On weighted graph homomorphisms

David Galvin and Prasad Tetali

ABSTRACT. For given graphs G and H, let $|Hom(G,H)|$ denote the set of graph homomorphisms from G to H. We show that for any finite, n-regular, bipartite graph G and any finite graph H (perhaps with loops), $|Hom(G,H)|$ is maximum when G is a disjoint union of $K_{n,n}$'s. This generalizes a result of J. Kahn on the number of independent sets in a regular bipartite graph. We also give the asymptotics of the logarithm of $|Hom(G,H)|$ in terms of a simply expressed parameter of H.

We also consider weighted versions of these results which may be viewed as statements about the partition functions of certain models of physical systems with hard constraints.

1. Introduction

Let G be an n-regular, N-vertex bipartite graph on vertex set $V(G)$, and let H be a fixed graph on vertex set $V(H)$ (perhaps with loops). We will always use u, v for the vertices of G and i,j for those of H. Set

$$Hom(G,H) = \{f : V(G) \rightarrow V(H) \ : \ u \sim v \Rightarrow f(u) \sim f(v)\}.$$

That is, $Hom(G,H)$ is the set of graph homomorphisms from G to H. (For graph theory basics, see e.g. [2], [5]).

When $H = H_{ind}$ consists of one looped and one unlooped vertex connected by an edge, an element of $Hom(G, H_{ind})$ can be thought of as a specification of an independent set (a set of vertices spanning no edges) in G. Our point of departure is the following result of Kahn [7], bounding the number of independent sets in regular bipartite graphs. For any graph G, write $\mathcal{I}(G)$ for the set of independent sets of G.

THEOREM 1.1. *For any n-regular, N-vertex bipartite graph G,*

$$|\mathcal{I}(G)| \leq (2^{n+1} - 1)^{N/2n}.$$

An approximate version of Theorem 1.1 — $\log|\mathcal{I}(G)| \leq (1/2 + o(1))N$, where $o(1) \rightarrow 0$ as $n \rightarrow \infty$ — for general n-regular, N-vertex G was earlier proved by Alon [1]. Note that $|Hom(K_{n,n}, H_{ind})| = 2^{n+1} - 1$ (where $K_{n,n}$ is the complete

bipartite graph with n vertices on each side), so we may paraphrase Theorem 1.1 by saying that $|Hom(G, H_{ind})|$ is maximum when G is a disjoint union of $K_{n,n}$'s. Our main result is a generalization of this statement (and our proof is a generalization of Kahn's).

PROPOSITION 1.2. *For any n-regular, N-vertex bipartite G, and any H,*

$$|Hom(G, H)| \leq |Hom(K_{n,n}, H)|^{N/2n}.$$

Somewhat surprisingly, we can also exhibit a lower bound that is good enough to allow us to obtain the asymptotics of $\log|Hom(G, H)|$ for fixed H as $n \to \infty$ (here, and throughout the rest of the paper, we use log for the base 2 logarithm). To state the result, it is convenient to introduce a parameter of H that is very closely related to $|Hom(K_{n,n}, H)|$, but is easier to work with. Set

$$\eta(H) = \max\{|A||B| : A, B \subseteq V(H), i \sim j \; \forall i \in A, j \in B\}.$$

(When H is loopless, this is the maximum number of edges in a complete bipartite subgraph of H. Peeters [10] has recently shown that determining $\eta(H)$, even when H is bipartite, is NP-complete.)

PROPOSITION 1.3. *For any n-regular, N-vertex bipartite G, and any H,*

$$\frac{\log \eta(H)}{2} \leq \frac{\log |Hom(G, H)|}{N} \leq \frac{\log \eta(H)}{2} + \frac{|V(H)|}{2n}.$$

We use the example of $H = K_k$, the complete graph on k vertices, to illustrate the definition of η. It is easy to see that for any $A, B \subseteq V(K_k)$, we have $i \sim j \; \forall i \in A, j \in B$ iff A and B are disjoint, and so $|A||B|$ is maximum when $|A|$ and $|B|$ are as close as possible to $k/2$. Hence $\eta(K_k) = \lfloor k/2 \rfloor \lceil k/2 \rceil$. Since an element of $Hom(G, K_k)$ is exactly a proper k coloring of G, we get as a corollary of Proposition 1.3 an approximate count of the number of k-colorings of a regular bipartite graph.

COROLLARY 1.4. *For any n-regular, N-vertex bipartite G,*

$$|Hom(G, K_k)| = (\lfloor k/2 \rfloor \lceil k/2 \rceil)^{N(1/2+o(1))}.$$

We now consider weighted versions of Propositions 1.2 and 1.3. Following [3], we put a measure on $Hom(G, H)$ as follows. To each $i \in V(H)$ assign a positive "activity" λ_i, and write Λ for the set of activities. Give each $f \in Hom(G, H)$ weight

$$w^\Lambda(f) = \prod_{v \in V(G)} \lambda_{f(v)}.$$

The constant that turns this assignment of weights on $Hom(G, H)$ into a probability distribution is

$$Z^\Lambda(G, H) = \sum_{f \in Hom(G,H)} w^\Lambda(f).$$

When all activities are 1, we have $Z^\Lambda(G, H) = |Hom(G, H)|$, and so the following is a generalization of Proposition 1.2.

PROPOSITION 1.5. *For any n-regular, N-vertex bipartite G, any H, and any system Λ of positive activities on $V(H)$,*

$$Z^\Lambda(G, H) \leq \left(Z^\Lambda(K_{n,n}, H)\right)^{N/2n}.$$

It was observed in [3] that $Z^{\Lambda}(G, H)$ may be related to $|Hom(G, H')|$ for an appropriate modification H' of H. That observation (which will be discussed in more detail in Section 3) is central to the proof of Proposition 1.5.

Proposition 1.3 also generalizes. For a set of activities Λ on $V(H)$, set

$$\eta^{\Lambda}(H) = \max \left\{ \left(\sum_{i \in A} \lambda_i \right) \left(\sum_{j \in B} \lambda_j \right) : A, B \subseteq V(H), i \sim j \ \forall i \in A, j \in B \right\}.$$

PROPOSITION 1.6. *For any n-regular, N-vertex bipartite G, any H, and any system Λ of positive activities on $V(H)$,*

$$\frac{\log \eta^{\Lambda}(H)}{2} \leq \frac{\log Z^{\Lambda}(G, H)}{N} \leq \frac{\log \eta^{\Lambda}(H)}{2} + \frac{|V(H)|}{2n}.$$

We may put these results in the framework of a well-known mathematical model of physical systems with "hard constraints" (see [3]). These are systems with strictly forbidden configurations. An example is the hard-core lattice gas model, in which a legal configuration of particles on a lattice is precisely one in which no two adjacent lattice sites are occupied. (By way of contrast, consider the ferromagnetic Ising model, where adjacent particles are discouraged from having opposing spins, but not forbidden — this is a "soft constraint".)

We think of the vertices of G as particles and the edges as bonds between pairs of particles, and we think of the vertices of H as possible "spins" that particles may take. Pairs of vertices of G joined by a bond may have spins i and j only when i and j are adjacent in H (in particular, they may both have spin i only when i has a loop in H). Thus the legal spin configurations on the vertices of G are precisely the homomorphisms from G to H. We think of the activities on the vertices of H as a measure of the likelihood of seeing the different spins; the probability of a particular spin configuration is proportional to the product over the vertices of G of the activities of the spins. Propositions 1.5 and 1.6 concern the "partition function" of this model — the normalizing constant that turns the above-described system of weights on the set of legal configurations into a probability measure.

The results we actually prove are in a slightly more general weighted model. Write $\mathcal{E}_G$ and $\mathcal{O}_G$ for the partition classes of G, and to each $i \in V(H)$ assign a positive *pair* of activities (λ_i, μ_i). Write (Λ, M) for the set of activities. Give each $f \in Hom(G, H)$ weight

$$w^{(\Lambda, \mathrm{M})}(f) = \prod_{v \in \mathcal{E}_G} \lambda_{f(v)} \prod_{v \in \mathcal{O}_G} \mu_{f(v)}.$$

The constant that turns this assignment of weights on $Hom(G, H)$ into a probability distribution is

$$\text{(1)} \qquad Z^{(\Lambda, \mathrm{M})}(G, H) = \sum_{f \in Hom(G, H)} w^{(\Lambda, \mathrm{M})}(f).$$

A special case of this model was considered by Kahn [8] (see also [6]), where Theorem 1.1 was extended to

THEOREM 1.7. *For any n-regular, N-vertex bipartite G, and any $\lambda, \mu \geq 1$,*

$$\sum_{I \in \mathcal{I}(G)} \prod_{v \in \mathcal{E}_G} \lambda^{|I \cap \mathcal{E}_G|} \prod_{v \in \mathcal{O}_G} \mu^{|I \cap \mathcal{O}_G|} \leq ((1 + \lambda)^n + (1 + \mu)^n - 1)^{N/2n}.$$

It was conjectured in [8] that the assumption $\lambda, \mu \geq 1$ may be relaxed to $\lambda, \mu \geq 0$. We show that this is indeed true, by generalizing Proposition 1.5 to:

PROPOSITION 1.8. *For any n-regular, N-vertex bipartite G, any H, and any system (Λ, M) of positive activities on $V(H)$,*

$$Z^{(\Lambda,\mathrm{M})}(G, H) \leq \left(Z^{(\Lambda,\mathrm{M})}(K_{n,n}, H) \right)^{N/2n}.$$

We also generalize Proposition 1.6 to this setting. Set

$$\eta^{(\Lambda,\mathrm{M})}(H) = \max \left\{ \left(\sum_{i \in A} \lambda_i \right) \left(\sum_{j \in B} \mu_j \right) : A, B \subseteq V(H), i \sim j \; \forall i \in A, j \in B \right\}.$$

PROPOSITION 1.9. *For any n-regular, N-vertex bipartite G, any H, and any system (Λ, M) of positive activities on $V(H)$,*

$$\frac{\log \eta^{(\Lambda,\mathrm{M})}(H)}{2} \leq \frac{\log Z^{(\Lambda,\mathrm{M})}(G, H)}{N} \leq \frac{\log \eta^{(\Lambda,\mathrm{M})}(H)}{2} + \frac{|V(H)|}{2n}.$$

Proposition 1.8 generalizes to the case of biregular G (a bipartite graph G with partition classes $\mathcal{E}_G$ and $\mathcal{O}_G$ is (a, b)-*biregular* if all vertices in $\mathcal{E}_G$ have degree a and all in $\mathcal{O}_G$ have degree b). The proof of the following proposition, which is a straightforward modification of the proof of Proposition 1.8, is omitted.

PROPOSITION 1.10. *For any (a, b)-biregular, N-vertex, bipartite G, any H, and any system (Λ, M) of positive activities on $V(H)$,*

$$Z^{(\Lambda,\mathrm{M})}(G, H) \leq \left(Z^{(\Lambda,\mathrm{M})}(K_{a,b}, H) \right)^{N/(a+b)}.$$

It was conjectured in [7] that Theorem 1.1 remains true without the assumption that G is bipartite. We similarly conjecture that bipyticity is unnecessary in Proposition 1.8, and hence also in Propositions 1.2 and 1.5. (Proposition 1.3, and hence also Propositions 1.6 and 1.9, is easily seen to fail for non-bipartite G.)

The proof of Proposition 1.8 requires entropy considerations; these are reviewed in Section 2. The proofs are then given in Section 3.

2. Entropy

Here we briefly review the relevant entropy material. Our treatment is mostly copied from [7]. For a more thorough discussion, see e.g. [9]. In what follows $\mathbf{X}$, $\mathbf{Y}$ etc. are discrete random variables, which in our usage are allowed to take values in any finite set.

The *entropy* of the random variable $\mathbf{X}$ is

$$H(\mathbf{X}) = \sum_x p(x) \log \frac{1}{p(x)},$$

where we write $p(x)$ for $\mathbf{P}(\mathbf{X} = x)$ (and extend this convention in natural ways below). The *conditional entropy* of $\mathbf{X}$ given $\mathbf{Y}$ is

$$H(\mathbf{X}|\mathbf{Y}) = \mathbf{E}H(\mathbf{X}|\{\mathbf{Y} = y\}) = \sum_y p(y) \sum_x p(x|y) \log \frac{1}{p(x|y)}.$$

Notice that we are also writing $H(\mathbf{X}|Q)$ with Q an event (in this case $Q = \{\mathbf{Y} = y\}$):

$$H(\mathbf{X}|Q) = \sum p(x|Q) \log \frac{1}{p(x|Q)}.$$

When we condition on a random variable and an event simultaneously, we use ";" to separate the two.

For a random vector $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_n)$ (note this is also a random variable), we have

$$(2) \qquad H(\mathbf{X}) = H(\mathbf{X}_1) + H(\mathbf{X}_2|\mathbf{X}_1) + \cdots + H(\mathbf{X}_n|\mathbf{X}_1, \dots, \mathbf{X}_{n-1}).$$

We will make repeated use of the inequalities

$$(3) \qquad H(\mathbf{X}) \leq \log |\text{range}(\mathbf{X})| \text{ (with equality if } \mathbf{X} \text{ is uniform)},$$

$$H(\mathbf{X}|\mathbf{Y}) \leq H(\mathbf{X}),$$

and more generally,

$$(4) \qquad \text{if } \mathbf{Y} \text{ determines } \mathbf{Z} \text{ then } H(\mathbf{X}|\mathbf{Y}) \leq H(\mathbf{X}|\mathbf{Z}).$$

Note that (2) and (4) imply

$$H(\mathbf{X}) \leq H(\mathbf{Y}) + H(\mathbf{X}|\mathbf{Y})$$

and

$$(5) \qquad H(\mathbf{X}_1, \dots, \mathbf{X}_n) \leq \sum H(\mathbf{X}_i).$$

We also have a conditional version of (5):

$$H(\mathbf{X}_1, \dots, \mathbf{X}_n|\mathbf{Y}) \leq \sum H(\mathbf{X}_i|\mathbf{Y}).$$

We will also need the following lemma of Shearer (see [4, p. 33]). For a random vector $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_m)$ and $A \subseteq [m]$, set $\mathbf{X}_A = (\mathbf{X}_i : i \in A)$.

LEMMA 2.1. *Let* $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_m)$ *be a random vector and* $\mathcal{A}$ *a collection of subsets (possibly with repeats) of* $[m]$, *with each element of* $[m]$ *contained in at least* t *members of* $\mathcal{A}$. *Then*

$$H(\mathbf{X}) \leq \frac{1}{t} \sum_{A \in \mathcal{A}} H(\mathbf{X}_A).$$

3. Proofs

For a regular bipartite graph G, we write $\mathcal{E}_G$ and $\mathcal{O}_G$ for the partition classes. For ease of notation, we write $\mathcal{E}_n$ for $\mathcal{E}_{K_{n,n}}$ and $\mathcal{O}_n$ for $\mathcal{O}_{K_{n,n}}$. For a partition $U \cup L$ of $V(H)$, set

$$Hom^{U,L}(G, H) = \{f \in Hom(G, H) : f(\mathcal{E}_G) \subseteq U, f(\mathcal{O}_G) \subseteq L\}.$$

(For a set X we write $f(X)$ for $\{f(x) : x \in X\}$.)

We begin by deriving an expression for $|Hom^{U,L}(K_{n,n}, H)|$. For $A \subseteq L$ set

$$\mathcal{H}(A) = \{f \in Hom^{U,L}(K_{n,n}, H) : f(\mathcal{O}_n) = A\},$$

$$T(A) = \{g : [n] \to A \; : \; g \text{ surjective}\}$$

and

$$C^U(A) = \{j \in U : j \sim i \; \forall i \in A\}.$$

(Note that $C^U(A)$ is the set of possible images of $v \in \mathcal{E}_G$ under a member of $Hom^{U,L}(G, H)$, given that the image of $N(v)$ is A.) It is easy to see that $\{\mathcal{H}(A) :$

$A \subseteq L\}$ forms a partition of $Hom^{U,L}(K_{n,n}, H)$, and also that for each A, $|\mathcal{H}(A)| = |T(A)||C^U(A)|^n$. Thus we have

$$(6) \qquad |Hom^{U,L}(K_{n,n}, H)| = \sum_{A \subseteq L} |T(A)||C^U(A)|^n.$$

The following is the central lemma in the proofs of Propositions 1.8 and 1.9.

LEMMA 3.1. *For any n-regular, N-vertex bipartite G, and any H with $U \cup L$ a partition of $V(H)$,*

$$|Hom^{U,L}(G, H)| \leq |Hom^{U,L}(K_{n,n}, H)|^{N/2n}.$$

Proof: The proof is based on [**7**, Thm. 1.9]. Let $\mathbf{f}$ be chosen uniformly from $Hom^{U,L}(G, H)$. For $v \in V(G)$, write $\mathbf{f}_v$ for $\mathbf{f}(v)$, $\mathbf{N}_v$ for $\mathbf{f}|_{N(v)}$ and $\mathbf{M}_v$ for $\{\mathbf{f}_w : w \in N(v)\}$. For $v \in \mathcal{E}_G$ and $A \subseteq L$, write $m_v(A)$ for $\mathbf{P}(\mathbf{M}_v = A)$. (Note that $\sum_A m_v(A) = 1$.) We have (with the main inequalities justified below; the remaining steps follow in a straightforward way from the material of Section 2)

$$
\begin{aligned}
\log|Hom^{U,L}(G, H)| &= H(\mathbf{f}) \\
&= H(\mathbf{f}|_{\mathcal{O}_G}) + H(\mathbf{f}|_{\mathcal{E}_G} \mid \mathbf{f}|_{\mathcal{O}_G}) \\
&\leq H(\mathbf{f}|_{\mathcal{O}_G}) + \sum_{v \in \mathcal{E}_G} H(\mathbf{f}_v \mid \mathbf{f}|_{\mathcal{O}_G}) \\
&\leq H(\mathbf{f}|_{\mathcal{O}_G}) + \sum_{v \in \mathcal{E}_G} H(\mathbf{f}_v \mid \mathbf{N}_v) \\
(7) \qquad &\leq \frac{1}{n} \sum_{v \in \mathcal{E}_G} H(\mathbf{N}_v) + \sum_{v \in \mathcal{E}_G} H(\mathbf{f}_v \mid \mathbf{N}_v) \\
&\leq \frac{1}{n} \sum_{v \in \mathcal{E}_G} [H(\mathbf{M}_v) + H(\mathbf{N}_v|\mathbf{M}_v)] + \sum_{v \in \mathcal{E}_G} H(\mathbf{f}_v \mid \mathbf{N}_v) \\
&\leq \frac{1}{n} \sum_{v \in \mathcal{E}_G} [H(\mathbf{M}_v) + H(\mathbf{N}_v|\mathbf{M}_v) + nH(\mathbf{f}_v|\mathbf{N}_v)] \\
&\leq \frac{1}{n} \sum_{v \in \mathcal{E}_G} \sum_{A \subseteq L} \Big[m_v(A) \log \frac{1}{m_v(A)} + \\
& \qquad\qquad m_v(A) H(\mathbf{N}_v|\{\mathbf{M}_v = A\}) + \\
& \qquad\qquad nm_v(A) H(\mathbf{f}_v|\mathbf{N}_v; \{\mathbf{M}_v = A\}) \Big] \\
&\leq \frac{1}{n} \sum_{v \in \mathcal{E}_G} \sum_{A \subseteq L} \Big[m_v(A) \log \frac{1}{m_v(A)} + \\
(8) \qquad & \qquad\qquad m_v(A) \log |T(A)| + nm_v(A) \log |C^U(A)| \Big] \\
&= \frac{1}{n} \sum_{v \in \mathcal{E}_G} \sum_{A \subseteq L} m_v(A) \log \frac{|T(A)||C^U(A)|^n}{m_v(A)} \\
(9) \qquad &\leq \frac{1}{n} \sum_{v \in \mathcal{E}_G} \log \left[\sum_{A \subseteq L} |T(A)||C^U(A)|^n \right] \\
(10) \qquad &= \frac{N}{2n} \log |Hom^{U,L}(K_{n,n}, H)|.
\end{aligned}
$$

The main inequality (7) involves an application of Lemma 2.1, with $\mathcal{A} = \{N(v) : v \in \mathcal{E}_G\}$, and (9) is an application of Jensen's inequality. In (8), we use (3), noting that conditioning on the event $\{\mathbf{M}_v = A\}$ there are $|T(A)|$ possible values for $\mathbf{N}_v$, and $|C^U(A)|$ possible values for $\mathbf{f}_v$. Finally, (10) follows from (6). $\qquad\square$

It is worth noting at this point that Lemma 3.1 easily implies Proposition 1.2. Let H' be the graph on vertex set $\cup_{i \in V(H)} \{v_i, w_i\}$ with v_i and w_j adjacent exactly when i and j are adjacent in H. Set $U = \{v_1, \ldots, v_{|V(H)|}\}$ and $L = \{w_1, \ldots, w_{|V(H)|}\}$. It is easy to check that $|Hom(G, H)| = |Hom^{U,L}(G, H')|$, from which Proposition 1.2 follows via an application of Lemma 3.1.

This idea of "doubling" H, combined with the construction of [3] that relates $Z^\Lambda(G, H)$ to $|Hom(G, H')|$ for an appropriate modification H' of H, allows us to pass from Proposition 1.2 to Proposition 1.8. The details are as follows.

Recall that our aim is to upper bound the partition function $Z^{(\Lambda,\mathrm{M})}(G, H)$ (see (1)). By continuity, we may assume that all activities are rational. Let C be the least positive integer such that $C\lambda_i$ and $C\mu_i$ are integers for each $i \in V(H)$. Let $H^{(\Lambda,\mathrm{M})}$ be the graph whose vertex set is obtained from H by replacing each $i \in V(H)$ by two sets, $D_i^U = \{i_1^U, \ldots, i_{C\lambda_i}^U\}$ and $D_i^L = \{i_1^L, \ldots, i_{C\mu_i}^L\}$ of $C\lambda_i$ and $C\mu_i$ vertices. For each $i, j \in V(H)$ (not necessarily distinct), $i' \in D_i^U$ and $j' \in D_j^L$, join i' to j' exactly when i and j are adjacent in H. Set $U = U(H^{(\Lambda,\mathrm{M})}) = \cup_{i \in V(H)} D_i^U$ and $L = L(H^{(\Lambda,\mathrm{M})}) = \cup_{i \in V(H)} D_i^L$.

We now show that $Z^{(\Lambda,\mathrm{M})}(G, H)$ is related to $|Hom^{U,L}(G, H^{(\Lambda,\mathrm{M})})|$. Say that $g \in Hom^{U,L}(G, H^{(\Lambda,\mathrm{M})})$ is a *lift* of $f \in Hom(G, H)$ if for all $v \in V(G)$,

$$g(v) = \begin{cases} f(v)_k^U, & \text{some } 1 \le k \le C\lambda_{f(v)} & \text{if } v \in \mathcal{E}_G, \\ f(v)_k^L, & \text{some } 1 \le k \le C\mu_{f(v)} & \text{if } v \in \mathcal{O}_G. \end{cases}$$

Set

$$\mathcal{G}(f) = \{g \in Hom^{U,L}(G, H^{(\Lambda,\mathrm{M})}) : g \text{ is a lift of } f\}.$$

It is easy to check that $|\mathcal{G}(f)| = w^{(\Lambda,\mathrm{M})}(f)C^N$ for each $f \in Hom(G, H)$, and that $\{\mathcal{G}(f) : f \in Hom(G, H)\}$ forms a partition of $Hom^{U,L}(G, H^{(\Lambda,\mathrm{M})})$. It follows that

$$(11) \qquad Z^{(\Lambda,\mathrm{M})}(G, H) = \frac{|Hom^{U,L}(G, H^{(\Lambda,\mathrm{M})})|}{C^N}.$$

We now have all we need to prove Propositions 1.8 and 1.9.
Proof of Proposition 1.8: Applying (11) with $G = K_{n,n}$ we get

$$(12) \qquad Z^{(\Lambda,\mathrm{M})}(K_{n,n}, H) = \frac{|Hom^{U,L}(K_{n,n}, H^{(\Lambda,\mathrm{M})})|}{C^{2n}}.$$

Proposition 1.8 now follows from (11), (12) and Lemma 3.1. $\qquad\square$

Proof of Proposition 1.9: For each $A \subseteq V(H)$, set $C(A) = \{j \in H : j \sim i \ \forall i \in A\}$ and

$$\mathcal{D}(A) = \{f \in Hom(K_{n,n}, H) : f(\mathcal{E}_n) \subseteq A, \ f(\mathcal{O}_n) \subseteq C(A)\}.$$

By Proposition 1.8 we have

$$
\begin{aligned}
\left(Z^{(\Lambda,\mathrm{M})}(G,H) \right)^{2n/N} &\leq Z^{(\Lambda,\mathrm{M})}(K_{n,n},H) \\
&\leq \sum_{A \subseteq V(H)} \sum_{f \in \mathcal{D}(A)} w^{(\Lambda,\mathrm{M})}(f) \\
&= \sum_{A \subseteq V(H)} \left(\sum_{i \in A} \lambda_i \right)^n \left(\sum_{j \in C(A)} \mu_j \right)^n \\
&\leq 2^{|V(H)|} \left(\eta^{(\Lambda,\mathrm{M})}(H) \right)^n .
\end{aligned}
$$

(13)

This gives the upper bound. For the lower bound, let $A, B \subseteq V(H)$ satisfying $i \sim j \ \forall i \in A, j \in B$ be such that $\eta^{(\Lambda,\mathrm{M})}(H) = \left(\sum_{i \in A} \lambda_i \right) \left(\sum_{j \in B} \mu_j \right)$. We have

$$
\begin{aligned}
Z^{(\Lambda,\mathrm{M})}(G,H) &\geq \sum \{ w^{(\Lambda,\mathrm{M})}(f) : f(\mathcal{E}_G) \subseteq A, f(\mathcal{O}_G) \subseteq B \} \\
&= \left(\sum_{i \in A} \lambda_i \right)^{N/2} \left(\sum_{j \in B} \mu_j \right)^{N/2} \\
&= \eta^{(\Lambda,\mathrm{M})}(H)^{N/2} .
\end{aligned}
$$

$\square$

References

[1] N. Alon, Independent sets in regular graphs and sum-free subsets of finite groups, *Israel J. Math.* **73** (1991), 247–256.

[2] B. Bollobás, *Modern Graph Theory*, Springer, New York, 1998.

[3] G. Brightwell and P. Winkler, Graph homomorphisms and phase transitions, *J. Combin. Theory Ser. B* **77** (1999), 221–262.

[4] F.R.K. Chung, P. Frankl, R. Graham and J.B. Shearer, Some intersection theorems for ordered sets and graphs, *J. Combin. Theory Ser. A.* **48** (1986), 23–37.

[5] R. Diestel, *Graph Theory*, Springer, New York, 1997.

[6] O. Häggström, Ergodicity of the hard-core model on $\mathbf{Z}^2$ with parity-dependent activities, *Ark. Mat.* **35** (1997), 171–184.

[7] J. Kahn, An entropy approach to the hard-core model on bipartite graphs, *Combin. Prob. Comp.* **10** (2001), 219–237.

[8] J. Kahn, Entropy, independent sets and antichains: a new approach to Dedekind's problem, *Proc. Amer. Math. Soc.* **130** (2002), 371–378

[9] R.J. McEliece, *The Theory of Information and Coding*, Addison-Wesley, London, 1977.

[10] R. Peeters, The maximum edge biclique problem is NP-complete, *Research Memorandum 789*, Faculty of Economics and Business Administration, Tilberg University (2000).

MICROSOFT RESEARCH, 1 MICROSOFT WAY, REDMOND WA 98052, USA
E-mail address: galvin@microsoft.com

SCHOOL OF MATHEMATICS AND COLLEGE OF COMPUTING, GEORGIA TECH, ATLANTA GA 30332-0160, USA
E-mail address: tetali@math.gatech.edu