On the independent set sequence of a tree

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Abstract

Alavi, Malde, Schwenk and Erdős asked whether the independent set sequence of every tree is unimodal. Here we make some observations about this question. We show that for the uniformly random (labelled) tree, asymptotically almost surely (a.a.s.) the initial approximately 49% of the sequence is increasing while the terminal approximately 38% is decreasing. We also present a generalization of a result of Levit and Mandrescu, concerning the final one-third of the independent set sequence of a König-Egerváry graph.

1 Introduction

A sequence \((a_0, a_1, \ldots, a_m)\) of positive terms is unimodal if there is \(k\) such that

\[ a_0 \leq a_1 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots a_{m-1} \geq a_m. \]

Unimodality is ubiquitous in combinatorics and algebra, see e.g. the survey papers [7, 8, 26].

It is well-known that the matching sequence of any finite graph (the sequence whose \(k\)th term is the number of matchings with \(k\) edges in the graph) is unimodal; this follows from the seminal theorem of Heilmann and Lieb [17] that the generating polynomial of the matching sequence has all real roots. In contrast, the independent set sequence of a graph \(G\) — the sequence whose \(k\)th term \(i_k = i_k(G)\) is the number of independent sets (sets of mutually non adjacent vertices) of size \(k\) in \(G\) — is not in general unimodal. Alavi, Malde, Schwenk and Erdős [1] showed, in fact, that it can be arbitrarily far from unimodal, in a precise sense (see also [3]).

There are families of graphs for which the independent set sequence is known to be unimodal — for example, claw-free graphs (graphs without an induced \(K_{1,3}\)), as first shown by Hamidoune [15]. In 1987 Alavi, Malde, Schwenk and Erdős [1] posed a question about another very basic family:

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**Question 1.1.** Is the independent set sequence of every tree unimodal? And what about every forest?

There have been numerous partial results — mostly exhibiting families of trees with unimodal independent set sequences, see e.g. [2, 5, 12, 19, 20, 21, 27, 29, 30, 31, 32] — but the general question remains stubbornly open. The best general result to date is due to Levit and Mandrescu. A König-Egerváry graph is one in which the number of vertices is \( \alpha + \mu \), where \( \alpha \) is the size of the largest independent set and \( \mu \) is the size of the largest matching (measured by number of edges). Bipartite graphs, and so in particular trees and forests, are König-Egerváry. Levit and Mandrescu [18] show:

**Theorem 1.2.** For a König-Egerváry graph \( G \),

\[
i_{\lceil(2\alpha-1)/3\rceil} \geq i_{\lceil(2\alpha-1)/3+1\rceil} \geq \cdots \geq i_{\alpha-1} \geq i_{\alpha}.
\]

So the (non-zero part of the) independent set sequence of a tree is decreasing for its last one-third. Theorem 1.2 is easily seen to be tight: the graph consisting of \( \alpha \) vertex disjoint edges (and no other vertices) has independent set sequence which is decreasing from exactly \( i_{\lceil(2\alpha-1)/3\rceil} \) on.

In this note we make a number of observations around Question 1.1, the first of which is a generalization of Theorem 1.2, showing that the theorem is more about graphs with independent sets of size at least half the number of vertices than about König-Egerváry graphs.

**Theorem 1.3.** Let \( G \) be a graph (not necessarily a tree or a König-Egerváry graph) with \( n \) vertices and maximum independent set size \( \alpha \). The sequence \( (i_k)_{k=\ell}^{\alpha} \) is weakly decreasing, where

\[
\ell = \left\lfloor \frac{\alpha(n-1)}{\alpha+n} \right\rfloor.
\]

If \( \kappa \) satisfies \( \alpha \geq \kappa n \) then

\[
\ell \leq \left\lfloor \frac{\alpha}{1+\kappa} - \frac{\kappa}{1+\kappa} \right\rfloor.
\]

(See Section 2.1 for the proof). The second part of Theorem 1.3 follows quickly from the first: if \( \alpha \geq \kappa n \) then

\[
\frac{\alpha(n-1)}{\alpha+n} \leq \frac{\alpha(n-1)}{(1+\kappa)n} \leq \frac{\alpha}{1+\kappa} - \frac{\alpha}{(1+\kappa)n} \leq \frac{\alpha}{1+\kappa} - \frac{\kappa}{1+\kappa}.
\]

Every \( n \)-vertex graph satisfies \( \mu \leq n/2 \), so every König-Egerváry graph satisfies \( \alpha \geq n/2 \) (the converse of this is not true; e.g. \( K_3 \) together with two isolated vertices has \( 3 = \alpha \geq 5/2 = n/2 \) but is not König-Egerváry). Thus, taking \( \kappa = 1/2 \) in (1) we recover Theorem 1.2.

Theorem 1.3 gives no new information on Question 1.1, the status of the independent set sequence of all trees, because there are trees with \( \alpha = \lceil n/2 \rceil \). But for trees with
If \( \alpha \) is larger than \( n/2 \), it gives a decreasing tail longer than one-third of the length of the sequence.

One obvious place to exploit this is in the study of the independent set sequence of the random uniform tree. Our model here is to select \( T \) uniformly from among the \( n^{n-2} \) labelled trees on vertex set \( \{1, \ldots, n\} \), and to consider the sequence \((X_0, X_1, \ldots, X_n)\) where \( X_k \) is the number of independent sets of size \( k \) in \( T \). To gain some evidence in favor of an affirmative answer to Question 1.1, it is natural to ask whether \((X_0, X_1, \ldots, X_n)\) is a.a.s. (asymptotically almost surely — with probability tending to 1 as \( n \) tends to infinity) unimodal.

This seemingly simple question turns out to be quite intricate. It is easy, via the Matrix Tree Theorem, to establish

\[
E(X_k) = \binom{n}{k} (1 - \frac{k}{n})^{n-1}
\]

(this was probably first observed by Bedrosian [4]), so that the sequence \((E(X_0), E(X_1), \ldots, E(X_n))\) is unimodal. One might then try to establish that with high probability the \( X_k \)'s fall in disjoint intervals centered around the \( E(X_k) \)'s, leading to a.a.s. unimodality. Unfortunately the variance of \( X_k \) (which can also be explicitly calculated via the matrix tree theorem) turns out to be very large, typically much larger than \( E(X_k)^2 \), precluding a straightforward application of the second moment method.

Nonetheless, Theorem 1.3 allows us to say something about the decreasing tail of the independent set sequence of almost all trees, beyond what is given by Theorem 1.2. Pittel [24], tightening an earlier result of Meir and Moon [22] established that for any \( f(n) = \omega(1) \), a.a.s.

\[
\alpha(T) \in (\rho n - f(n)\sqrt{n}, \rho n + f(n)\sqrt{n})
\]

where \( \alpha(T) \) is the size of the largest independent set in \( T \), and \( \rho \approx 0.5671 \) is the unique real solution to \( \rho e^\rho = 1 \). This leads us to conclude that a.a.s. the (non-zero part of the) independent set of the uniform labelled tree is decreasing for its last about 36%, or from about \( 0.37n \) on. Here we improve this.

**Theorem 1.4.** Let \( T \) be a uniformly random labelled tree on \( n \) vertices, and let \( X_k \) be the number of independent sets of size \( k \) in \( T \). A.a.s. the sequence \((X_\ell, X_1, \ldots, X_n)\) is weakly decreasing, where \( \ell = 0.34668n \).

So a.a.s. the (non-zero part of the) independent set of the uniform labelled tree is decreasing for its terminal 38.8%. See Section 2.3 for the proof of Theorem 1.4. With some further computation it is likely that we could improve to \( \ell = 0.34667n \), but not to \( 0.34666n \).

Our second observation around Question 1.1 concerns the start of the independent set sequence. Again, we begin with a general statement:

**Theorem 1.5.** Let \( G \) be a graph in which every maximal (by inclusion) independent set has size at least \( \lambda \). Then the initial portion \((i_0, i_1, \ldots, i_{\lfloor \lambda/2 \rfloor})\) of the independent set sequence of \( G \) is non-decreasing.

This is a straightforward generalization (see Section 2.2 for the short proof) of a result of Michael and Traves [23], who showed that if every independent set in \( G \) is contained in an independent set of size \( \alpha \) (\( G \) is well-covered) then \( i_0 \leq i_1 \leq \cdots \leq i_{\lfloor \alpha/2 \rfloor} \).
To connect this to the independent set sequence of a tree, we show:

**Theorem 1.6.** Let $T$ be a tree with $n$ vertices and maximum independent set size $\alpha$. Every maximal independent set in $T$ has size at least $\left\lceil \frac{n - \alpha + 1}{2} \right\rceil$, and so the initial portion $(i_0, i_1, \ldots, i_\ell)$ of the independent set sequence of $T$ is non-decreasing, where

$$\ell = \left\lceil \frac{n - \alpha + 1}{2} \right\rceil = \left\lceil \frac{n - \alpha + 1}{4} \right\rceil.$$

(See Section 2.2 for the proof.) For example, if we know that $\alpha = \left\lceil \frac{n}{2} \right\rceil$ (its smallest possible value) then we get that the independent set sequence is increasing up to about $n/8$ or $0.25\alpha$. On the other hand, if we know that $\alpha = n - 1$ (its largest possible value) then we get no information from Theorem 1.6.

Recalling (2), from Theorem 1.6 we can immediately say that a.a.s. the (non-zero part of the) independent set sequence of the uniform labelled tree is increasing for its initial about 19%, or up to about $0.1n$. By modifying the idea that goes into the proof of Theorem 1.5, we can improve this substantially.

**Theorem 1.7.** Let $T$ be a uniformly random labelled tree on $n$ vertices, and let $X_k$ be the number of independent sets of size $k$ in $T$. A.a.s. the sequence $(X_0, X_1, \ldots, X_\ell)$ is weakly increasing, where $\ell = 0.28096n$.

So a.a.s. the (non-zero part of the) independent set of the uniform labelled tree is increasing for its initial 49.5%. See Section 2.3 for the proof of Theorem 1.7. With some further computation it is likely that we could improve to $\ell = 0.28098n$, but not to $0.28099n$. Using quite different methods Heilman [16] has recently shown that the independent set sequence of $T$ is a.a.s. weakly increasing up to $0.26543n$, or for the initial about 46% of its non-zero part.

We end the introduction with a few further remarks about generalizations of Question 1.1. Recall that there is a sequence of ever-stronger (first is implied by second, et cetera, but no reverse implications) conditions on a sequence $(a_0, \ldots, a_m)$ of positive terms:

- **Unimodality:** $a_0 \leq a_1 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_m$.
- **Log-concavity:** $a_k^2 \geq a_{k-1}a_{k+1}$ for $k = 1, \ldots, m - 1$.
- **Ordered log-concavity:**

$$a_k^2 \geq \left(1 + \frac{1}{k}\right)a_{k-1}a_{k+1}$$

for $k = 1, \ldots, m - 1$. (We say “ordered” because ordered log-concavity corresponds to the sequence $(k!a_k)_k^n$ being log-concave, and when $a_k$ counts objects each consisting of $k$ unordered elements, $k!a_k$ counts the same objects when also an order is put on the elements).
• **Ultra log-concavity:**

\[
a_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{m-k}\right) a_{k-1}a_{k+1}
\]

for \(k = 1, \ldots, m - 1\) (corresponding to the sequence \((a_k/\binom{m}{k})^m\) \(_{k=0}\), or equivalently \((k!(m-k)!a_k)^m\) \(_{k=0}\) being log-concave).

• **Real roots:** \(\sum_{k=0}^m a_kx^k\) has all real roots.

Chudnovsky and Seymour [9] showed that the independent set sequence of a claw-free graph satisfies not just unimodality but the real roots property; on the other hand, the independent set sequence of trees does not in general satisfy ultra-log concavity, as witnessed by the star on four vertices. It is plausible, however that there is an affirmative answer to the following question:

**Question 1.8.** Is the independent set sequence of every tree ordered log-concave?

Radcliffe [25] has verified that every tree on up to 25 vertices has ordered log-concave independent set sequence.

One reason to think about ordered log-concavity is that it has a very nice reformulation. For a graph \(G\) with maximum independent set size \(\alpha\), let \(\mathcal{I}\) and \(\mathcal{I}_k\) be the set of independent sets of \(G\), and the set of independent sets of size \(k\), respectively. For \(I \in \mathcal{I}\), denote by \(e(I)\) the number of extensions of \(I\) to an independent set of size \(|I| + 1\) (or: \(e(I)\) is the number of vertices in \(G\) that are neither in \(I\) nor adjacent to anything in \(I\)). Denote by \(e_k\) the average number of extensions of an independent set of size \(k\), that is

\[
e_k = \frac{\sum_{I \in \mathcal{I}_k} e(I)}{i_k}.
\]

**Claim 1.9.** The sequence \((i_k)_{k=0}^\alpha\) is ordered log-concave if and only if the sequence \((e_k)_{k=0}^{\alpha-1}\) is weakly decreasing.

(See Section 2.2 for the quick proof.) So Question 1.8 is equivalent to:

**Question 1.10.** For every tree, is the sequence \((e_k)_{k=0}^{\alpha-1}\) weakly decreasing?

Before turning to proofs of Theorems 1.3, 1.4, 1.5, 1.6 and 1.7 and Claim 1.9 (in Section 2), we make a remark concerning the difference between Question 1.1 for trees versus forests. If \(G\) has components \(G_1, \ldots, G_k\), and component \(G_\ell\) has independent set sequence \(i^\ell = (i^\ell_1, i^\ell_2, \ldots)\), then the independent set sequence of \(G\) is the convolution of the sequences \(i^\ell\) — that is, it is the coefficient sequence of the polynomial \(\prod_{\ell=1}^k \sum_{j \geq 0} i^\ell_j x^j\). It is not in general the case that the convolution of unimodal sequences is unimodal, which means that Question 1.1 for trees is distinct from Question 1.1 for forests. On the other hand, it is the case that the convolution of log-concave sequences is log-concave [10], which means that to establish the log-concavity of the independent set sequence of an arbitrary forest, it is sufficient to do so for an arbitrary tree. We do not at the moment know whether the convolution of ordered log-concave sequences is ordered log-concave.
2 Proofs

2.1 Proof of Theorem 1.3

The proof follows from two old results. First, a theorem of Fisher and Ryan [11]:

**Theorem 2.1.** For any graph $G$ with maximum independent set size $\alpha$, we have

\[
\left( \frac{i_1}{\binom{\alpha}{1}} \right)^{\frac{1}{1}} \geq \left( \frac{i_2}{\binom{\alpha}{2}} \right)^{\frac{1}{2}} \geq \left( \frac{i_3}{\binom{\alpha}{3}} \right)^{\frac{1}{3}} \geq \cdots \geq \left( \frac{i_{\alpha-1}}{\binom{\alpha}{\alpha-1}} \right)^{\frac{1}{\alpha-1}} \geq \left( \frac{i_\alpha}{\binom{\alpha}{\alpha}} \right)^{\frac{1}{\alpha}}.
\]

Second, a theorem of Zykov [33]:

**Theorem 2.2.** For any graph $G$ with $n$ vertices and with maximum independent set size $\alpha$, and any $0 \leq k \leq \alpha$, we have

\[
i_k \leq \binom{\alpha}{k} \left( \frac{n}{\alpha} \right)^k.
\]

(This is a corollary of a more general result that among all graphs on $n$ vertices with maximum independent set size $\alpha$, the one which maximizes the number of independent sets of size $k$ for each $0 \leq k \leq \alpha$ is the balanced union of $\alpha$ cliques.)

**Proof (of Theorem 1.3):** From Theorem 2.1 we see that for each $k \leq \alpha - 1$ we have

\[
\left( \frac{i_k}{\binom{\alpha}{k}} \right)^{\frac{1}{k}} \geq \left( \frac{i_{k+1}}{\binom{\alpha}{k+1}} \right)^{\frac{1}{k+1}},
\]

so that if $i_{k+1} > i_k$ then

\[
\left( \frac{i_{k+1}}{\binom{\alpha}{k+1}} \right)^{\frac{1}{k+1}} > \left( \frac{i_k}{\binom{\alpha}{k}} \right)^{\frac{1}{k}},
\]

or

\[
i_{k+1} > \binom{\alpha}{k+1} \left( \frac{\alpha}{k+1} \right)^{\frac{1}{k+1}} = \binom{\alpha}{k+1} \left( \frac{k+1}{\alpha - k} \right)^{\frac{k+1}{k+1}}.
\]

Now Theorem 2.2 says

\[
i_{k+1} \leq \binom{\alpha}{k+1} \left( \frac{n}{\alpha} \right)^{k+1}
\]

from which we deduce

\[
\frac{n}{\alpha} > \frac{k+1}{\alpha - k}.
\]

In summary, $i_{k+1} > i_k$ forces $k < (\alpha n - \alpha)/(\alpha + n)$, which implies Theorem 1.3. □
2.2 Proofs of Theorems 1.5 and 1.6, and of Claim 1.9

The proofs of Theorems 1.4, 1.5 and 1.7, and of Claim 1.9, all have an element in common, which we introduce now.

Given a graph $G$ with maximum independent size $\alpha$, for $0 \leq j \leq \alpha - 1$ denote by $B_j$ the bipartite graph with classes $I_j$ (the set of independent sets of size $j$ in $G$) and $I_{j+1}$, with an edge joining $I \in I_j$, $J \in I_{j+1}$ if and only if $I \subseteq J$.

$B_j$ has $(j + 1)i_{j+1}$ edges, since each independent set of size $j + 1$ is an extension of exactly $j + 1$ independent sets of size $j$. It also has $\sum_{I \in I_j} e(I)$ edges, where as before $e(I)$ is the number of extensions of $I$ to an independent set of size $|I| + 1$. So we have the identity

$$\sum_{I \in I_j} e(I) = (j + 1)i_{j+1}$$

for $j = 0, \ldots, \alpha - 1$.

**Proof (of Theorem 1.5):** For $k \leq \lambda$, each $I \in I_{k-1}$ has $e(I) \geq \lambda - (k - 1)$, since each such $I$ is in at least one independent set of size $\lambda$. From (3) it follows that $(\lambda - (k - 1))i_{k-1} \leq ki_k$, so that if $k \leq \lceil \lambda/2 \rceil$ then $i_{k-1} \leq i_k$.

**Proof (of Theorem 1.6):** Let $K$ be a maximal independent set in $T$, of size $|K|$.

Each of the $n - |K|$ vertices of $T - K$ must have at least one edge to $K$, so the subgraph induced by $T - K$ is a forest with $n - |K|$ vertices and at most $|K| - 1$ edges, and so at least $n - 2|K| + 1$ components. It follows that $T - K$, and hence $T$, has an independent set of size at least $n - 2|K| + 1$. The result follows from $n - 2|K| + 1 \leq \alpha$.

**Proof (of Claim 1.9):** From (3) we have

$$e_j = \frac{(j + 1)i_{j+1}}{i_j},$$

so that monotonicity of $(e_k)_{k=0}^{\alpha - 1}$ is equivalent to

$$\frac{i_1}{i_0} \geq \frac{2i_2}{i_1} \geq \cdots \geq \frac{ki_k}{i_{k-1}} \geq \frac{(k + 1)i_{k+1}}{i_k} \geq \cdots \geq \frac{\alpha i_\alpha}{i_{\alpha-1}},$$

which is in turn equivalent to ordered log-concavity of $(i_k)_{k=0}^{\alpha}$.

2.3 Proofs of Theorems 1.4 and 1.7

Theorem 1.5 hinged on the observation that if every independent set of size $k$ has more than $k$ extensions to an independent set of size $k + 1$, then $i_k \leq i_{k+1}$. For Theorem 1.7 we modify this to: if all but a vanishing proportion of independent sets of size $k$ have more than $k$ extensions to an independent set of size $k + 1$, then $i_k \leq i_{k+1}$. Theorem 1.4 depends on a similar statement, that if all but a vanishing proportion of independent sets of size $k$ have fewer than $k$ extensions to an independent set of size $k + 1$, then $i_k \geq i_{k+1}$. In Section 2.3.1 below we make these ideas precise. In Section 2.3.2 we do the necessary analysis on the inequalities presented in Section 2.3.1. In Section 2.3.3 we use the Matrix Tree Theorem to establish a key identity used throughout.
2.3.1 Key claim

Let $e(n,k,t)$ denote the probability that, in a uniformly chosen labelled tree on $[n] := \{1, \ldots, n\}$, a particular set of size $k$ is independent and has exactly $t$ extensions to an independent set of size $k + 1$. In Section 2.3.3 we use the Matrix Tree Theorem (and inclusion-exclusion) to establish

$$e(n,k,t) = \left( \frac{n-k}{n} \right)^{n-k-t} \sum_{\ell=0}^{t} (-1)^{\ell} \left( \frac{n-k-t}{\ell} \right) \left( 1 - \frac{k}{n} \right)^{t+\ell-1} \left( 1 - \frac{(k+t+\ell)}{n} \right)^{k}. \quad (4)$$

Let $g_1(n,k)$ denote the expected number of independent sets of size $k$ that have no more than $k + 1$ extensions to an independent set of size $k + 1$, and let $g_2(n,k)$ denote the expected number of independent sets of size $k$ that have $k + 1$ or more extensions to an independent set of size $k + 1$. By linearity of expectation we have

$$g_1(n,k) = \left( \frac{n}{k} \right)^{k+1} \sum_{t=0}^{k} e(n,k,t) \quad \text{and} \quad g_2(n,k) = \left( \frac{n}{k} \right)^{n-k} \sum_{t=k+1}^{n-k} e(n,k,t).$$

**Claim 2.3.** Suppose that $n$ and $k$ with $k + 2 \leq n$ satisfy

$$g_1(n,k) \leq \left( \frac{n-k+1}{k} \right)^{n}. \quad (5)$$

Then all but a proportion $1/(n \log n)$ of trees on $[n]$ satisfy $i_k \leq i_{k+1}$. And if

$$g_2(n,k) \leq \left( \frac{n-k}{k+1} \right)^{n-k} \frac{1}{n \log n}, \quad (6)$$

then all but a proportion $1/(n \log n)$ of trees on $[n]$ satisfy $i_{k+1} \leq i_k$.

**Proof:** By Markov’s inequality, under (5) all but a proportion at most $1/(n \log n)$ of trees on $[n]$ have no more than $(n-k+1)/n$ independent sets of size $k$ with no more than $k + 1$ extensions to an independent set of size $k + 1$. In what follows we work inside in this set $T_1$ of trees.

As in the proofs of Theorem 1.5 and Claim 1.9, for $T \in T_1$ consider the bipartite graph $B_k$ with classes $\mathcal{I}_k$ (the set of independent sets of size $k$ in $T$) and $\mathcal{I}_{k+1}$, with an edge joining $I \in \mathcal{I}_k$, $J \in \mathcal{I}_{k+1}$ if and only if $I \subseteq J$. Recalling (3) we have

$$\sum_{I \in \mathcal{I}_k} e(I) = (k+1)i_{k+1} \quad (7)$$

where $e(I)$ denotes the number of extensions of $I$ to an independent set of size $k + 1$.

Now lower bounding $e(I)$ by 0 if $I$ has no more than $k+1$ extensions to an independent set of size $k + 1$, and by $k + 2$ otherwise, we get

$$\sum_{I \in \mathcal{I}_k} e(I) \geq (k+2) \left( i_k - \left( \frac{n-k+1}{k} \right) \right).$$
Inserting into (7) and using \( k + 2 \leq n \) yields
\[
(k + 1)(i_{k+1} - i_k) \geq i_k - (k + 2) \left( \frac{n-k+1}{n} \right)
\]
\[
\geq i_k - \binom{n-k+1}{k}.
\]

That \( i_k \geq \binom{n-k+1}{k} \) (completing the proof of the first part of the claim) follows from the fact that the number of independent sets of size \( k \) in any tree is not smaller than the number of independent sets of size \( k \) in the path on \( n \) vertices, which is \( \binom{n-k+1}{k} \). (This fact was possibly first observed by Wingard [28, Theorem 5.1]).

The proof of the second part of the claim is similar. By Markov, under (6) all but a proportion at most \( 1/(n \log n) \) of trees on \([n]\) have no more than \( \binom{n-k}{k+1}/n \) independent sets of size \( k \) with \( k + 1 \) or more extensions to an independent set of size \( k + 1 \). In what follows we work inside in this set \( \mathcal{T}_2 \) of trees.

For \( T \in \mathcal{T}_2 \) we again consider the bipartite graph \( B_k \). Upper bounding \( e(I) \) by \( n \) if \( I \) has \( k + 1 \) or more extensions to an independent set of size \( k + 1 \), and by \( k \) otherwise, we get
\[
(k + 1)i_{k+1} = \sum_{I \in \mathcal{I}_k} e(I) \leq ki_k + \binom{n-k}{k+1}
\]
or
\[
k(i_{k+1} - i_k) \leq \binom{n-k}{k+1} - i_{k+1} \leq 0,
\]
the last inequality following from Wingard’s bound applied to independent sets of size \( k + 1 \).

\[ \square \]

2.3.2 Analysis

To complete the proofs of Theorems 1.4 and 1.7, it remains to verify (4) (which we do in Section 2.3.3), and to show that for all sufficiently large \( n \), for all \( k \geq 0.34668n \) (6) holds (so that, by a union bound, all but a proportion at most \( 1/\log n \) of trees on \([n]\) satisfy \( i_k \geq i_{k+1} \) for all \( k \geq 0.34668n \)), while for all \( k \leq 0.28096n \) (5) holds. This section furnishes that verification.

It will be convenient to introduce \( f(n, k, t) \), the expected number of independent sets
of size $k$ that have exactly $t$ extensions to an independent set of size $k + 1$; we have

$$f(n, k, t) = \binom{n}{k} e(n, k, t)$$

$$= \binom{n}{k} \binom{n-k}{t} \times \sum_{\ell=0}^{n-k-t} (-1)^\ell \binom{n-k-t}{\ell} \left(1 - \frac{k}{n}\right)^{t+\ell-1} \left(1 - \frac{(k+t+\ell)}{n}\right)^k$$

$$= a(n, k, t) \sum_{\ell=0}^{n-k-t} \binom{n-k-t}{\ell} \left(\frac{k}{n} - 1\right)^\ell \left(1 - \frac{\ell}{n} \right)^k$$  \hspace{0.5cm} (8)

where

$$a(n, k, t) = \binom{n}{k} \binom{n-k}{t} \left(1 - \frac{k}{n}\right)^{t-1} \left(1 - \frac{(k+t)}{n}\right)^k.$$

The sum in (8) is of the form

$$\sum_{\ell=0}^N \binom{N}{\ell} (x - 1)^\ell \left(1 - \frac{\ell}{N}\right)^k = \frac{1}{N^k} \sum_{\ell=0}^N \binom{N}{\ell} \left(\frac{1}{x-1}\right)^\ell \ell^k$$

$$= \frac{1}{N^k} \sum_{j=1}^k \binom{k}{j} \binom{N}{j} x^{N-j}$$

where in the second line we use symmetry of the binomial coefficients and in the last line we use the identity

$$\sum_{\ell=0}^N \binom{N}{\ell} \ell^k z^k = \sum_{j=0}^k \binom{k}{j} \binom{N}{j} (1 + z)^{N-j} z^j$$

(see, e.g. [6, Proposition 2.5]) with $z = 1/(x - 1)$. Here $\binom{a}{b}$ is a Stirling number of the second kind and $(a)_b$ is a falling power. With $N = n - k - t$ and $x = k/n$ this yields

$$f(n, k, t) = \frac{a(n, k, t)(k/n)^{n-k-t}}{(n-k-t)^k} \sum_{j=1}^k \binom{k}{j} (n-k-t)_j \left(\frac{n}{k}\right)^j.$$  \hspace{0.5cm} (9)

We first give a heuristic analysis of the right-hand side of (9). Setting $\kappa = k/n$ and $\tau = t/n$, and ignoring polynomial factors of $n$ in the approximations below, we have

$$\frac{a(n, k, t)(k/n)^{n-k-t}}{(n-k-t)^k} \approx \left(\frac{2^{\left(H(\kappa) + (1-\kappa)H(\frac{\tau}{\kappa})\right)}(1-\kappa)^\tau (1-\kappa-\tau)^{1-\kappa-\tau}}{(1-\kappa-\tau)^{\kappa n^\kappa}}\right)^n.$$

(Here we use $\binom{n}{k} \approx 2^{nH(k/n)}$, where $H$ is the binary entropy function.)
To estimate the sum in (9) we start with the identity
\[
\sum_{m \geq 0} \binom{m}{i} \frac{z^m}{m!} = \frac{(e^z - 1)^i}{i!},
\]
(10)
from which we deduce
\[
\sum_{j=1}^{k} \binom{k}{j} (n - k - t)_j \left( \frac{n}{k} \right)^j = k! \left[ z^k \right] \left( 1 + \left( \frac{n}{k} \right) (e^z - 1) \right)^{n-k-t},
\]
(11)
where \( \left[ z^k \right] \) is the operation that extracts the coefficient of \( z^k \) from a power series in variable \( z \).

We now appeal to a result of Good [14, Theorem 6.1] (see also [13, Theorem 2]) concerning the asymptotics of a coefficient of a high power of \( z \) in the power series expansion of a high power of a power series. Suppose that \( f(z) = \sum_{k=0}^{\infty} f_k z^k \) is a power series with positive coefficients and with infinite radius of convergence. Suppose that \( N = N(r) \) (\( r \) a natural number) is such that \( N/r \) is bounded away from 0 and from infinity as \( r \to \infty \). Then the implicit equation
\[
\rho f'(\rho) = \frac{N}{r}
\]
defines a unique positive real \( \rho = \rho(r) \), and
\[
\left[ z^N \right] (f(z))^r = \frac{f(\rho)^r}{\sigma \rho^N \sqrt{2\pi r}} (1 + o_r(1))
\]
as \( r \to \infty \), where \( \sigma = \sigma(t) > 0 \) is defined by \( \sigma^2 = \rho^2 \left( \frac{f''(\rho)}{f(\rho)} - \frac{f'(\rho)^2}{f(\rho)^2} + \frac{f'(\rho)}{f(\rho)} \right) \). As observed in [13], \( \rho f'(\rho)/f(\rho) \) and \( \sigma \) have probabilistic interpretations: \( \rho f'(\rho)/f(\rho) \) is the expectation of the probability distribution \( X \), supported on the natural numbers, given by \( P(X = k) \propto f_k \rho^k \), while \( \sigma^2/\rho^2 \) is the variance of \( X \).

Taking \( f(z) = 1 + (1/\kappa)(e^z - 1) \), defining \( \rho \) implicitly via \( \rho f'(\rho)/f(\rho) = \kappa/(1 - \kappa - \tau) \), and using \( k! \approx (k/e)^k \) we get from (11) that
\[
\sum_{j=1}^{k} \binom{k}{j} (n - k - t)_j \left( \frac{n}{k} \right)^j \approx \left( \frac{n^\kappa f(\rho)^{1-\kappa-\tau}}{e^\kappa \rho^\kappa} \right)^n.
\]
It follows that \( f(n, k, t) \approx C(\kappa, \tau)^n \) where
\[
C(\kappa, \tau) = \frac{2^{H(\kappa)+(1-\kappa)H(\tau)}}{e^\kappa \rho^\kappa},
\]
so that (using \( \binom{n-k-1}{k} \approx 2^{(1-\kappa)H(\kappa/(1-\kappa))} \)) we get that (5) holds as long as
\[
\sup_{\tau \in [0, \kappa]} \frac{C(\kappa, \tau)}{2^{(1-\kappa)H(\kappa/(1-\kappa))}} < 1
\]
(12)
and (6) holds as long as
\[ \sup_{\tau \in [\kappa,1]} \frac{C(\kappa, \tau)}{2^{(1-\kappa)H(\kappa/(1-\kappa))}} < 1. \] (13)

We can computationally verify that (12) holds for \( \kappa \leq 0.28098 \) (but not for \( \kappa = 0.28099 \)), and that (13) holds for \( \kappa \geq 0.34667 \) (but not for \( \kappa = 0.34666 \)), heuristically justifying the comments after the statements of Theorems 1.4 and 1.7.

To make the analysis rigorous, we break \([0,n]\) into finitely many equal intervals, and for each \( k \) and \( t \) we upper bound the various terms that comprise \( f(n,k,t) \) (and lower bound \( \binom{n-k+1}{k}, \binom{n-k}{k} \)) in terms of the upper and lower endpoints of the intervals in which \( k \) and \( t \) lie. This reduces the computation to a finite one.

We start with the analysis for Theorem 1.7, by analyzing (9) for \( k \leq 0.28096n \) and \( t \leq k + 1 \) (the range of values relevant for that theorem). Note that in proving Theorem 1.7 we may assume \( k \geq \frac{n}{10} \) (since we already know, from Theorem 1.6, that a.a.s. the independent set sequence of the random tree is increasing up to \( 0.108n \)). Similarly for Theorem 1.4 we may assume \( k \leq 0.362n \).

Let \( M \) be some large positive integer. Let \( n, k \) and \( t \) be given, with \( 0.1n \leq k \leq 0.28096n \) and \( 0 \leq t \leq k + 1 \). Let \( 1 \leq p \leq M \) be that integer such that \( (p-1)n/M \leq k < pn/M \) and let \( 1 \leq q \leq M \) be that integer such that \( (q-1)n/M \leq t < qn/M \). We have the following straightforward bounds:

- \( \binom{n}{k} \leq \exp_2 \left\{ nH \left( \frac{k}{n} \right) \right\} \leq \exp_2 \left\{ nH \left( \frac{p}{M} \right) \right\} := A(p,M)^n \), where \( H \) is the binary entropy function;
- \( \binom{n-k}{t} \leq \exp_2 \left\{ (n-k)H \left( \frac{t}{n-k} \right) \right\} \leq \exp_2 \left\{ \left( 1 - \frac{(p-1)}{M} \right) nH \left( \frac{q}{M-p} \right) \right\} := B(p,q,M)^n \);
- \( (1 - \frac{k}{n})^{t-1} \leq \left( \frac{M}{M-(p-1)} \right)^{\frac{(q-1)n}{M}} := C(p,q,M)^n \);
- \( \left( 1 - \frac{(k+t)}{n} \right)^k \leq \left( 1 - \frac{(p+q-2)}{M} \right)^{\frac{(p-1)n}{M}} := D(p,q,M)^n \);
- \( \left( \frac{k}{n} \right)^{n-k-t} \leq \left( \frac{p}{M} \right)^{n(1-\frac{(p+q)}{M})} := E(p,q,M)^n \); and
- \( (n-k-t)^k \geq n^k \left( 1 - \frac{(p+q)}{M} \right)^{\frac{kn}{M}} := n^k F(p,q,M)^n \).
Now we deal with the sum in (9). Using (10) we have
\[
\sum_{j=1}^{k} \left\{ \binom{k}{j} (n - k - t)_j \left( \frac{n}{k} \right)^j \right\} \leq \sum_{j=1}^{k} \left\{ \binom{k}{j} (n - k - t)_j \left( \frac{M}{p-1} \right)^j \right\} = k! \left[ z^k \right] \left( 1 + \left( \frac{M}{p-1} \right) (e^z - 1) \right)^{n-k-t} \leq \left( \frac{k}{e} \right)^k \left[ z^k \right] \left( 1 + \left( \frac{M}{p-1} \right) (e^z - 1) \right)^{n-k-t} \leq \frac{k^k}{e^{(p-1)n}} \left[ z^k \right] \left( 1 + \left( \frac{M}{p-1} \right) (e^z - 1) \right)^{n-k-t}.
\]

We now use [14, Theorem 6.1] (described earlier), with \( N = k \) and \( r = n - k - t \). For the \( n, k, t \) we are considering we have \( n - k - t \to \infty \), and we have that \( k/(n - k - t) \) is confined to the constant interval
\[
\left( \frac{p-1}{M-p-q+2}, \frac{p}{M-p-q} \right).
\]
(14)

So defining \( \rho = \rho(n, k, t) \) implicitly by
\[
\frac{k}{n-k-t} = \frac{\left( \frac{M}{p-1} \right) \rho^e}{1 + \left( \frac{M}{p-1} \right) (e^\rho - 1)} := s_{M,p}(\rho)
\]
we have
\[
\left[ z^k \right] \left( 1 + \left( \frac{M}{p-1} \right) (e^z - 1) \right)^{n-k-t} = \frac{1 + \left( \frac{M}{p-1} \right) (e^\rho - 1)}{\sigma \rho^k \sqrt{2\pi(n-k-t)}} \left( 1 + o_{n-k-t}(1) \right) .
\]

Evidently \( 1 + (M/(p-1)) (e^\rho - 1) \geq 1 \). Also, since \( M/(p-1) > 1 \) we have \( \rho < s_{M,p}(\rho) = \frac{k}{n-k-t} < 1 \)
(the first inequality is simple algebra, and the second follows from the specific bounds on \( k \) and \( t \)). It follows that for every \( \varepsilon > 0 \) and \( n \) sufficiently large, we can bound
\[
\left[ z^k \right] \left( 1 + \left( \frac{M}{p-1} \right) (e^z - 1) \right)^{n-k-t} \leq \left( \frac{1 + \left( \frac{M}{p-1} \right) (e^{\rho_{\max}} - 1)}{(\rho_{\min})^{\frac{M}{p-1}}} + \varepsilon \right)^n := (G(p, q, M) + \varepsilon)^n \tag{15}
\]
where \( \rho_{\max} \) and \( \rho_{\min} \) are the maximum and minimum values of \( \rho \) on the interval in (14). For \( \rho > 0 \), the function \( s_{M,p}(\rho) \) is increasing on the interval \((0, \infty)\). It follows that \( \rho_{\min} \) is
achieved at the beginning of the interval in (14), and \( \rho_{\text{max}} \) at the end, and so in particular \( \rho_{\text{min}} \) and \( \rho_{\text{max}} \) are functions only of \( p, q \) and \( M \), and not of \( n, k \) and/or \( t \).

Note that in (15) we implicitly assumed that \( \sigma \) is bounded below by a positive constant depending only on \( p, q \) and \( M \). This can be verified directly, but also follows from the fact that, as observed earlier, \( \sigma \) is the variance of a distribution that is not almost surely constant. We may lower bound \( \sigma \) by the minimum value it attains on the interval in (14). (This minimum is not 0, since for the range of values of \( k \) and \( t \) under consideration, the interval in (14) is bounded away from 0.)

For any fixed \( \varepsilon > 0 \) (and \( M \)) we can choose \( n \) large enough that (15) holds for all valid choices of \( p \) and \( q \), since there are only finitely many such choices (depending on \( M \)). So for all large enough \( n \), and for all \( 0 < \frac{1}{n} \leq k \leq \frac{1}{n} + 1 \) we have

\[
f(n, k, t) \leq \left( \frac{ABCDE(G + \varepsilon)H}{F} \right)^n
\]

where

\[
H(p, M) := \left( \frac{p-1}{eM} \right)^{p-1}.\]

We can now upper bound \( g_1(n, k) \): for any particular \( k \), there is an associated \( p \), and

\[
g_1(n, k) = \sum_{t=0}^{k+1} f(n, k, t) \leq n \left( \max \left\{ \frac{ABCDE(G + \varepsilon)H}{F} : 0 \leq q \leq p + 1 \right\} \right)^n
\]

(taking \( q \) up to \( q = p + 1 \) is necessary since \( t \) ranges up to \( k + 1 \)).

On the other hand, for all large enough \( n \)

\[
\binom{n-k+1}{k} \geq \exp_2 \left\{ \frac{(n-k+1)H \left( \frac{k}{n-k+1} \right)}{n} \right\}
\]

\[
\geq \exp_2 \left\{ \frac{n \left( 1 - \frac{p}{M} \right) H \left( \frac{p-1}{M-2} \right)}{n} \right\} := I(p, M)^n
\]

If we can exhibit an \( M \) such that

\[
\max \left\{ \frac{ABCDEGH}{F} : 0 \leq q \leq p + 1 \right\} \leq \frac{1}{I}
\]

for all \( M/10 \leq p \leq 0.28096M \), then, by choosing \( \varepsilon \) sufficiently small, we obtain (5) for all sufficiently large \( n \), completing the proof of Theorem 1.7 (that the quantity on the left-hand side of (16) is strictly less than 1 allows us to absorb the various terms in (5) that grow at most polynomially with \( n \)).

To make the computation manageable, we proceed in stages. We can begin, for example, by showing that with \( M = 100 \), (16) holds for \( 10 \leq p \leq 23 \), yielding that the independent set sequence of the uniform labelled tree on \([n]\) is a.a.s. increasing up to \( 0.23n \). So from here on we may restrict attention to \( p \geq 0.23M \). With \( M = 1000 \), (16) holds for \( 230 \leq p \leq 274 \), allowing us in the sequel to restrict to \( p \geq 0.274M \).
Bootstrapping in this way, we eventually get to $M = 400000$, at which value (16) holds for $p \leq 112384$, yielding the bound claimed in Theorem 1.7. All computations were performed on Mathematica.

The analysis in the proof of Theorem 1.4 is almost identical, with just a few changes needed in the bounds.

- In $A(p, M)$: we replace $p$ with $p - 1$ (because now $p/M, (p - 1)/M \geq 1/2$).

- In $B(p, q, M)$: inside the entropy term, we replace $q$ with $q - 1$ and $p$ with $p - 1$ (since the argument $t/(n - k)$ is bigger than $1/2$, we need to make it smaller to get an upper bound).

- All of the analysis that goes into determining $G(p, q, M)$ goes through unchanged, except that in the range of $k, t$ under consideration, we no longer have $\rho_{\min} \leq 1$ (for all $k$ in the relevant range, $\rho_{\min}$ starts out below 1, but eventually flips to being greater than 1). Rather than attempting to determine exactly where the flip happens, we just replace $\rho_{\min}^{p/M}$ in the denominator of $G$ with

$$\min\{\rho_{\min}^{p/M}, \rho_{\min}^{(p-1)/M}\}$$

(so if $\rho_{\min}$ happens to be smaller than 1, this will pick out the larger power to get the upper bound, while if $\rho_{\min}$ happens to be bigger than 1, this will pick out the smaller power).

- In $I(p, M)$: since the argument in the entropy function is now greater than $1/2$, we replace it with $(p + 1)/(M - p)$.

We omit the further details of the analysis.

### 2.3.3 Deriving (4)

Here we use the Matrix Tree Theorem to find an explicit expression for $e(n, k, t)$, the probability that, in a uniformly chosen labelled tree on $[n]$, a particular set of size $k$ is independent and has exactly $t$ extensions to an independent set of size $k + 1$.

Given two disjoint subsets $K, L$ of $[n]$ with $|K| = k \geq 1$ and $|L| = \ell$, denote by $T_{K,L}$ the set of trees on $[n]$ with $K$ an independent set and with $L$ having no edges to $K$.

**Claim 2.4.**

$$|T_{K,L}| = n^{n-2} \left[ \left(1 - \frac{k}{n}\right)^{\ell-1} \left(1 - \frac{(k + \ell)}{n}\right)^k \right].$$

**Proof:** $T_{K,L}$ is exactly the set of spanning trees of the graph $G(K, L)$ obtained from $K[n]$ by deleting all the edges inside $K$, as well as all edges from $L$ to $K$. The Laplacian of $G(K, L)$, with the rows and columns indexed first by vertices in $K$, then $L$, then the rest of the vertices (call this set $M$), is a block matrix.
• The block with rows indexed by $K$, columns indexed by $K$, has 0’s off the diagonal, and $n - k - \ell$’s down the diagonal.

• The block with rows indexed by $K$, columns indexed by $L$, is all 0.

• The block with rows indexed by $K$, columns indexed by $M$, is all $-1$.

• The block with rows indexed by $L$, columns indexed by $L$, has $-1$’s off the diagonal, and $n - k - 1$’s down the diagonal.

• The block with rows indexed by $L$, columns indexed by $M$, is all $-1$.

• The block with rows indexed by $M$, columns indexed by $M$, has $-1$’s off the diagonal, and $n - 1$’s down the diagonal.

(No other blocks need be specified — the matrix is symmetric). This matrix has
• 0 as an eigenvalue with geometric multiplicity at least 1 (all row sums are 0);

• $n - k - \ell$ as an eigenvalue with geometric multiplicity at least $k$ (on subtracting $n - k - \ell$ from each diagonal entry, the first $k$ rows become identical, and the sum of the rows indexed by $L$ is a multiple of this common value);

• $n - k$ as an eigenvalue with geometric multiplicity at least $\ell - 1$ (on subtracting $n - k$ from each diagonal entry, the $\ell$ rows indexed by $L$ become identical); and

• $n$ as an eigenvalue with geometric multiplicity at least $n - k - \ell$ (on subtracting $n$ from each diagonal entry, the $n - k - \ell - 1$ rows indexed by $M$ become identical, and the sum of the remaining rows is a multiple of this common value).

Since $1 + k + (\ell - 1) + (n - k - \ell) = n$ it follows that these lower bounds on geometric multiplicities are equalities, and that the algebraic multiplicities of all the eigenvalues coincide with their geometric multiplicities. So, from the Matrix Tree Theorem we get

$$|T_{K,L}| = n^{n-k-\ell-1}(n-k)^{\ell-1}(n-k-\ell)^k$$

$$= n^{n-2} \left[ \left(1 - \frac{k}{n}\right)^{\ell-1} \left(1 - \frac{(k+\ell)}{n}\right)^k \right].$$

Now let $\emptyset \neq K \subseteq [n]$ be given, as well as $T \subseteq [n] \setminus K$ ($T$ might be empty). Set $|K| = k$ and $|T| = t$.

Claim 2.5. The number of trees on $[n]$ with $K$ as an independent set, and with $T$ as the exact set of vertices that extend $K$ to an independent set of size $k + 1$, is

$$n^{n-2} \sum_{\ell=0}^{n-k-t} (-1)^\ell \binom{n-k-t}{\ell} \left(1 - \frac{k}{n}\right)^{\ell+t-1} \left(1 - \frac{(k+t+\ell)}{n}\right)^k.$$
**Proof:** Let $U_{K,T}$ be the set of trees with $K$ as an independent set and with $T$ among the set of vertices that extend $K$ to an independent set of size $k + 1$; we know from Claim 2.4 that

$$|U_{K,T}| = n^{n-2} \left(1 - \frac{k}{n}\right)^{t-1} \left(1 - \frac{(k+t)}{n}\right)^k.$$ 

Let the vertices of $[n] \setminus (K \cup T)$ be $v_1, \ldots, v_{n-k-t}$. Let $A_j$ be the set of trees in $U_{K,T}$ in which there is no edge from $v_j$ to $K$. Then the number of trees on $[n]$ with $K$ as an independent set, and with $T$ as the exact set of vertices that extend $K$ to an independent set of size $k + 1$, is

$$U_{K,T} \setminus (A_1 \cup \cdots \cup A_{n-k-t}).$$

If $L$ is any subset of $\{1, \ldots, n - k - t\}$ then $\cap_{i \in L} A_i$ is exactly the set of trees with $K$ independent, and with $T \cup L$ among the set of vertices that extend $K$ to an independent set of size $k + 1$, so by Claim 2.4 we have

$$|\cap_{i \in L} A_i| = n^{n-2} \left(1 - \frac{k}{n}\right)^{t+\ell-1} \left(1 - \frac{(k+t+\ell)}{n}\right)^k.$$ 

So, by inclusion-exclusion, the number of trees on $[n]$ with $K$ as an independent set, and with $T$ as the exact set of vertices that extend $K$ to an independent set of size $k + 1$, is

$$n^{n-2} \left(1 - \frac{k}{n}\right)^{t-1} \left(1 - \frac{(k+t)}{n}\right)^k$$

$$-n^{n-2} \sum_{\ell=1}^{n-k-t} (-1)^{\ell-1} {n-k-t \choose \ell} \left(1 - \frac{k}{n}\right)^{t+\ell-1} \left(1 - \frac{(k+t+\ell)}{n}\right)^k$$

or more compactly

$$n^{n-2} \sum_{\ell=0}^{n-k-t} (-1)^{\ell} \left(1 - \frac{k}{n}\right)^{t+\ell-1} \left(1 - \frac{(k+t+\ell)}{n}\right)^k.$$ 

The claimed expression (4) for $e(n, k, t)$ (the probability that in a uniformly chosen labelled tree on $[n]$ a given set of size $k$ is independent and has exactly $t$ extensions to an independent set of size $k + 1$) follows from Claim 2.5 by first summing over all possible choices for $T$ (the set of extensions) and then using Cayley’s formula.

**References**


