use the Local Discharging Lemma to prove exact values of $\overline{\alpha}(S)$ for several families of sets S in Section 6. We determine the independence ratio for a range of graphs with generator sets of size 3. This extends work of Zhu [42], who considered the fractional chromatic number of $G(S)$ in distance graphs with generator sets of size 3. Finally in Section 7 we discuss the algorithm we used to compute values of $\overline{\alpha}(S)$ for specific finite sets S. The computed values of $\overline{\alpha}(\{1, 1+k, 1+k+i\})$ are given as a table in Appendix A, while more values are given in data available online.[1]

Our notation is standard. Throughout the paper we consider S to be a finite, nonempty set of positive integers. For a positive integer n, we write $[n] = \{1, \dots, n\}$, and similarly $[a, b] = \{a, a+1, \dots, b-1, b\}$. When $d \ge 1$, we let $d \cdot S = \{d \cdot s : s \in S\}$ and $S + d = \{s + d : s \in S\}$.

2. Periodic sets of extremal value

In this section, we prove Theorem 2. As a consequence, we give an alternative proof of Theorem 1. We start by proving a lemma that implies that periodic extremal sets exist for several different extremal problems on distance graphs, such as dominating sets and r-identifying codes.

Consider a finite directed graph G where every vertex v is given a weight $w(v)$. Let $W = (v_i)_{i\in\mathbb{Z}}$ be a doubly infinite walk on G. Then the *upper average weight* $\overline{w}(W)$ of W is defined as $\limsup_{N\to\infty} \frac{\sum_{i=-N}^{N} w(v_i)}{2N+1}$, and the *lower average weight* $\underline{w}(W)$ of W is defined as $\liminf_{N\to\infty} \frac{\sum_{i=-N}^{N} w(v_i)}{2N+1}$. Given a simple cycle C in G, define the infinite walk W_C by infinitely repeating C. Observe that $\overline{w}(W_C) = \underline{w}(W_C) = \frac{\sum_{v\in V(C)} w(v)}{|C|}$.

[1] See http://www.math.iastate.edu/dstolee/r/distance.htm for all data files.

Lemma 3. *Let G be a finite, vertex-weighted digraph. The supremum of upper average weights (or infimum of lower average weights) of infinite walks on G is equal to the maximum upper average weight (or minimum lower average weight, respectively) of some infinite walk W_C where C is a simple cycle.*

Proof. We prove that every infinite walk W has $\overline{w}(W)$ bounded above by the maximum upper average weight of a simple cycle. In the case of minimizing $\underline{w}(W)$, we can simply negate the weights of the vertices and apply the maximization case.

Fix $W = (v_i)_{i\in\mathbb{Z}}$. Let $n = |V(G)|$ and let N be large. We will approximate the fraction $\frac{\sum_{i=-N}^{N} w(v_i)}{2N+1}$ by a convex combination of upper average weights of simple cycles of G. This approximation improves as N grows, so we find the limit definition of $\overline{w}(W)$ is bounded above by the maximum upper average weight of a simple cycle.

Let $X = (x_i)_{i\in[a,b]}$ be any finite walk. If $x_i, x_{i+1}, \ldots, x_{i+\ell}$ are distinct vertices where $x_{i+\ell+1} = x_i$, then we say $C = (x_i, x_{i+1}, \ldots, x_{i+\ell})$ is a simple cycle within X (note that a simple cycle may have one or two vertices). Define X' to be the *contraction of* C *in* X by removing the subwalk $(x_i, x_{i+1}, \ldots, x_{i+\ell})$ from X.

Let $W_N = (v_i)_{i\in[-N,N]}$. Starting with $W^{(1)} = W_N$, we iteratively construct a list of walks $W^{(1)}, \ldots, W^{(t+1)}$ and a list of simple cycles $C_1, \ldots, C_t$ such that C_i is a simple cycle in $W^{(i)}$ and $W^{(i+1)}$ is the contraction of C_i in $W^{(i)}$. This iterative construction stops when $W^{(t+1)}$ does not contain a simple cycle. Such a walk does not repeat any vertices, so $|W^{(t+1)}| \le n \ll N$. Observe that

$$\sum_{i=-N}^{N} w(v_i) = \sum_{j=1}^{t}\left[\sum_{v_i\in C_j} w(v_i)\right] + \sum_{v_i\in W^{(t+1)}} w(v_i)$$
$$= \sum_{j=1}^{t} |C_j|\overline{w}(C_j) + \sum_{v_i\in W^{(t+1)}} w(v_i).$$

Let C be a simple cycle of G that maximizes $\overline{w}(C)$ and u be a vertex of G that maximizes $w(u)$, so $\sum_{j=1}^{t} |C_j|\overline{w}(C_j) + \sum_{v_i\in W^{(t+1)}} w(v_i) \le |W_N|\overline{w}(C) + nw(u)$ and hence

$$\overline{w}(W) = \limsup \frac{\sum_{i=-N}^{N} w(v_i)}{2N+1} \le \limsup \frac{(2N+1)\overline{w}(C) + nw(u)}{2N+1} = \overline{w}(C).$$

Therefore, if W_C is the infinite walk given by repeating the cycle C, then $\overline{w}(W_C) \ge \overline{w}(W)$. ■

We now proceed to use Lemma 3 to prove that there is a periodic independent set that attains the independence ratio. Our technique is a general approach that shows, for any type of object that is a subset of vertices in distance graphs, the extremal density is attained by periodic objects.

Let $X \subseteq \mathbb{Z}$ be an object satisfying a property $\mathcal{P}$ in a distance graph $G(S)$. We consider the intersection of X with disjoint consecutive intervals of fixed length ℓ:

$$\ldots, X \cap [-\ell+1, 0],\ X \cap [1, \ell],\ X \cap [\ell+1, 2\ell], \ldots.$$

There are only a finite number of possibilities for $X \cap [k\ell+1, (k+1)\ell]$, considered up to translation by multiples of ℓ. We call such patterns "states", and encode in a "state graph" which states can follow a given state. Since there are a finite number of states, we can apply Lemma 3 to the state graph to obtain an extremal periodic object.

More formally, a *state* is a set $T \subseteq [\ell]$. An object X satisfying property $\mathcal{P}$ has state T on the interval $[k\ell+1, (k+1)\ell]$, where $k \in \mathbb{Z}$, if $X \cap [k\ell+1, (k+1)\ell] = T + k\ell$. An *admissible state* is a state T such that there exists an object X with property $\mathcal{P}$ such that X has state T on some interval. A *transition* occurs between two admissible states T and T' if there exists an object X with property $\mathcal{P}$ and an integer k such that X has state T on interval $[k\ell+1, (k+1)\ell]$ and state T' on interval $[(k+1)\ell+1, (k+2)\ell]$. Let the *state graph* be the directed graph where the vertices are the admissible states, and the directed edges are the transitions. The weight of a state is determined by the type of objects, but is usually the density $|T|/\ell$.

Given an object X with property $\mathcal{P}$, the states $T_i = X \cap [i\ell+1, (i+1)\ell]$ give rise to an infinite walk in the state graph. Every doubly infinite walk $W = (T_i)_{i\in\mathbb{Z}}$ in the state graph gives rise to a subset Y of $\mathbb{Z}$ by $Y = \bigcup_{i\in\mathbb{Z}}(T_i + i\ell)$ for $i \in \mathbb{Z}$. Suppose that every subset Y created from an infinite walk in this way also has property $\mathcal{P}$. For the properties that we are interested in, this fact can usually be guaranteed by choosing ℓ large enough. By applying Lemma 3 to the state graph, there is a cycle C whose weight is extremal. Let W be the walk formed by infinitely repeating C, and let Y be the corresponding object with property $\mathcal{P}$. Note that Y is periodic with period ℓ times the length of C. Since there are at most 2^ℓ states, the period of Y is at most $\ell 2^\ell$.

A type is *admissible* if T is an independent set in $G(S)$. Recall that S is finite and $s = \max S$, so there are at most 2^s admissible types. An ordered pair of admissible types (T, T') is *compatible* if $T \bigcup (T' + s)$ is independent. Let the *state graph* be the directed graph where the admissible types are vertices, and the compatible pairs are the edges.

Proof of Theorem 2. We consider independent sets in $G(S)$. Let $s = \max S$, and choose interval lengths of $\ell = s$. The states are subsets of $[s]$, the admissible states are subsets of $[s]$ that are independent in $G(S)$, and the weight of a state T is $|T|/s$. State T can transition to state T' if $T \cup (T' + s)$ is an independent set. Given any independent set X in $G(S)$, the states $T_i = X \cap [is + 1, (i + 1)s]$ give rise to an infinite walk W in the state graph since the intersection of X with two consecutive intervals remains independent. Note that the density $\delta(X)$ is equal to $\overline{w}(W)$.

Next we show that every subset $Y = \bigcup_{i\in\mathbb{Z}}(T_i + is)$ created from an infinite walk on the state graph is an independent set in $G(S)$. If not, there are $j_1, j_2 \in Y$ such that $|j_1 - j_2| \in S$. Since $\ell = s$ both j_1 and j_2 either belong to the same $T_i + is$ or two consecutive states, $T_i + is$ and $T_{i+1} + (i + 1)s$. Both j_1 and j_2 cannot belong to the same $T_i + is$, since T_i is admissible and hence independent in $G(S)$. If $j_1 \in T_i + is$ and $j_2 \in T_{i+1} + (i + 1)s$, then $(T_i + is) \bigcup (T_{i+1} + (i + 1)s)$ is not independent, which contradicts that there is a transition between those states.

By Lemma 3, the weight of an infinite walk in the state graph is maximized by a simple cycle C. The independent set Y created from the infinite walk repeating C has density $\overline{\alpha}(S)$ and is periodic with a period of length $s|C|$, which is at most $s2^s$. ■

To demonstrate the versatility of Lemma 3, we also prove that other problems on distance graphs admit periodic extremal sets.

A set of vertices D is *dominating* if every vertex in the graph is either in D or adjacent to a vertex in D.

Theorem 4. *Let S be a finite set of positive integers and set $s = \max S$. The minimum density of a dominating set in $G(S)$ is achieved by a periodic set with period at most $(2s)2^{2s}$.*

Proof. We consider dominating sets in $G(S)$. Let $s = \max S$, and choose interval lengths of $\ell = 2s$. The states are subsets of $[2s]$, and all states are admissible. The weight of a state T is $|T|/(2s)$. State T can transition to state T' if every vertex in $[s + 1, 3s]$ either is in $T \cup (T' + 2s)$ or has a neighbor in $T \cup (T' + 2s)$. Given any dominating set X in $G(S)$, the states $T_i = X \cap [i(2s) + 1, (i + 1)2s]$ give rise to an infinite walk W in the state graph since every integer in the interval $[i(2s) + s + 1, i(2s) + 3s]$ is either in, or has a neighbor in, the set $(T_i + i(2s)) \cup (T_{i+1} + (i + 1)(2s))$. Note that the density $\delta(X)$ is equal to $\overline{w}(W)$.

Next we show that every subset $Y = \bigcup_{i\in\mathbb{Z}}(T_i + i(2s))$ created from an infinite walk on the state graph is a dominating set in $G(S)$. If not, then there exists $j \notin Y$ that has no neighbor in Y. The integer j is in some interval $[i(2s) + 1, (i + 1)2s]$. If $j \in [i(2s)+1, (i)2s+s]$, then since there is a transition from T_{i-1} to T_i, j must have a neighbor in $Y \cap [(i-1)(2s)+1, (i+1)(2s)]$. If $j \in [i(2s)+s+1, (i+1)2s]$, then since there is a transition from T_i to T_{i+1}, j must have a neighbor in $Y \cap [(i)(2s)+1, (i+2)(2s)]$.

By Lemma 3, the weight of an infinite walk in the state graph is minimized by a simple cycle C. The dominating set Y created from the infinite walk repeating C has minimum periodic with a period of length $2s|C|$, which is at most $(2s)2^{2s}$. ■

Define the *ball $B_r(u)$ of radius r centered at u* to be the set of all vertices in $G(S)$ that are distance at most r from u. A set A of vertices is an *r-identifying code* if for every pair of distinct vertices u and v in $G(S)$, the sets $A \cap B_r(u)$ and $A \cap B_r(v)$ are nonempty and distinct.

Theorem 5. *Let S be a finite set of positive integers and set $s = \max S$. The minimum density of a 1-identifying code in $G(S)$ is achieved by a periodic set with period at most $(6s)2^{6s}$.*

Proof. Let $s = \max S$, and choose intervals of length $\ell = 6s$. The states are subsets of $[6s]$; hence there are at most 2^{6s} admissible states. The weight of a state T is $T/(6s)$. State T can transition to state T' if in $T \cup (T' + 6s)$ every distinct vertices $u \in [3s + 1, 9s]$ and $v \in [s + 1, 11s]$ have the property $N[u] \cap (T \cap T' + 6s) \neq N[v] \cap (T \cap T' + 6s)$.

Next we show that every subset $Y = \bigcup_{i\in\mathbb{Z}}(T_i + i(6s))$ created from an infinite walk on the state graph is a 1-identifying code in $G(S)$. If not, then there exist distinct $u, v \in \mathbb{Z}$ where $N[u] \cap Y = N[v] \cap Y$. The integer u is in some interval $[i6s + 3s + 1, (i + 1)6s + 3s]$ for some i. Since $N[u] \cap N[v] \neq \emptyset$, then u and v must be within distance $2s$, and hence $v \in [i6s + s + 1, (i + 1)6s + 5s]$. Thus, u and v are both in $(T_i + i6s) \cup (T_{i+1} + (i + 1)6s)$, where there is a transition from T_i to T_{i+1}.

By Lemma 3, the weight of an infinite walk in the state graph is minimized by a simple cycle C. The 1-identifying code Y created from the infinite walk repeating C has minimum density and is periodic with a period of length $6s|C|$, which is at most $(6s)2^{6s}$. ■

Observe that an r-identifying code in a graph G corresponds to a 1-identifying code in G^r, where G^r is the graph with the same vertex set as G but two vertices u, v are adjacent in G^r if they have distance at most r in G. Further, $G(S)^r$ is a distance graph with distance set $S' = \bigcup_{t=1}^{r}\left\{\sum_{i=1}^{t} a_i : |a_i| \in S\right\}$, and $\max S' = r \max S$. Thus, we have the following corollary.

Corollary 6. *Let S be a finite set of positive integers and set $s = \max S$. The minimum density of an r-identifying code in $G(S)$ is achieved by a periodic set with period at most $(6sr)2^{6sr}$.*

For completeness, we demonstrate a similar proof for chromatic number, using an unweighted statement analogous to Lemma 3. This result was previously shown by Eggleton, Erdő, and Skilton [16].

Theorem 7. *Let S be a finite set of positive integers and set $s = \max S$. For $k = \chi(G(S))$, there exists a periodic proper k-coloring c with minimum period at most sk^s.*

Proof. A state is a coloring $c : [s] \to \{1, \dots, k\}$. A state c is admissible if the partial coloring induced on $G(S)$ is proper. State c can transition to state c' if the partial coloring

$$c''(i) = \begin{cases} c(i) & \text{if } i \in [s] \\ c'(i-s) & \text{if } i \in [s+1, 2s] \end{cases}$$

is proper in $G(S)$. The state graph has admissible states as vertices and transitions as edges. As in the case of independent sets, there is a correspondence between proper k-colorings of $G(S)$ and infinite walks in the state graph. Since there is a proper k-coloring of $G(S)$, there is some infinite walk in the state graph, and hence the state graph contains at least one cycle. Infinitely repeating this cycle corresponds to a periodic proper k-coloring with period at most sk^s. ■

To finish this section, we present an alternate proof of one inequality of Theorem 1 using an extremal periodic independent set. The original proof by Lih, Liu, and Zhu [31] first showed equality between $\chi_f(G(n, S))$ and $\alpha(G(n, S))/n$ and then took the limit.

Theorem 1 (*Lih, Liu, and Zhu [31]*)**.** *Let S be a finite set of positive integers. Then*

$$\chi_f(S) = \overline{\alpha}(S)^{-1}.$$

Proof. First we show $\chi_f(S) \le \overline{\alpha}(S)^{-1}$. Let $\mathrm{Ind}(S)$ be the family of independent sets in $G(S)$. By Theorem 2, there exists a periodic independent set A with period p and $\delta(A) = \overline{\alpha}(S)$. Form the fractional chromatic coloring c by assigning $c(A+i) = \frac{1}{p\delta(A)}$ for all $i \in \{0, \dots, p-1\}$ and $c(I) = 0$ for all other independent sets I. Since A is periodic and contains $p\delta(A)$ elements within any interval of length $[p]$, a vertex x appears in exactly $p\delta(A)$ sets $A + i$. Thus, $\sum_{\mathrm{Ind}(S): v \in I} c(I) = 1$, so c is a fractional coloring. The value of this fractional coloring is $p \cdot \frac{1}{p\delta(A)} = \overline{\alpha}(S)^{-1}$.

Next we show $\chi_f(S) \ge \overline{\alpha}(S)^{-1}$, which is the analogue for infinite graphs of the well known fact that $\chi_f(G) \ge \frac{1}{\alpha(G)}$ for a finite graph G. Let $G(S)[-n, n]$ be the finite subgraph of $G(S)$ induced by the vertices $[-n, n]$. We know

$$\frac{2n+1}{\alpha(G(S)[-n,n])} \le \chi_f(G(S)[-n,n]) \le \chi_f(S)$$

for all $n \in \mathbb{Z}$. We show $\overline{\alpha}(S) \ge \limsup_{n\to\infty} \frac{\alpha(G(S)[-n,n])}{2n+1}$, which with the above inequality implies $\chi_f(S) \ge \overline{\alpha}(S)^{-1}$.

Let $L := \limsup_{n\to\infty} \frac{\alpha(G(S)[-n,n])}{2n+1}$, and let A_n be a maximum independent set in $G(S)[-n, n]$. For $n \ge s$, $X_n := \bigcup_{z\in\mathbb{Z}}(A_n \cap [-n, n-s] + 2nz)$ is an infinite independent set in $G(S)$ of density at least $\frac{\alpha(G(S)[-n,n])-s}{2n}$, which approaches L as $n \to \infty$. Thus there exists a sequence $(X_n)_{n\ge s}$ of independent sets in $G(S)$ with densities that approach L, and so $\overline{\alpha}(S) \ge L$. ■

3. Previous results

We outline here the known results concerning $\overline{\alpha}(S)$. The results were all phrased in terms of $\chi_f(S)$, but by Theorem 1 we know the following versions are equivalent.

Theorem 8 (*Gao and Zhu [19]*)**.** *Let k and k' be positive integers such that $k \le k'$.*

1. $\overline{\alpha}([1, k']) = \overline{\alpha}([k']) = \frac{1}{k'+1}$.
2. *If* $k' \ge (5/4)k$, *then* $\overline{\alpha}([k, k']) = \frac{k}{k+k'}$. □

Theorem 9 (*Chang, Liu, and Zhu [9]; Liu and Zhu [34]*)**.** *For positive integers m, k, s such that $sk \le m$, let $D_{m,k,s} = [m] \setminus k[s]$.*

1. *If* $2k > m$, *then* $\overline{\alpha}(D_{m,k,1}) = \frac{1}{k}$.
2. *If* $2k \le m$, *then* $\overline{\alpha}(D_{m,k,1}) = \frac{2}{m+k+1}$.
3. *If* $m \ge (s+1)k$, *then* $\overline{\alpha}(D_{m,k,s}) = \frac{s+1}{m+sk+1}$. □

Theorem 10 (*Lam and Lin [29]*)**.** *For positive integers $m \ge k' \ge k \ge 1$, let $D_{m,[k,k']} = [m] \setminus [k, k']$.*

1. *If* $m < 2k$, *then* $\overline{\alpha}(D_{m,[k,k+i]}) = \frac{1}{k}$.
2. *If* $2k \le m < 2k + 2i$, *and* $1 \le i \le k - 1$, *then* $\overline{\alpha}(D_{m,[k,k+i]}) = \frac{2}{m+1}$.

3. *If* $m \geq 2k + 2i$ *and* $1 \leq i \leq k-1$, *then* $\overline{\alpha}(D_{m,[k,k+i]}) = \frac{2}{m+k+1}$.
4. *If* $m < (s+1)k$, *then* $\overline{\alpha}(D_{m,[k,sk+i]}) = \frac{1}{k}$.
5. *If* $1 \leq i \leq k-1$ *and* $(s+1)k \leq m < (s+1)k + i$, *then* $\overline{\alpha}(D_{m,[k,sk+i]}) = \frac{s+1}{m+1}$. *In particular, if* $k > 1$ *then*

$$\overline{\alpha}([k,m]) = \begin{cases} \dfrac{1}{k} & \text{if } m \not\equiv 0 \pmod{k}, \\ \dfrac{s+1}{k+1} & \text{if } m = (s+1)k. \end{cases} \quad \square$$

Theorem 11 (*Liu, Zhu [35]*)**.** *Let* $0 < a < b$, $m \geq 2$, *and* $\gcd(a,b) = 1$.

1. $\overline{\alpha}(\{a, 2a, \ldots, (m-1)a, b\}) = \begin{cases} \dfrac{k}{km+1} & \text{if } b = km \text{ for some } k \\ \dfrac{1}{m} & \text{otherwise} \end{cases}$.
2. $\overline{\alpha}(\{a, b, a+b\}) = \begin{cases} \dfrac{1}{3} & \text{if } b-a = 3k \\ \dfrac{a+k}{3a+3k+1} & \text{if } b-a = 3k+1 \\ \dfrac{a+2k+1}{3a+6k+4} & \text{if } b-a = 3k+2 \end{cases}$.
3. *if* $a \not\equiv b \pmod 2$, *then* $\overline{\alpha}(\{a, b, b-a, a+b\}) = \frac{1}{4}$.
4. $\overline{\alpha}(\{1, 2m, 2m+1, 2m+2\}) = \frac{m}{4m+1}$.

Chang, Huang, and Zhu [8] and Collins [13] determined the exact values of $\chi_f(S)$ for all sets S of size two.

Theorem 12 (*Chang, Huang, and Zhu [8]; Collins [13]*)**.** *Let* $S = \{a, b\}$ *with* $1 \leq a < b$ *and* $\gcd(a,b) = 1$.

1. *If* a *and* b *are both odd, then* $\overline{\alpha}(S) = \frac{1}{2}$.
2. *If at least one of* a *and* b *is even,* $\overline{\alpha}(S) = \frac{a+b-1}{2a+2b}$. *In particular,* $\overline{\alpha}(\{1, 2k\}) = \frac{k}{2k+1}$. $\square$

Zhu [42] investigated the case where $|S| = 3$ and determined bounds on $\chi_f(S)$ and the circular chromatic number for $G(S)$. These bounds are sufficient to determine $\chi(S)$ exactly.

Theorem 13 (*Zhu [42]*)**.** *Let* $S = \{a, b, c\}$ *where* $1 \leq a < b < c$.

1. *If* a, b, *and* c *are odd, then* $\overline{\alpha}(S) = \frac{1}{2}$.
2. *If* $S = \{1, 2, 3k\}$ *where* $k \geq 1$, *then* $\overline{\alpha}(S) = \frac{k}{3k+1}$.
3. *If* $b = a + 3k$ *and* $c = 2a + 3k$ *for* $k \geq 1$, *then* $\overline{\alpha}(S) = \frac{1}{3}$.
4. *If* $b = a + 3k + 1$ *and* $c = 2a + 3k + 1$ *for* $k \geq 1$, *then* $\frac{a+k}{3(a+k)+1} \leq \overline{\alpha}(S) \leq \frac{a+2k}{3(a+2k)+1}$.
5. *If* $b = a + 3k + 2$ *and* $c = 2a + 3k + 2$ *for* $k \geq 1$, *then* $\frac{a+2k+1}{3(a+2k+2)+1} \leq \overline{\alpha}(S) \leq \frac{a+2k+2}{3(a+2k+2)+1}$.
6. *If* a, b, c, *are not all odd,* $c \neq a + b$, *and* $(a, b, c) \neq (1, 2, 3k)$ *for any* $k \geq 1$, *then* $\frac{1}{3} \leq \overline{\alpha}(S) < \frac{1}{2}$.
7. *If* a, b, c, *are not all odd,* $c \neq 2b$, $b \neq 2a$, $c \neq 2a$ *and* $c \neq a + b$, *then* $\frac{3}{8} \leq \overline{\alpha}(S) < \frac{1}{2}$ *with a finite number of exceptional triples* (a, b, c). $\square$

While Theorem 13 determines the exact value of $\overline{\alpha}(S)$ for several classes of sets of size three, it leaves many triples undetermined.

4. Relations between generating sets

The following observations are very easy to prove.

Observation 14. If S is nonempty and contains only odd numbers, then $\overline{\alpha}(S) = \frac{1}{2}$.

Observation 15. If $S \subseteq T$, then $\overline{\alpha}(S) \geq \overline{\alpha}(T)$.

Lemma 16. $\overline{\alpha}(\{1, 2, \ldots, \ell\}) = \frac{1}{\ell+1}$

Proof. The set $X = (\ell+1) \cdot \mathbb{Z}$ is independent in $G(\{1, 2, \ldots, \ell\})$ with density $\frac{1}{\ell+1}$. Let A be any independent set of $G(S)$. For each element $a \in A$ the integers $a+1, \ldots, a+\ell$ cannot be in A. Therefore $\delta(A) \leq \frac{1}{\ell+1}$ for every independent set A of $G(S)$. ∎

Next we prove several lemmas that are useful in determining the independence density.

Lemma 17. *The density of an independent set A of $G(S)$ equals $\limsup_{n\to\infty} \frac{|A\cap[-nd,nd]|}{2nd+1}$ for any fixed positive integer d.*

Proof. Recall $\delta(A) = \limsup_{m\to\infty} \frac{|A\cap[-m,m]|}{2m+1}$, where we can write $m = nd + \ell$ for $\ell \in [0, d-1]$. Taking the limsup as $m \to \infty$ (which implies $n \to \infty$) of the following bounds

$$\left(\frac{2nd+1}{2m+1}\right)\frac{|A\cap[-nd,nd]|}{2nd+1} \le \frac{|A\cap[-m,m]|}{2m+1} \le \left(\frac{2(n+1)d+1}{2m+1}\right)\frac{|A\cap[-(n+1)d,(n+1)d]|}{2(n+1)d+1}$$

gives $\delta(A) = \limsup_{n\to\infty} \frac{|A\cap[-nd,nd]|}{2nd+1}$. ∎

Lemma 18. *Let A be a periodic independent set in $G(S)$ with period p, and set $q = |A \cap [0, p-1]|$. Then $\delta(A) = q/p$.*

Proof. Note that for every integer $z \in \mathbb{Z}$, $|A \cap [z, z+p-1]| = q$. We have the following bounds

$$\frac{2nq}{2np+1} \le \frac{|A\cap[-np,np]|}{2np+1} \le \frac{2nq+1}{2np+1}.$$

Taking the limsup as $n \to \infty$, we obtain $\frac{q}{p} \le \delta(A) \le \frac{q}{p}$ by Lemma 17. ∎

Lemma 19. *For $d \ge 1$, $\overline{\alpha}(S) = \overline{\alpha}(d\cdot S)$.*

Proof. Let A be an independent set in $G(d\cdot S)$ with maximum density. Define $A_\ell := A \cap (d\cdot Z + \ell)$, where $\ell \in [0, d-1]$, and note that the disjoint union of the A_ℓ sets is A. Since A_ℓ is independent in $G(d\cdot S)$, the set $X_\ell := (A_\ell - \ell)/d$ is independent in $G(S)$.

If $dz + \ell \in A_\ell \cap [-nd, nd]$, where $z \in \mathbb{Z}$, then $z \in [-n, n]$. Hence, $|A_\ell \cap [-nd, nd]| \le |X_\ell \cap [-n, n]|$ and we have the following

$$\begin{aligned}\frac{|A\cap[-nd,nd]|}{2nd+1} &= \frac{\sum_{\ell=0}^{d-1}|A_\ell\cap[-nd,nd]|}{2nd+1}\\ &\le \frac{\sum_{\ell=0}^{d-1}|X_\ell\cap[-n,n]|}{2nd+1}\,\frac{(2nd+1)+(d-1)}{2nd+d}\\ &= \left(1+\frac{d-1}{2nd+1}\right)\frac{1}{d}\sum_{\ell=0}^{d-1}\left(\frac{|X_\ell\cap[-n,n]|}{2n+1}\right).\end{aligned}$$

Taking the limsup as $n \to \infty$, we have $\overline{\alpha}(d\cdot S) = \delta(A) \le \frac{1}{d}\sum_{\ell=0}^{d-1}\delta(X_\ell) \le \overline{\alpha}(S)$.

Next we show that $\overline{\alpha}(d\cdot S) \ge \overline{\alpha}(S)$. Let X be an independent set in $G(S)$ with maximum density; by Theorem 2, we can assume that X is periodic with period p. We claim that the set $A = \bigcup_{\ell=0}^{d-1}(d\cdot X+\ell)$ is independent in $G(d\cdot S)$. If not, then there exist integers i, j such that $i - j = ds$ for some $s \in S$ and $i = dk_1 + \ell_1$ and $j = dk_2 + \ell_2$, where $k_1, k_2 \in X$ and $\ell_1, \ell_2 \in [0, d-1]$. Therefore, $d(k_1 - k_2) + (\ell_1 - \ell_2) = ds$ which implies $\ell_1 = \ell_2$ and $k_1 - k_2 = s$, which contradicts the assumption that X is independent.

Note that A is periodic with period dp and $|A \cap [1, dp]| = d|X \cap [1, p]|$. Therefore, $\overline{\alpha}(d\cdot S) \ge \delta(A) = \delta(X) = \overline{\alpha}(S)$. ∎

By Lemma 19, if the greatest common divisor of S is not 1, we can factor out the greatest common divisor. The following corollary is similar to, but not implied by Theorem 11 [35].

Corollary 20. *Let $k \ge 2$ and $\ell \ge 2$. Then $\overline{\alpha}(\{1, k, 2k, \ldots, \ell k\}) = \frac{1}{\ell+1}$.*

Proof. By Lemmas 16 and 19 we know that

$$\overline{\alpha}(\{1,k,2k,\ldots,\ell k\}) \le \overline{\alpha}(\{k,2k,\ldots,\ell k\}) = \overline{\alpha}(\{1,2,\ldots,\ell\}) = \frac{1}{\ell+1}.$$

Let $X = (\ell+1)\cdot\mathbb{Z}$, and for $0 \le j \le k-1$ let X_j denote $k\cdot X + 2j$. If k is odd, let $X' = \bigcup_{j=0}^{k-1} X_j$. Note that X_{j_1} and X_{j_2} are disjoint for $j_1 \ne j_2$. Let $a = k(\ell+1)x_1 + 2j_1$ and $b = k(\ell+1)x_2 + 2j_2$ be distinct elements of X'. If $|a-b| \equiv 0 \pmod{k}$, then $2|j_1 - j_2| \equiv 0 \pmod{k}$. Since k is odd, 2 is invertible modulo k, and so $j_1 = j_2$. Since a and b are both in X_j for some j, $|a-b| \ge k(\ell+1)$. Thus $|a-b| \notin \{k, 2k, \ldots, \ell k\}$. Moreover, since each element from X is translated by at most $2(k-1)$ positions, and $2(k-1) < k(\ell+1) - 1$ we have that no two elements of X' have distance 1. Therefore, X' is independent and has density $\frac{1}{\ell+1}$.

If k is even, let $h = k/2$. Let $X' = \bigcup_{j=0}^{h-1} X_j \cup \bigcup_{j=h}^{k-1}(X_j+1)$. Note that X_j contains only even numbers for $0 \le j \le h-1$ and $X_j + 1$ contains only odd numbers for $h \le j \le k-1$. So X_{j_1} and X_{j_2} are disjoints for $j_1 \ne j_2$. Let $a = k(\ell+1)x_1 + 2j_1 + e_1$

and $b = k(\ell+1)x_2 + 2j_2 + e_2$ be distinct elements of X', where $e_1, e_2 \in \{0, 1\}$. If $|a-b| \equiv 0 \pmod k$, then $e_1 = e_2$ and $2|j_1 - j_2| \equiv 0 \pmod k$. Thus either $j_1, j_2 < h$ and so $j_1 = j_2$, or $h \le j_1, j_2 \le k-1$ and so $j_1 = j_2$. Thus a and b are both in X_j for some j, and hence $|a-b| \ge k(\ell+1)$. Thus $|a-b| \notin \{k, 2k, \ldots, \ell k\}$. Moreover, since each element from X is translated by at most $2(k-1)+1$ positions, and $2(k-1)+1 < k(\ell+1)-1$ we have that no two elements of X' have distance 1. Therefore, X' is independent and has density $\frac{1}{\ell+1}$. ■

5. The Local Discharging Lemma

In this section, we define our process of using discharging to show upper bounds on $\overline{\alpha}(S)$. We begin by defining objects, called blocks and frames, that are crucial to our process.

Fix a distance set S and an independent set $X \subseteq \mathbb{Z}$. Index the elements of X by $\mathbb{Z}$, so $X = \{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\}$. The ith *block* B_i is the set $B_i = \{x_i, x_i + 1, \ldots, x_{i+1} - 1\}$. Hence, each block contains exactly one element of X, and the blocks partition $\mathbb{Z}$. Observe that $|B_i| = x_{i+1} - x_i$, and since X is independent, $|B_i| \notin S$.

Let t be a positive integer. A *frame of length* t is a set of t consecutive blocks. For $j \in \mathbb{Z}$, let $F_j = \{B_j, \ldots, B_{j+t-1}\}$ be the jth frame of length t. For a set F of consecutive blocks, let $\sigma(F) = \sum_{B\in F} |B|$. Observe that $\sigma(F)$ is the distance from the first element of X in F to the first element of X following F. Since X is an independent set, $\sigma(F) \notin S$. Throughout the rest of the paper we refer to blocks of length i as i-blocks and frames of length t as t-frames.

We can describe the structure of a frame F by listing the sizes of its blocks in order. We denote such a list of sizes using *block notation*, which is defined recursively. First, any list of integers $b_1\ b_2\ \cdots\ b_k$ corresponds to k consecutive blocks with sizes $b_1, \ldots, b_k$. For block structure π, the block structure π^e corresponds to e consecutive sets of blocks matching block structure π. Finally, for two block structures π_1 and π_2, the block structure $\pi_1\pi_2$ corresponds to consecutive sets of blocks first matching block structure π_1 then matching the block structure π_2. For example, the block structure $(2\ 3)^5\ 7\ (3\ 4)^2$ corresponds to 15 consecutive blocks, first with ten blocks alternating between 2- and 3-blocks, then a 7-block, then four blocks alternating between 3- and 4-blocks. Every block structure also defines an infinite, periodic set given by repeating the block structure infinitely in both directions.

We are now prepared to discuss our discharging method. Generally, discharging is a technique that interfaces between local structure and global averages. We use our knowledge of local structure (the distance set) to demonstrate an upper bound on the global average (the density of an independent set).

Let $\mathcal{B}$ be the set of blocks and $\mathcal{F}$ be the set of frames. Fix a *charge function* $\mu : \mathcal{B} \to \mathbb{Z}$, which is any assignment of integers to the blocks of X.[2] A *discharging rule* is a function $d : \mathcal{B} \times \mathcal{B} \to \mathbb{Z}$ such that $d(B_i, B_j) = -d(B_j, B_i)$. We say that a discharging rule d is *m-local* when if $|x_i - x_j| > m$, then $d(B_i, B_j) = 0$, if $|x_i - x_j| \le m$, then $d(B_i, B_j)$ depends only on the block structure of the blocks $B_{i-m}, \ldots, B_{i+m}$. The discharging rule d defines a new charge function $\mu^* : \mathcal{B} \to \mathbb{Z}$ given by

$$\mu^*(B_i) = \mu(B_i) + \sum_{B_j \in \mathcal{B}} d(B_j, B_i).$$

That is, positive values of $d(B_j, B_i)$ are considered to be charge sent from B_j to B_i and negative values of $d(B_j, B_i)$ are considered to be charge received by B_j from B_i. In the second stage, we discharge on frames. Define $\nu^*(F_j) = \sum_{i=j}^{j+t-1} \mu^*(B_i)$. A *second-stage discharging rule* is a function $d' : \mathcal{F} \times \mathcal{F} \to \mathbb{Z}$ such that $d(F_i, F_j) = -d(F_j, F_i)$. We similarly define d' to be *m-local* when if $|i-j| > m$, then $d'(F_i, F_j) = 0$, and if $|i-j| \le m$, then $d'(F_i, F_j)$ depends only on the block structure of the frames $F_{i-m}, \ldots, F_{i+m}$. Finally, perform the second-stage discharging rule by defining the charge function ν' on the frames as

$$\nu'(F_i) = \nu^*(F_i) + \sum_{F_j \in \mathcal{F}} d'(F_j, F_i).$$

The following lemma allows us to relate discharging functions and densities of independent sets.

Lemma 21 (*Local Discharging Lemma*). *Let S be a finite, nonempty set of positive integers. Fix integers $a, b, t \ge 1$ and $c \ge 0$. Let X be a periodic independent set of $G(S)$. Initialize the charge function μ to be $\mu(B_i) = a|B_i| - b$. Let there be an m-local discharging rule $d : \mathcal{B} \times \mathcal{B} \to \mathbb{Z}$ that defines $\mu^*(B_i)$. On the family $\mathcal{F}$ of all t-frames, define $\nu^*(F_j) = \sum_{B_i \in F_j} \mu^*(B_j)$, where F_j is a t-frame, and let d' be an m-local discharging rule $d' : \mathcal{F} \times \mathcal{F} \to \mathbb{Z}$ that defines $\nu'(F_j)$. If $\nu'(F_j) \ge c$ for all j, then $\delta(X) \le \frac{at}{bt+c}$.*

We present two proofs of the Local Discharging Lemma, and both require the set X to be periodic. In the first proof, we reduce to a finite circulant graph where the discharging rules are equivalent to the periodic set. In the second proof, we use the limit definition to observe that the local nature causes only a finite amount of perturbation about the boundary during the density calculation.

[2] Note that we could relax the definition of a charge function to be fractional as in other contexts, such as coloring of planar graphs. However, we only use integral charges in our proofs as the parameters a, b, and t of the Local Discharging Lemma allow us to avoid fractional values.

Proof 1. Let p be a period of the independent set X. Set $q = 2mpt$ and observe that q is a period for X. Thus, $X_q = X \cap [q]$ is an independent set in the circulant graph $G(q, S)$. Let $r = |X_q|$ and let $x_1 < \cdots < x_r$ be the elements of X_q. Observe that for all $i, i' \in \mathbb{Z}$, we have $x_i \equiv x_{i'} \pmod q$ if and only if $i \equiv i' \pmod r$. Thus when $i \equiv i' \pmod r$ the block structure surrounding B_i is equivalent to the block structure surrounding $B_{i'}$. Further, when $i \equiv i' \pmod r$ and $j \equiv j' \pmod r$, the discharging rules d and d' satisfy $d(B_i, B_j) = d(B_{i'}, B_{j'})$ and $d'(F_i, F_j) = d'(F_{i'}, F_{j'})$. Therefore, when $i \equiv i' \pmod r$, the charge functions all satisfy

$$\mu(B_i) = \mu(B_{i'}), \qquad \mu^*(B_i) = \mu^*(B_{i'}), \qquad \nu^*(F_i) = \nu^*(F_{i'}), \quad \text{and} \quad \nu'(F_i) = \nu'(F_{i'}).$$

Observe also that

$$t(aq - br) = t\sum_{j=1}^{r} \mu(B_j) = t\sum_{j=1}^{r} \mu^*(B_j) = \sum_{j=1}^{r} \nu^*(F_j) = \sum_{j=1}^{r} \nu'(F_j) \geq cr.$$

From this, we have the inequality $taq \geq (tb + c)r$ and hence $\frac{ta}{tb+c} \geq \frac{r}{q} = \frac{|X_q|}{q} = \delta(X)$. ■

Proof 2. We assume that X is a maximal independent set, which implies that the maximum length of a block is at most $2\max S$. By the locality of the first stage discharging rule d, there are a finite number of combinations of $2m + 1$ consecutive blocks and thus a finite number of values to $d(B_i, B_j)$. Thus, there exists a number υ such that $|d(B_i, B_j)| \leq \upsilon$ for all pairs of blocks B_i, B_j.

Since both discharging rules d and d' are m-local, the absolute differences

$$\left|\sum_{i=1-N}^{N} \mu(B_i) - \sum_{i=1-N}^{N} \mu^*(B_i)\right| \quad \text{and} \quad \left|\sum_{i=1-N}^{N} \nu^*(F_i) - \sum_{i=1-N}^{N} \nu'(F_i)\right|$$

are bounded by a constant C_1. Also, since t is a fixed constant, the absolute difference

$$\left|\sum_{i=1-N}^{N} \nu^*(F_i) - \sum_{i=1-N}^{N} t\mu^*(B_i)\right|$$

is bounded by a constant C_2. Let $C = \max\{C_1, C_2\}$.

$$\begin{aligned} c &\leq \frac{\sum_{i=1-N}^{N} \nu'(F_i)}{2N} \leq \frac{\sum_{i=1-N}^{N} \nu^*(F_i)}{2N} + \frac{C}{2N} \leq \frac{t\sum_{i=1-N}^{N} \mu^*(B_i)}{2N} + \frac{2C}{2N} \\ &\leq \frac{t\sum_{i=1-N}^{N} \mu(B_i)}{2N} + \frac{3C}{2N} \\ &= \frac{ta\sum_{i=1-N}^{N} |B_i| - 2Ntb}{2N} + \frac{3C}{2N}. \end{aligned}$$

Recall that X is a periodic independent set with maximum density. Let p be the period of X. Let $q = |X \cap [0, p-1]|$ and let $N = kq + r$, where $r \in [0, q-1]$. Note that $\sum_{i=a}^{a+q-1} |B_i| = p$ for any integer $a \in \mathbb{Z}$. Therefore, we have the inequalities

$$\begin{aligned} \frac{c + tb}{ta} &\leq \frac{\sum_{i=1-N}^{N} |B_i|}{2N} + \frac{3C}{2Nta} \leq \frac{\sum_{i=-kq}^{kq-1} |B_i|}{2kq} + \frac{2q}{2N} + \frac{3C}{2Nta} \\ &= \frac{2kp}{2kq} + \frac{2q}{2N} + \frac{3C}{2Nta} \\ &= \delta(X)^{-1} + \frac{2q}{2N} + \frac{3C}{2Nta}. \end{aligned}$$

Taking the limit as $N \to \infty$, we have $\frac{c+tb}{ta} \leq \delta(X)^{-1}$. ■

We use the Local Discharging as part of our method for determining independence ratios. Suppose that we want to determine the independence ratio for some family of generator sets parameterized by k. We compute values for $\overline{\alpha}(S)$ for some explicit values of k using the computational techniques outlined in Section 7. This leads us to a conjectured value, τ say, for $\overline{\alpha}(S)$. (Obviously τ will usually also depend on k, though we do not make it explicit in our notation here.) To prove that $\overline{\alpha}(S) = \tau$ for each set S in the family, we use the approach outlined below.

1. Construct a periodic independent set with density τ. This proves that $\overline{\alpha}(S) \geq \tau$.
2. Determine parameters a, b, c, and t such that $\frac{at}{bt+c} = \tau$.

3. Let X be an independent set in $G(S)$ with maximum density. By Theorem 2 we may assume that X is periodic.
4. Construct Stage 1 discharging rules on such that $\mu^*(B) \geq 0$ for all blocks B. Requiring that all blocks have nonnegative μ^* charge makes creation of Stage 2 discharging rules easier.
5. Construct Stage 2 discharging rules such that $\nu'(F) \geq c$ for every t-frame F.
6. Deduce from the Local Discharging Lemma that $\delta(X) \leq \frac{at}{bt+c}$, i.e., that $\overline{\alpha}(S) \leq \tau$.

The following theorem is a short example using the Local Discharging Lemma. It provides an alternative proof of (a generalization of) part 2 of Theorem 13.

Theorem 22. *Let $\ell \geq 2$ and $k > \ell$. Then*

$$\overline{\alpha}(\{1, \ldots, \ell-1, k\}) = \begin{cases} \dfrac{1}{\ell} & k \not\equiv 0 \pmod{\ell} \\ \dfrac{k}{\ell(k+1)} & k \equiv 0 \pmod{\ell}. \end{cases}$$

Proof. Since $\{1, \ldots, \ell-1\}$ is a subset of the generators, every block has size at least ℓ. So, $\overline{\alpha}(\{1, \ldots, \ell-1, k\}) \leq \frac{1}{\ell}$. If $k \not\equiv 0 \pmod{\ell}$, then the periodic set of all ℓ-blocks is independent and equality holds.

Otherwise, $k = \ell t$ for some integer t. The periodic set (given in block notation) $\ell^{t-1}\ (\ell+1)$ is independent with density $\frac{k/\ell}{\ell t+1} = \frac{k}{\ell(k+1)}$. To prove the upper bound we use Lemma 21, where $a = 1$, $b = \ell$, $t = k/\ell$, and $c = 1$. Thus, $\frac{at}{bt+c} = \frac{k/\ell}{k+1} = \frac{k}{\ell(k+1)}$.

There are no discharging rules in this case. So $\mu(B_i) = \mu^*(B_i)$, and $\nu^*(F_j) = \sum_{B_i \in F_j} \mu(B_i) = \nu'(F_j)$. Since every block is at least an ℓ-block, and receives initial charge $a|B_i| - b$, all blocks have non-negative charge. Any block of size at least $\ell+1$ has charge at least 1. Since no frame has $\sigma(F) = \ell t = k$, every frame contains a block of size at least $\ell+1$. Hence each frame has at least $c = 1$ unit of charge. ■

6. Discharging arguments

In this section we use the Local Discharging Lemma to prove exact values of $\overline{\alpha}(S)$ for several families of sets S of size 3 where $1 \in S$. Recall that if k, ℓ are both odd integers, then $\overline{\alpha}(\{1, k, \ell\}) = \frac{1}{2}$.

We begin by determining the asymptotic behavior of $\overline{\alpha}(\{1, k, k+i\})$ and $\overline{\alpha}(\{1, i, k\})$ for constants i and growing k. We then determine the exact values for these infinite families when i is a small constant. Finally, we list some conjectures for values of the next few values of i.

6.1. Asymptotic results

Consider sets $S = S(k)$ determined by $S(k) = \{1, f(k), g(k)\}$. We determine the limit of $\overline{\alpha}(S(k))$ for certain functions $f(k)$ and $g(k)$.

Theorem 23. *For $i \geq 1$, $\lim_{k\to\infty} \overline{\alpha}(\{1, 2i+1, 2k\}) = \frac{1}{2}$.*

Proof. Since $\overline{\alpha}(\{1, 2i+1, 2k\}) \leq \overline{\alpha}(\{1\}) = \frac{1}{2}$, we have the upper bound immediately.

Observe that the periodic set with block structure $2^{k-1}\ (2i+3)$ is independent in $G(\{1, 2i+1, 2k\})$ and has density $\frac{k}{2k+2i+1}$, which tends to $\frac{1}{2}$ as k grows. ■

Theorem 24. *For $i \geq 1$, $\lim_{k\to\infty} \overline{\alpha}(\{1, 2i, k\}) = \frac{i}{2i+1}$.*

Proof. By Theorem 12 and Observation 15 we know $\overline{\alpha}(\{1, 2i, k\}) \leq \overline{\alpha}(\{1, 2i\}) = \frac{i}{2i+1}$.

Let $k = (2i+1)q + r$, where $1 \leq r < 2i+1$ and $q \geq 1$. Observe that the periodic set with block structure $(2^{i-1}\ 3)^{q-1}\ (2i+2+r)$ is independent in $G(\{1, 2i, k\})$, with density $\frac{iq-i+1}{(2i+1)q+r+1}$, which tends to $\frac{i}{2i+1}$ as k grows (q grows). ■

Theorem 25. *For $i \geq 0$, we have $\lim_{k\to\infty} \overline{\alpha}(\{1, k, k+2i+1\}) = \frac{i+1}{2i+3}$.*

Proof. Let $k + 2i + 2 = (2i+3)q + r$, where $0 \leq r < 2i+3$. The lower bound is given by the periodic set with block structure $(2^i\ 3)^{q-1}(2i+3+r)$. The density is given by $\frac{(i+1)(q-1)+1}{(2i+3)q+r} \geq \frac{(i+1)(q-1)}{(2i+3)(q+1)} = \frac{i+1}{2i+3}\frac{q-1}{q+1}$. Note that as k goes to infinity the lower bound density approaches the value $\frac{i+1}{2i+3}$.

For the upper bound, let X be a periodic independent set in $G(S)$ with maximum density, and let $a = 1$, $b = 2$, $t = i+1$, and $c = 1$. Every block B_j has nonnegative charge $\mu(B_j)$, so no Stage 1 discharging is required. For a t-frame F_j, if $\nu^*(F_j) = 0$, then F_j consists entirely of 2-blocks and $\sigma(F_j) = 2i + 2$. Then, the elements $x_j, x_{j+1}, \ldots, x_{j+i}, x_{j+i+1}$ have consecutive pair distances of 2, and hence the generators k and $k + 2i + 1$ have

$$x_j + k + 2i + 1 = (x_{j+i} + k) + 1, \qquad x_{j+1} + k + 2i + 1 = (x_{j+i+1} + k) + 1.$$

Thus, the consecutive elements $x_{j+i} + k, x_j + (k + 2i + 1), x_{j+i+1} + k, x_{j+1} + (k + 2i + 1)$ are not in X and so are contained in a single block $B_{j'}$. The block $B_{j'}$ has size at least five, and hence $\mu^*(B_{j'}) \geq 3$. Let φ_2 be the function from frames F_j with $\sigma(F_j) = 2i + 2$ to the block $B_{j'}$ that contains $x_j + k + 2i + 1$. Observe that if $\varphi_2^{-1}(B_{j'}) \neq \varnothing$, then $|B_{j'}| \geq 2|\varphi_2^{-1}(B_{j'})| + 3$. Thus, our Stage 2 discharging rule is as follows:

Stage 2: Every frame F_j with $\sigma(F_j) = 2i + 2$ pulls 1 unit of charge from the frame $F_{j'}$ where $B_{j'} = \varphi_2(F_j)$.

If a frame F_j has $\sigma(F_j) = 2i+2$, then $\nu'(F_j) = 1$. Otherwise, $\sigma(F_j) > 2i+2$ and F_j contains at least one block of size at least three, so $\nu^*(F_j) \geq 1$. If F_j loses charge in Stage 2, then $\varphi_2^{-1}(B_j) \neq \varnothing$ and $|B_j| \geq 2|\varphi_2^{-1}(B_j)|+3$, and so $\mu^*(B_j) \geq 2|\varphi_2^{-1}(B_j)|+1$. Since F_j loses at most one unit of charge for each frame in $\varphi_2^{-1}(B_j)$, F_j retains at least $|\varphi_2^{-1}(B_j)| + 1$ units of charge, giving $\nu'(F_j) \geq 1$. By the Local Discharging Lemma, $\overline{\alpha}(\{1, k, k + 2i + 1\}) \leq \frac{at}{bt+c} = \frac{i+1}{2i+3}$. ∎

6.2. $S = \{1, b, 2i\}$

Theorem 12 states that for an odd number b, $\overline{\alpha}(\{1, b\}) = \frac{1}{2}$ and the maximum independent set consists entirely of 2-blocks. For $i \geq 1$, $\overline{\alpha}(\{1, 2i\}) = \frac{i}{2i+1}$ and the maximum independent set has block structure $2^{i-1}\,3$. We consider the union of these generators.

Conjecture 26. *Fix $\ell \geq 3$ where ℓ is odd and let $2i \geq 3\ell$. Then $\overline{\alpha}(\{1, \ell, 2i\}) = \frac{i}{2i+\ell}$.*

This conjecture is sharp, since the periodic set with block structure $2^{i-1}\,(\ell + 2)$ matches this density and is independent in $G(\{1, \ell, 2i\})$. The bound $2i \geq 3\ell$ may not be sharp in all cases, but it is required for this set to be extremal, since there are independent sets of higher density even for the case $\ell = 5$ when i is small.

We prove the first few cases of this conjecture. We do not perform any discharging, so our technique is really a *charging* method. Essentially the proofs boil down to determining that $\sigma(F) \geq 2i + \ell$ for all frames of length i, but it is helpful to use the discharging perspective to instead show that $\nu^*(F) \geq \ell$.

Theorem 27. *Let $i \geq 2$. Then $\overline{\alpha}(\{1, 3, 2i\}) = \frac{i}{2i+3}$.*

Proof. The lower bound is achieved by the periodic set with block structure $2^{i-1}\,5$.

Let X be a periodic independent set in $G(S)$ with maximum density. Observe that since $3 \in S$ there are no 3-blocks in X. Let $a = 1, b = 2, t = i$, and $c = 3$ and perform no discharging. Consider a frame F of length t. Since $\sigma(F) \neq 2i$, not all blocks in F are 2-blocks. Thus, there is a block in F of size at least 4. If there is a block of size at least 5 in F, then $\nu^*(F) \geq 3$. If there are two 4-blocks in F, then $\nu^*(F) \geq 4$. Thus, if $\nu^*(F) < 3$ there must be $i - 1$ 2-blocks in F and exactly one 4-block. In this case, either the first block in F or the last block in F is a 2-block. Removing this 2-block results in a set of $i - 1$ consecutive blocks spanning $2i$ elements, a contradiction.

Thus $\nu'(F) \geq 3$ for all frames, and by the Local Discharging Lemma $\overline{\alpha}(\{1, 3, 2i\}) \leq \frac{i}{2i+3}$. ∎

Theorem 28. *Let $i \geq 5$. Then $\overline{\alpha}(\{1, 5, 2i\}) = \frac{i}{2i+5}$.*

Proof. The lower bound is achieved by the periodic set with block structure $2^{i-1}\,7$.

Let X be a periodic independent set in $G(S)$ with maximum density. Observe that since $5 \in S$ there are no 5-blocks in X. Also, if there is a 3-block B in X, then the blocks preceding and following B are not 2-blocks.

Let $a = 1, b = 2, t = i$, and $c = 5$ and perform no discharging. Consider a frame F of length t. Since $\sigma(F) \neq 2i$, not all blocks in F are 2-blocks. Thus, there is a block in F of size at least 3. If there is a block of size at least 7 in F, then $\nu^*(F) \geq 5$. We now assume there are no blocks of size at least 7 in F.

Suppose there is no 3-block in F. If there are two 6-blocks in F or three 4-blocks in F, then $\nu^*(F) \geq 6$. If there is a 4-block and a 6-block in F, then $\nu^*(F) \geq 5$. If there are exactly two 4-blocks and $i - 2$ 2-blocks in F, then $\nu^*(F) = 4$ and $\sigma(F) = 2i + 4$. However, if either the first or the last block B of F is a 4-block, then $\sigma(F - B) = 2i$, a contradiction. Thus the first and last blocks of F are 2-blocks, but removing these two blocks leaves a set of $i - 2$ consecutive blocks covering $2i$ elements, a contradiction.

Therefore, there is a 3-block in F. If there are at least 5 3-blocks in F, then $\nu^*(F) \geq 5$. If there is at least one 3-block and one 6-block in F, then $\nu^*(F) \geq 5$. If there are at least one 3-block and two 4-blocks in F, then $\nu^*(F) \geq 5$.

Hence if $\nu^*(F) < 5$, then F consists of 2-blocks, at most four 3-blocks, and at most one 4-block. Note that a 3-block and 2-block cannot be consecutive, so a 4-block must be between the 3-blocks and the 2-blocks. Thus, there exists exactly one 4-block in F that separates the 2-blocks from the 3-blocks. The cases are symmetric whether the 2-blocks or 3-blocks come before the 4-block, so we assume that F has block structure $3^q\,4\,2^{i-q-1}$, where $1 \leq q \leq i - 2$. If $q \geq 3$, then $\nu^*(F) \geq 5$. If

$q = 1$, then the $i - 1$ blocks starting at the 4-block cover exactly $2i$ elements, a contradiction. If $q = 2$, then there are $i - 3$ 2-blocks and $\sigma(F) = 2i + 4$. Since $i - 3 \geq 2$, removing the last two 2-blocks from F results in $i - 2$ consecutive blocks covering exactly $2i$ elements, a contradiction.

Therefore, we have $\nu^*(F) \geq 5$ for all frames F. ■

It is not difficult to also prove that $\overline{\alpha}(\{1, 7, 2i\}) = \frac{i}{2i+7}$ using similar techniques to the proofs above. However, the case analysis becomes long and tedious, and so we do not include the proof.

6.3. $S = \{1, 2k, 2k + 2\ell\}$

Let S be a set of three generators including 1 where the difference between the other two generators is even. Since $\overline{\alpha}(S) = \frac{1}{2}$ when S contains no even numbers, we assume the generators other than 1 are even. For a fixed even difference between these generators, we form the following conjecture.

Conjecture 29. *Let $k \geq 1$ and $\ell \geq 1$. Then, $\overline{\alpha}(\{1, 2k, 2k + 2\ell\}) = \frac{2k}{4k+2\ell}$.*

The following lemma shows that this conjecture is sharp for all possible values of $k \geq 2$ and $\ell \geq 1$.

Lemma 30. *Let $1 \leq \ell \leq k$. Then $\overline{\alpha}(\{1, 2k, 2k + 2\ell\}) \geq \frac{2k}{4k+2\ell}$.*

Proof. When $k = \ell = 1$, $\overline{\alpha}(\{1, 2, 4\}) = \frac{2}{4+2} = \frac{1}{3}$ by Corollary 20.

For $k \geq 2$, consider the periodic set X with block structure $2^{k-1}\ 3\ 2^{k-1}\ (2\ell + 1)$. Clearly there are no two elements in X distance 1 apart. Let X_1 denote the elements in X described by the first k blocks in the description, i.e. arising from $2^{k-1}\ 3$, and let X_2 denote $X \setminus X_2$, the elements in X arising from $2^{k-1}\ (2\ell + 1)$. Notice that all the integers in X_1 have the same parity, and all the integers in X_2 have the opposite parity to those in X_1. Thus the distance between an element of X_1 and element of X_2 is odd.

Thus if the set X is not independent, then there must be two elements from X_1, or two from X_2 that are distance $2k$ or $2k + 2\ell$ apart. No pair of elements in $2^{k-1}\ 3$ are distance $2k$ or $2k + 2\ell$ apart, and $2^{k-1}(2\ell + 1)$ has length $2k + 2\ell + 1$. No pair of elements in $2^{k-1}\ (2\ell + 1)$ are distance $2k$ or $2k + 2\ell$ apart, and $(2\ell + 1)\ 2^{k-1}\ 3$ has length $2k + 2\ell + 2$. ■

7. Computational methods

We obtained the values of $\overline{\alpha}(S)$ given in the theorems and conjectures of Section 6 by computing the independence ratio and looking for patterns. For a given family of sets of generators parameterized by k, we computed $\alpha(S)$ for enough fixed sets S in the family until we had enough data to conjecture a formula for $\overline{\alpha}(S)$ in terms of k. As $G(S)$ is an infinite graph and computing the independence number of a graph is in general difficult, we describe here our approach.

To compute $\overline{\alpha}(S)$ for a fixed set S, we recall that for integers n and m, we have the inequalities

$$\frac{\alpha(G(n, S))}{n} \leq \overline{\alpha}(S) \leq \frac{\alpha(G(S)[m])}{m},$$

where $G(S)[m]$ is the subgraph of $G(S)$ induced on the interval $[m]$. Thus, we will find maximum independent sets in $G(n, S)$ and $G(S)[m]$ for n and m growing until the largest lower bound matches the smallest upper bound.

Finding independent sets in a graph G is equivalent to finding cliques in the complement of G. Bašić and Ilić [3,25] previously computed some clique numbers and chromatic numbers for certain classes of circulant graphs using a backtracking search. In [25], they used Niskanen and Östergård's *cliquer* [38] as part of their implementation, but gave no other details. We use a slight modification of the *cliquer* algorithm to compute lower and upper bounds on $\overline{\alpha}(S)$.

The *cliquer* algorithm greatly depends on the ordering of the vertices of the input graph. For $G(n, S)$ and for $G(S)[n]$, we will use the ordering $1, 2, \ldots, n$ in order to exploit the vertex-transitivity of $G(n, S)$ and $G(S)$, respectively. We will focus first on the distance subgraphs $G(S)[n]$; a similar algorithm can be applied to the circulant graphs $G(n, S)$. Define $\alpha(n)$ to be the largest size of an independent set in $G(S)[n]$. Observe that for each $i \in [n]$, the subgraph $G(S)[i]$ is a subgraph of $G(S)[n]$. Thus, we will compute the values of $\alpha(i)$ in increasing order of i, and use previous values in our later computation. Also observe that $\alpha(i) \leq \alpha(i + 1) \leq \alpha(i) + 1$. Thus, in order to compute $\alpha(i + 1)$, we must only search for an independent set of size $\alpha(i) + 1$. We can terminate the search once one is found.

We use a recursive, backtracking search where we attempt to construct a large independent set A in $G(S)[n]$ in *decreasing* order. Initialize $A = \varnothing$ and $B = [n]$. At every step, we are given sets A and B, where A is an independent set and B consists of the vertices b such that $b < \min A$ and b is not adjacent to any vertex in A. Thus, the vertices in B are possible next choices for growing the independent set A. If $|A| > \alpha(n - 1)$, then we have determined $\alpha(n)$, we report the set A, and we terminate the algorithm. If $|B| + |A| \leq \alpha(n - 1)$, then there is no independent set $A' \supset A$ with size at least $\alpha(n - 1) + 1$, and we can backtrack. If these simple termination conditions fail, we attempt to add a new element to A from B, but use our previous calculations of $\alpha(i)$ to assist. The standard use of $\alpha(i)$ given by the *cliquer* algorithm is to check if $\alpha(b) + |A|$ is at least the size of our goal independent set size. We also use the structure of $G(S)$ to our advantage for an additional pruning mechanism.

Let A be an independent set in $G(S)[n]$, and let B be a set of vertices that are not adjacent to any vertices in A with $\max B < \min A$. Represent B as disjoint intervals $[x_i, y_i]$ where $B = \bigcup_{i=1}^{t}[x_i, y_i]$ and define $\beta(B) = \sum_{i=1}^{t} \alpha(y_i - x_i + 1)$. If there is an independent set A' with $A \subseteq A' \subseteq A \cup B$, then $A' \cap [x_i, y_i]$ is also an independent set. Further, $(A' \cap [x_i, y_i]) - x_i$ is an independent set in $G(S)[y_i - x_i + 1]$ by the vertex transitivity of $G(S)$. Thus, $|A'| \le |A| + \sum_{i=1}^{t} \alpha(y_i - x_i + 1) = |A| + \beta(B)$. Therefore, if $|A| + \beta(B)$ is below our target size of an independent set, we can backtrack.

Algorithm 1 defines the recursive algorithm FindIndependentSet(α, n, S, A, B) to find the largest size of an independent set A' in $G(S)[n]$ with $A \subseteq A' \subseteq A \cup B \subseteq [n]$ and $|A'| > \alpha(n-1)$. To compute $\alpha(n)$, call FindIndependentSet($\alpha, n, S, \varnothing, [n]$) to initialize the recursive algorithm.

Algorithm 1 FindIndependentSet(α, n, S, A, B)

```
if |A| > α(n − 1) then
  α(n) ← |A|
  return A
else if |A| + β(B) ≤ α(n − 1) then
  return Null
end if
for all b ∈ B in decreasing order do
  if |A| + α(b) ≤ α(n − 1) then
    return Null
  end if
  A′ ← A ∪ {b}
  B′ ← (B ∩ [b − 1]) − N(b)
  A″ ← FindIndependentSet(α, n, S, A′, B′)
  if A″ ≠ Null then
    return A″
  end if
end for
return Null
```

Define $\alpha(n, i)$ to be the largest size of an independent set in the circulant graph $G(n, S)$ using only vertices in $\{1, \ldots, i\}$. We can define a similar algorithm, FindIndependentSet(α, n, i, S, A, B), that computes $\alpha(n, i)$ for $i \in [n]$. In order to determine $\alpha(G(n, S))$, we compute all values $\alpha(n, i)$ for $i \in [n]$ in increasing order.

Note that for a fixed set S, it may be less work to compute $\alpha(n)$ than to compute $\alpha(n, n)$ as n increases, since $\alpha(n')$ for $n' < n$ may be used in the computation of $\alpha(n)$, but $\alpha(n', i)$ is not helpful for computing $\alpha(n, i)$. However, early computations suggested that the value of n such that $\alpha(n, n)/n = \overline{\alpha}(S)$ is much smaller than the value m such that $\overline{\alpha}(S) = \alpha(m)/m$. Thus, we organized our computation as follows: for every $n \ge 1$, compute $\alpha(2n - 1)$ and $\alpha(2n)$ and if $n > \max S$ then compute $\alpha(n, n)$. We terminated our computation when the lower and upper bounds matched.

Our implementation and all computation data are available online.[3] The computed values of $\overline{\alpha}(\{1, 1 + k, 1 + k + i\})$ are given as a table in Appendix A.

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Appendix. Table of computed values $\overline{\alpha}(\{1, 1 + k, 1 + k + i\})$

See Table A.1.

[3] Code and data are available at http://www.math.iastate.edu/dstolee/r/distance.htm and http://www.github.com/derrickstolee/DistanceGraphs.

Table A.1
Computed values of $\overline{\alpha}(S)$ for $S = \{1, 1+k, 1+k+i\}$.

k \ i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1/4	1/3	1/3	2/7	1/3	1/3	3/10	1/3	1/3	4/13	1/3	1/3	5/16	1/3	1/3	6/19	1/3	1/3	7/22	1/3
2	2/7		1/3		4/11		5/13		2/5		7/17		8/19		3/7		10/23		11/25	
3	1/3	2/5	3/8	1/3	2/5	4/11	2/5	5/13	5/14	2/5	3/8	2/5	7/18	7/19	2/5	8/21	2/5	9/23	3/8	2/5
4	2/7		1/3		1/3		6/17		7/19		8/21		9/23		2/5		11/27		12/29	
5	4/13	3/7	2/5	3/8	5/12	1/3	3/7	2/5	3/7	7/17	9/23	8/19	2/5	3/7	9/22	3/7	5/12	2/5	11/26	11/27
6	1/3		4/11		3/8		1/3		6/17		7/19		3/8		11/29		12/31		13/33	
7	3/10	4/9	7/19	2/5	3/7	4/11	7/16	1/3	4/9	5/13	4/9	3/7	12/29	10/23	12/31	11/25	13/33	4/9	3/7	4/9
8	6/19		5/13		2/5		2/5		1/3		8/21		9/23		2/5		2/5		11/29	
9	1/3	5/11	5/14	5/12	2/5	5/13	4/9	4/11	9/20	1/3	5/11	3/8	5/11	7/17	3/7	4/9	15/37	13/29	5/13	14/31
10	4/13		2/5		7/17		5/12		12/31		1/3		2/5		11/27		12/29		5/12	
11	8/25	6/13	3/8	3/7	5/13	2/5	13/31	8/21	5/11	5/13	11/24	1/3	6/13	10/27	6/13	2/5	18/41	3/7	18/43	5/11
12	1/3		11/29		8/19		3/7		3/7		14/37		1/3		16/41		13/31		14/33	
13	5/16	7/15	7/18	7/16	2/5	7/17	2/5	9/23	16/37	2/5	6/13	2/5	13/28	1/3	7/15	12/31	7/15	13/33	21/47	5/12
14	10/31		7/19		3/7		10/23		7/16		17/41		3/8		1/3		18/47		3/7	
15	1/3	8/17	2/5	4/9	9/22	8/19	7/17	2/5	17/41	11/27	19/43	7/17	7/15	9/23	15/32	1/3	8/17	2/5	8/17	15/37
16	6/19		8/21		16/39		11/25		4/9		4/9		19/47		7/18		1/3			
17	12/37	9/19	5/13	9/20	5/12	3/7	8/19	9/22	2/5	12/29	20/47	13/31	22/49	8/19	8/17	5/13	17/36	1/3	9/19	11/28
18	1/3		9/23		2/5		4/9		13/29		9/20		22/51		2/5		2/5		1/3	
19		10/21	3/8	5/11	11/26	10/23	3/7	5/12	3/7	2/5	7/17	14/33	23/53	3/7	5/11	3/7	9/19	8/21	19/40	1/3
20	14/43		2/5		11/27		3/7		14/31		5/11		5/11		8/19		9/22		24/61	
21	1/3	11/23	5/13	11/24	3/7	11/25	13/30	11/26	10/23	11/27	2/5	3/7	8/19	16/37	26/59	10/23	28/61	13/31	10/21	9/23
22			19/49		12/29		5/12		5/11		16/35		11/24		27/61				5/12	
23	16/49	12/25	11/28	6/13	22/53	4/9	7/16	3/7	11/25	12/29	25/59	2/5	25/61	17/39	3/7	18/41	29/65	11/25	31/67	7/17
24	1/3				13/31		11/26		26/59		17/37		6/13		6/13		29/67		19/45	
25		13/27	2/5	13/28	13/32	13/29	15/34	13/30	4/9	13/31	4/9		2/5	14/33	28/67	19/43	10/23	4/9	32/71	4/9
26	18/55		12/31		14/33		3/7		3/7		6/13		19/41		13/28		32/71			
27	1/3	14/29	23/59	7/15	7/17	14/31	4/9	7/16	17/38	14/33	13/29	12/29	29/69	2/5		4/9	31/73	21/47	11/25	13/29
28			13/33		3/7		16/37		13/30		31/69		20/43		7/15		7/15		34/77	
29	20/61	15/31		15/32	5/12	5/11	29/67	15/34	9/20	3/7	14/31		32/73	13/31	2/5	8/19		22/49	34/79	23/51
30	1/3		2/5		28/67		17/39		7/16		32/73		7/15		22/47		15/32		37/81	
31		16/33	7/18	8/17	8/19	16/35		4/9	19/42	16/37	5/11		5/11	14/33	33/79	2/5		18/41		24/53
32	22/67		9/23				18/41		15/34		3/7		36/79		23/49		8/17		8/17	
33	1/3	17/35	15/38	17/36	17/40	17/37	3/7	17/38	5/11	17/39	21/46		16/35	3/7	36/83	3/7	2/5	18/43		5/11
34					17/41		19/43		4/9		4/9		37/83		8/17		25/53		17/36	
35	24/73	18/37	2/5	9/19	3/7	6/13	19/44	9/20	4/9	18/41	11/24	3/7	17/37	22/51	13/29	16/37	37/89	2/5	3/7	10/23
36	1/3		16/41		18/43		4/9		21/47		17/38		38/87		41/89		26/55		9/19	
37		19/39	31/79	19/40	34/81	19/41	10/23	19/42		19/43	23/50	19/44	6/13	23/53	6/13	17/39	40/93	20/47	2/5	5/12
38	26/79		17/43		19/45		37/85		22/49		9/20		3/7		14/31		9/19		28/59	
39	1/3	20/41		10/21		20/43	7/16	5/11	18/41	4/9	6/13	10/23	25/54	24/55	19/41	25/57	43/97	18/41		2/5
40			2/5		20/47		3/7		23/51		19/42		42/95		43/97		46/99		29/61	
41	28/85	21/43	9/23	21/44	5/12	7/15	11/25	21/46	19/43	21/47	43/95	7/16	13/28	3/7	20/43	26/59	41/90	19/43	41/96	11/26
42	1/3		35/89		3/7		22/51		24/53		5/11		5/11		27/62		26/57		10/21	
43		22/45	19/48	11/23	21/50	22/47	23/52	11/24	4/9	22/49	4/9	11/25	27/58	22/51	7/15	27/61	7/15	4/9	29/66	4/9
44	30/91				8/19		23/53		5/11		26/57		21/46		3/7		35/78		29/62	
45	1/3	23/47	2/5	23/48	11/26	23/49	4/9	23/50	25/56	23/51	21/47	23/52	7/15	23/53	29/62	4/9	22/47	29/65	41/91	21/47
46			20/51				24/55		25/56		27/59		11/24		37/84		34/77		29/63	
47		24/49	13/33	12/25	23/54	8/17	38/87	6/13	13/29	24/53	22/49	4/9	39/85	24/55	15/32	3/7	23/49	30/67	41/89	22/49
48	1/3		21/53		23/55		25/57				28/61		23/50		37/82		10/23		29/64	
49		25/51		25/52	3/7	25/53		25/54	9/20	5/11	23/51	25/56	37/82	25/57	31/66		8/17	31/69	8/17	32/71
50			2/5		8/19		26/59		23/52		29/63		6/13		6/13				29/65	

k \ i	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	1/3	8/25	1/3	1/3	9/28	1/3	1/3	10/31	1/3	1/3	11/34	1/3	1/3	12/37	1/3	1/3	13/40	1/3	1/3	14/43
2	4/9		13/29		14/31		5/11		16/35		17/37		6/13		19/41		20/43		7/15	
3	5/13	2/5	11/28	11/29	2/5	12/31	2/5	13/33	13/34	2/5	7/18	2/5	15/38	5/13	2/5	16/41	2/5	17/43	17/44	2/5
4	13/31		14/33		3/7		16/37		17/39		18/41		19/43		4/9		21/47		22/49	
5	3/7	12/29	3/7	13/31	13/32	14/33	7/17	3/7	5/12	3/7	8/19	16/39	17/40	17/41	3/7	18/43	3/7	19/45		20/47
6	2/5		15/37		16/39		17/41		18/43		19/45		20/47		3/7		22/51		23/53	
7	13/30	8/19	7/16	2/5	15/34	3/7	4/9	16/37	4/9	17/39	20/47	18/41	3/7	19/43	19/44	4/9	10/23	4/9	7/16	3/7
8	12/31		13/33		2/5		2/5		19/47		20/49		7/17		22/53		23/55		8/19	
9	16/41	5/11	18/43	5/11	4/9	10/23	17/38	5/12	9/20	2/5	19/42	11/26	5/11	4/9	5/11	21/47	25/57	22/49	25/59	23/51
10	17/43		2/5		15/37		16/39		17/41		5/12		2/5							
11	2/5	16/35	5/13	17/37	19/49	6/13	7/17	6/13	23/53	4/9	5/11	3/7	21/46	12/29	11/24	2/5	23/50	13/31	6/13	7/16
12	3/7		3/7		20/51		21/53		18/43		19/45		20/47		3/7		3/7		2/5	
13	3/7	11/25	7/17	6/13	2/5	19/41	2/5	20/43		7/15	24/59	7/15	26/61	14/31	4/9	7/16	6/13	14/33	25/54	7/17
14	16/37		17/39		7/16		8/19		23/59		24/61		3/7		22/51		23/53		24/55	
15	24/53		24/55	3/7	8/19	13/29	9/22	7/15	7/17	22/47	25/63	23/49	2/5	8/17		8/17	29/69	16/35	31/71	4/9
16	23/55		18/41		19/43		4/9		4/9		27/65				9/23		30/71		25/57	
17	9/19	17/41	27/59	18/43	27/61	8/19	3/7	7/16		5/11	21/50	8/17	8/19	25/53	28/71	26/55	29/73	9/19	12/29	9/19
18	16/41		25/61		4/9		21/47		22/49		9/20		31/71		30/73		2/5			
19	10/21	12/31	10/21	19/45	6/13	20/47	30/67	3/7	10/23	3/7		4/9	23/54	17/37	3/7	9/19	3/7	28/59	31/79	29/61
20	1/3		2/5				10/23		23/51		24/53		5/11		5/11		34/79		9/22	
21	21/44	1/3	11/23		11/23	3/7	33/71	22/51	33/73	23/53	11/25	10/23	3/7	17/39	25/58	9/20	13/30	19/41	10/23	10/21
22	26/67		1/3		28/71		7/17		32/75		5/11		26/57		27/59		11/24		38/85	
23	11/23	2/5	23/48	1/3	12/25	20/51	12/25	8/19	36/77	24/55	36/79	25/57	4/9	11/25	36/83	3/7	27/62	19/43	7/16	5/11
24	11/26				1/3		30/77		23/55				37/83		28/61		29/63		6/13	
25	34/73		12/25	15/38	25/52	1/3	13/27	2/5	13/27	17/41	39/83	26/59	39/85	27/61	13/29	4/9	39/89	4/9	3/7	10/23
26	3/7		3/7		11/28						25/59		26/61		39/89		6/13		31/67	
27	5/11	17/39	37/79	12/29	13/27	16/41	27/56	1/3	14/29	17/43	14/29		42/89	4/9	6/13	29/65	14/31	30/67	42/95	13/29
28	22/51		13/30		35/83		2/5		1/3		24/61		3/7		28/65				44/97	
29	4/9	14/31	38/83	3/7	8/17	13/31	14/29		29/60		15/31	9/23	15/31	27/65	9/19	7/16	45/97	31/69	5/11	32/71
30			24/55		7/16		37/89		36/91				2/5		41/97		10/23			
31	37/85	5/11	13/29	5/11	41/89		43/91	14/33	15/31	13/33	31/64		16/33		16/33	29/69	29/61	22/51	41/88	33/73
32	13/29		25/57		26/59		15/34				38/97						38/91		32/73	
33	3/7	26/57	40/91	16/35	14/31	21/47	44/95	3/7	46/97	3/7	16/33	2/5	33/68		17/35		17/35	31/73	41/86	
34	6/13		41/93		27/61		4/9		4/9		5/12									
35		27/59		28/61	43/97	17/37	5/11	11/25	27/58	16/37	39/82		17/35		35/72		18/37	2/5	18/37	3/7
36	9/19		44/97		4/9		29/65		17/38		39/89		8/19		15/38					
37	13/30	22/49		29/63		6/13	25/56	6/13	16/35		43/92	17/39	31/65		18/37		37/76		19/39	
38	19/40		27/58		25/56		30/67		31/69		9/20				17/40					
39	23/54	22/51	7/16	6/13	18/41	31/67	29/66	19/41	40/89	5/11	17/37	32/73	38/81	18/41	11/23		19/39		39/80	
40	10/21		10/21		38/83		37/84		32/71		33/73		19/42		3/7		3/7			
41	2/5		29/66	4/9	15/34	32/69	19/43	33/71	27/61	20/43	33/73		6/13	34/77		19/43	35/73	18/43	20/41	
42	31/65		21/44		37/79		37/82		43/97		34/75		5/11		5/11		19/44			
43	19/45	2/5	36/85	3/7	31/70	26/57	4/9	34/73	4/9	7/15	41/92	7/15			19/41	4/9		4/9		19/45
44	32/67		11/23		11/23						5/11		36/79		21/46				10/23	
45		8/19	2/5		4/9	26/59	33/74	7/15	21/47	36/77	18/41	22/47	43/96			37/83	20/43	21/47		
46	11/23		34/71		23/48		41/87				11/25		37/81		38/83		11/24			
47	17/39	26/59	3/7				17/38		35/78	37/79	22/49	38/81	19/43	23/49				13/29	7/15	22/49
48	8/17		35/73		12/25		12/25								39/85		40/87		23/50	
49	29/65	23/51		13/31		3/7	41/93	7/16	9/20		23/51	39/83	43/97	8/17	4/9	8/17				41/91
50			12/25		37/77		25/52						4/9				41/89		6/13	

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