Ultrafilters, with applications to analysis, social choice and combinatorics

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Abstract

We define the notion of an ultrafilter on a set, and present three applications. The first is an alternative presentation of the Banach limit of a bounded sequence. The second is a proof of Arrow's Theorem, with an amusing corollary for infinite societies. The third is Glazer's startling proof of Hindman's Theorem from Ramsey Theory.

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1 Introduction

What constitutes a "large" subset of a set? That is, if $\mathcal{F}$ is the collection large subsets of a set X, what properties might we expect $\mathcal{F}$ to satisfy?

Here are two quite natural properties:

- $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$, and
- If $F \in \mathcal{F}$ and $F' \supseteq F$ then $F' \in \mathcal{F}$.

Other properties might be open to debate. Two possibilities are

- There cannot be $F \in \mathcal{F}$ and $G \in \mathcal{F}$ with $F \cap G = \emptyset$, and
- If $F \in \mathcal{F}$ and $F' \notin \mathcal{F}$ for some $F' \subseteq F$, then $F \setminus F' \in \mathcal{F}$.

This last property proposes a "robustness": a large set cannot become non-large by the removal of a non-large set. A consequence is that if $F, G \in \mathcal{F}$ then so is $F \cap G$. For if $F \cap G \notin \mathcal{F}$ then both $F \setminus (F \cap G), G \setminus (F \cap G) \in \mathcal{F}$; but these sets are disjoint.

The notion of an *ultrafilter*, introduced by Riesz [11] in a talk at the 4th ICM, captures exactly the sense of largeness suggested by these four properties. In the next two sections we formally define and derive some properties, basic and otherwise, of ultrafilters. In the last three sections we present applications of ultrafilters to analysis, voting and combinatorics.

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2 Filters and Ultrafilters

Definition 2.1 *A* filter *on a set X is a set of subsets $\mathcal{F} \subseteq 2^X$ satisfying*

1. *$X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$*
2. *If $A \in \mathcal{F}$ and $B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$*
3. *If $A \in \mathcal{F}$ and $B \supseteq A$ then $B \in \mathcal{F}$.*

An ultrafilter *is a maximal filter.*

More generally, filters may be defined on any partially ordered set. There is a slight confusion of terminology here; many combinatorics textbooks define a filter to be an $\mathcal{F} \subseteq 2^X$ satisfying only the third of our conditions above.

Example 2.2 *For $\emptyset \neq S \subseteq X$, the set $\mathcal{F}(S) = \{A : S \subseteq A\}$ is a filter. It is not an ultrafilter unless $S = \{x\}$ is a singleton. In this case we write $\mathcal{F}(x)$ for $\mathcal{F}(S)$; $\mathcal{F}(x)$ is called the* principle ultrafilter *on x.*

Our first result is that every filter extends to an ultrafilter. This requires the axiom of choice and one definition.

Definition 2.3 *A non-empty family $\mathcal{I} \subseteq 2^X$ of subsets of X has the* finite intersection property *(FIP) if the intersection of any finite collection of elements from $\mathcal{I}$ is non-empty.*

For example, a filter has the FIP.

Lemma 2.4 *If $\mathcal{F}$ is a non-empty family of subsets of X with the FIP then there is an ultrafilter $\mathcal{F}' \supseteq \mathcal{F}$. In particular, every filter is contained in an ultrafilter.*

Proof: Let $\mathcal{X}$ be the collection of subsets of 2^X that contain $\mathcal{F}$ and have the FIP. $\mathcal{X}$ is a non-empty poset (partially ordered by inclusion). The union of the elements of a chain in this poset is also in the poset (for any finite collection of sets from the union, there must be some element of the chain that contains all of them, so they have non-empty intersection). So every chain has an upper bound and by Zorn's lemma there is a maximal element $\mathcal{F}'$, which we claim is an ultrafilter. We have $X \in \mathcal{F}'$ since if not we could add it, contradicting maximality of $\mathcal{F}'$, and since $\mathcal{F}'$ has the FIP, $\emptyset \notin \mathcal{F}'$. For any $A \in \mathcal{F}'$ and $B \supseteq A$ we must have $B \in \mathcal{F}'$, for if not we could add B without damaging the FIP, a contradiction since $\mathcal{F}'$ is maximal. For the same reason, for $A, B \in \mathcal{F}'$ we have $A \cap B \in \mathcal{F}'$. So $\mathcal{F}'$ is an filter. It is maximal (as a filter) since if we add any set we get something which does not have the FIP and so cannot be a filter. □

Remark 2.5 *We could have presented a simpler argument: if $\mathcal{X}$ is the set of filters that contain $\mathcal{F}$, partially ordered by inclusion, then $\mathcal{X}$ is non-empty and every chain in $\mathcal{X}$ has an upper bound (the union of an increasing sequence of filters is a filter) and so by Zorn's lemma there is a maximal filter containing $\mathcal{F}$. We give the more involved argument as it will be helpful when we want to derive some further properties of ultrafilters.*

Corollary 2.6 *Let $\mathcal{F}$ and $\mathcal{G}$ be ultrafilters on a set X.*

1. *If B is such that $A \cap B \neq \emptyset$ for all $A \in \mathcal{F}$ then $B \in \mathcal{F}$.*

2. *If A and B are such that $A \cup B \in \mathcal{F}$ then at least one of $A, B \in \mathcal{F}$.*

3. *If $\mathcal{F} \neq \mathcal{G}$ then there are $A \in \mathcal{F}$, $B \in \mathcal{G}$ with $A \cap B = \emptyset$.*

Proof: For the first statement, observe that $\mathcal{B} = \{A \cap B : A \in \mathcal{F}\}$ is non-empty and has the FIP, so extends to an ultrafilter $\mathcal{F}'$. We have $B \in \mathcal{B}$ so $B \in \mathcal{F}'$. But also $\mathcal{F} \subseteq \mathcal{F}'$ (for each $A \in \mathcal{F}$ we have $A \cap B \in \mathcal{B}$, so $A \cap B \in \mathcal{F}'$, so $A \in \mathcal{F}'$) and so $\mathcal{F} = \mathcal{F}'$ and $B \in \mathcal{F}$.

For the second statement, if we have both $A, B \notin \mathcal{F}$ then (by the first statement) there are $C, D \in \mathcal{F}$ with $A \cap C = \emptyset$ and $B \cap D = \emptyset$, so $(A \cup B) \cap (C \cap D) = \emptyset$, so $A \cup B \notin \mathcal{F}$ (since $C \cap D \in \mathcal{F}$).

For the last statement, there must be $B \in \mathcal{G}$ with $B \notin \mathcal{F}$ (else $\mathcal{G} \subseteq \mathcal{F}$) and so $A \cap B = \emptyset$ for some $A \in \mathcal{F}$ (by the first statement). □

By induction, we can extend the second statement above: if $\cup_i A_i \in \mathcal{F}$ (where the union is finite), then $A_i \in \mathcal{F}$ for some i. If the A_i are disjoint, then $A_i \in \mathcal{F}$ for exactly one i. Thus ultrafilters are "robust under partitioning".

Corollary 2.7 *If $\mathcal{F}$ is an ultrafilter and $A \in \mathcal{F}$, then whenever we write*

$$A = A_1 \cup \ldots \cup A_n$$

as a disjoint union of finitely many sets, exactly one of the A_i is in $\mathcal{F}$.

We now give some alternate characterizations of ultrafilters. We write A^c for $X \setminus A$.

Lemma 2.8 *A set $\mathcal{F} \subseteq 2^X$ is an ultrafilter if and only if it satisfies*

1. *$X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$*

2. *If $A \in \mathcal{F}$ and $B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$*

3. *For all $A \subseteq X$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.*

A filter $\mathcal{F}$ on X is an ultrafilter if and only if exactly one of $A, A^c \in \mathcal{F}$ for all $A \subseteq X$.

Proof: First characterization: if $\mathcal{F}$ is an ultrafilter then it is a filter and so satisfies the first two conditions. The last condition is a special case of the previous corollary.

Conversely, suppose $\mathcal{F}$ satisfies the given conditions, and pick $A \in \mathcal{F}$. If there is $B \supseteq A$ with $B \notin \mathcal{F}$, then $B^c \in \mathcal{F}$, a contradiction since then $A \cap B^c = \emptyset$. So $\mathcal{F}$ is a filter, and by the third condition it is maximal.

Second characterization: if a filter $\mathcal{F}$ is such that exactly one of $A, A^c \in \mathcal{F}$ for all A, then $\mathcal{F}$ satisfies all the conditions of the lemma above and so is an ultrafilter.

Conversely, if $\mathcal{F}$ is an ultrafilter then it is a filter, and exactly one of $A, A^c \in \mathcal{F}$ for all A by the lemma. □

Our third characterization captures the notion of an ultrafilter being the set of "large" subsets of a set, via the notion of a finitely-additive measure.

Definition 2.9 *A* 0-1 finitely-additive measure *on X is a* 0-1 *function $\mu : 2^X \to \{0,1\}$ that satisfies*

1. *$\mu(X) = 1$*
2. *If $A_1, \ldots, A_n$ are pairwise disjoint, then $\mu(\cup_i A_i) = \sum_i \mu(A_i)$.*

Lemma 2.10 *There is a bijection from the set of ultrafilters on X to the set of* 0-1 *finitely additive measures on X given by $\mathcal{F} \to \mu_{\mathcal{F}}$ where*

$$\mu_{\mathcal{F}}(A) = \begin{cases} 1 & \text{if } A \in \mathcal{F} \\ 0 & \text{if } A \notin \mathcal{F}. \end{cases}$$

Proof: We first verify that $\mu_{\mathcal{F}}$ is a finitely-additive measure. Certainly $\mu_{\mathcal{F}}(X) = 1$. For the second condition, it is enough to prove $\mu_{\mathcal{F}}(A \cup B) = \mu_{\mathcal{F}}(A) + \mu_{\mathcal{F}}(B)$ for disjoint A, B (the general statement follows by induction). If $A \in \mathcal{F}$ but $B \notin \mathcal{F}$, or vice versa, then both right and left hand side are 1. Since A and B are disjoint, we cannot have $A, B \in \mathcal{F}$. So there remains the case $A, B \notin \mathcal{F}$. In this case we have $A^c \cap B^c = (A \cup B)^c \in \mathcal{F}$ (since $\mathcal{F}$ is an ultrafilter) and so $A \cup B \notin \mathcal{F}$, and both right- and left-hand side are 0.

To see that the map is a bijection, note that the inverse is given by $\mu \to \mathcal{F}_\mu$ where

$$\mathcal{F}_\mu = \{A \subseteq X : \mu(A) = 1\}.$$

□

Remark 2.11 *The measure corresponding to an ultrafilter is not necessarily countably additive. We will see in a moment that when X is countably infinite there are ultrafilters on X that have no finite sets. For these ultrafilters, the corresponding measure μ satisfies $\mu\left(\cup_{x \in X}\{x\}\right) = \mu(X) = 1$ whereas $\sum_{x \in X} \mu\left(\{x\}\right) = 0$.*

We now distinguish between two very different types of ultrafilters: principle (trivial) and non-principle (highly non-trivial).

Definition 2.12 *An ultrafilter on X is called* non-principal *if it is not $\mathcal{F}(x)$ for some $x \in X$.*

Lemma 2.13 *Let $\mathcal{F}$ be an ultrafilter on X. The following are equivalent.*

1. *$\mathcal{F}$ is a non-principle ultrafilter*
2. *$\mathcal{F}$ has no finite sets.*
3. *$\mu_{\mathcal{F}}(A) = 0$ for finite A.*

Proof: The last two conditions are clearly equivalent. If $\mathcal{F}$ is an ultrafilter with no finite sets, then it must be non-principle. If there is $A \in \mathcal{F}$ with $|A| = 1$ then $\mathcal{F}$ is principle. If $|A| > 1$ and A is finite then we claim that there is $B \in \mathcal{F}$ with $|B| < |A|$, and so by repeating this observation we get that $\mathcal{F}$ is principle. To see the claim, note that if B is any proper subset of A then either $B, B^c \in \mathcal{F}$, and so also is either $A \cap B$ or $A \cap B^c = A \setminus B$. □

Are there any non-principle ultrafilters? Yes, in the presence of the Axiom of Choice.

Lemma 2.14 *Let X be infinite. Then there is a non-principle ultrafilter on X.*

Proof: For infinite X the collection of cofinite subsets (sets whose complement is finite) has the FIP and so is contained in a (clearly non-principle) ultrafilter. $\square$

3 The space $\beta\mathbb{N}$

We write βX for the set of all ultrafilters on X. If X is finite then $|X| = |\beta X|$ with a natural bijection: $x \to \mathcal{F}(x)$. If X is infinite, βX again contains a copy of X (the collection of principle ultrafilters $\{\mathcal{F}(x) : x \in X\}$), but is considerable richer. We'll just consider the simple case $X = \mathbb{N}$ (this is the one that is relevant for our discussion of Ramsey theory), although most of the discussion makes sense for more general spaces and in particular for discrete spaces.

We begin by putting a topological structure on $\beta\mathbb{N}$. For each $A \subseteq \mathbb{N}$ set

$$A^\star = \{\mathcal{F} \in \beta\mathbb{N} : A \in \mathcal{F}\}.$$

Lemma 3.1 *The set $\mathcal{B} = \{A^\star : A \subseteq \mathbb{N}\}$ is a basis for a compact Hausdorff topology on $\beta\mathbb{N}$.*

Proof: We first verify that $\mathcal{B}$ is a valid basis. Since each $\mathcal{F} \in \beta\mathbb{N}$ is non-empty, there is $A^\star \in \mathcal{B}$ with $\mathcal{F} \in A^\star$, so $\cup A^\star = \beta\mathbb{N}$. It remains to show that if $\mathcal{F} \in A^\star \cap B^\star$ then there is $C^\star$ with $\mathcal{F} \in C^\star$ and $C^\star \subseteq A^\star \cap B^\star$. We may take $C = A \cap B$, since in fact $(A \cap B)^\star = A^\star \cap B^\star$. For if $\mathcal{F}' \in (A \cap B)^\star$ then $A \cap B \in \mathcal{F}'$, so $A, B \in \mathcal{F}'$, so $\mathcal{F}' \in A^\star \cap B^\star$; if $\mathcal{F}' \in A^\star \cap B^\star$ then $A, B \in \mathcal{F}'$, so $A \cap B \in \mathcal{F}'$, so $\mathcal{F}' \in (A \cap B)^\star$.

For compactness, we will show that any collection of closed sets in $\beta\mathbb{N}$ with the FIP has a non-empty intersection. We begin by noting that each basic open set is also closed, since $A^\star = (A^c)^\star$. It follows that any closed set in $\beta\mathbb{N}$, being of the form $(\cup O_i)^c = \cap O_i^c$ for some collection of basis elements $\{O_i\}$, is an intersection of basis elements.

Let now $\{F_i\}$ be a collection of closed sets with the FIP. Each F_i consists of all ultrafilters that include all of a collection $\mathcal{F}_i$ of subsets of $\mathbb{N}$. The FIP is equivalent to the statement that for every finite collection of $\mathcal{F}_i$'s, there is an ultrafilter that extends $\cup_i \mathcal{F}_i$. Since this ultrafilter has the FIP, it follows that $\cup_i \mathcal{F}_i$ has the FIP and so $\cup \mathcal{F}_i$ (where the union is over the whole collection) has the FIP, and so there is an ultrafilter extending $\cup \mathcal{F}_i$ and $\cap F_i \neq \emptyset$.

To show that $\beta\mathbb{N}$ is Hausdorff we note that if $\mathcal{F} \neq \mathcal{G}$ then there is $A \in \mathcal{F}$, $B \in \mathcal{G}$ with $A \cap B = \emptyset$ and so $\mathcal{F} \in A^\star$, $\mathcal{G} \in B^\star$ and $A^\star \cap B^\star = \emptyset$. $\square$

Remark 3.2 *The topology we have put on $\beta\mathbb{N}$ is called the* Stone-Čech compactification *of $\mathbb{N}$. It is a compact extension of $\mathbb{N}$ (identified with the set of principle ultrafilters) in which $\mathbb{N}$ is dense. It is the unique compact Hausdorff space X extending $\mathbb{N}$ in which $\mathbb{N}$ is dense and for which every bounded real-valued function on $\mathbb{N}$ extends to a continuous function on X.*

We next put an algebraic structure on $\beta\mathbb{N}$. We define a binary operation $+ : \beta\mathbb{N} \times \beta\mathbb{N} \to 2^{2^{\mathbb{N}}}$ as follows. For $\mathcal{F}, \mathcal{G} \in \beta\mathbb{N}$,

$$\mathcal{F} + \mathcal{G} = \{A \subseteq \mathbb{N} : \{n \in \mathbb{N} : A - n \in \mathcal{G}\} \in \mathcal{F}\}$$

where $A - n = \{a - n : a \in A\}$. For example, consider two principle ultrafilters $\mathcal{F}(n_1)$ and $\mathcal{F}(n_2)$. We claim that $\mathcal{F}(n_1) + \mathcal{F}(n_2) = \mathcal{F}(n_1 + n_2)$. Indeed

$$\begin{aligned} A \in \mathcal{F}(n_1) + \mathcal{F}(n_2) &\iff \{n \in \mathbb{N} : A - n \in \mathcal{F}(n_2)\} \in \mathcal{F}(n_1) \\ &\iff n_1 \in \{n \in \mathbb{N} : A - n \in \mathcal{F}(n_2)\} \\ &\iff A - n_1 \in \mathcal{F}(n_2) \\ &\iff n_2 \in A - n_1 \\ &\iff n_2 + n_1 \in A \\ &\iff A \in \mathcal{F}(n_1 + n_2). \end{aligned}$$

Lemma 3.3 *For each $\mathcal{F}, \mathcal{G} \in \beta\mathbb{N}$, $\mathcal{F} + \mathcal{G} \in \beta\mathbb{N}$ (so $+$ is a binary operation on $\beta\mathbb{N}$). It is an associative operation.*

Proof: We have $\{n : X - n \in \mathcal{G}\} = \{n : X \in \mathcal{G}\} = X \in \mathcal{F}$ so $X \in \mathcal{F} + \mathcal{G}$, and similarly $\{n : \emptyset - n \in \mathcal{G}\} = \{n : \emptyset \in \mathcal{G}\} = \emptyset \notin \mathcal{F}$ so $\emptyset \notin \mathcal{F} + \mathcal{G}$.

Suppose $A, B \in \mathcal{F} + \mathcal{G}$. Then $A' := \{n : A - n \in \mathcal{G}\}, B' := \{n : B - n \in \mathcal{G}\} \in \mathcal{F}$ so $A' \cap B' \in \mathcal{F}$. We claim that $A' \cap B' \subseteq \{n : A \cap B - n \in \mathcal{G}\}$ (so that $A \cap B \in \mathcal{F} + \mathcal{G}$). Indeed,

$$\begin{aligned} m \in A' \cap B' &\Rightarrow A - m, B - m \in \mathcal{G} \\ &\Rightarrow A \cap B - m \in \mathcal{G} \qquad \text{(because } A - m \cap B - m = A \cap B - m) \\ &\Rightarrow m \in \{n : A \cap B - n \in \mathcal{G}\}. \end{aligned}$$

Suppose $A \notin \mathcal{F} + \mathcal{G}$. Then $\{n : A - n \in \mathcal{G}\} \notin \mathcal{F}$ and so $\{n : A - n \notin \mathcal{G}\} \in \mathcal{F}$. Since $(A - n)^c = A^c - n$ this is the same as $\{n : A^c - n \in \mathcal{G}\} \in \mathcal{F}$ and so $A^c \in \mathcal{F} + \mathcal{G}$.

This shows that $\mathcal{F} + \mathcal{G}$ is an ultrafilter. To see that $+$ is associative, note that

$$A \in \mathcal{F} + (\mathcal{G} + \mathcal{H}) \iff \{n : \{m : A - n - m \in \mathcal{H}\} \in \mathcal{G}\} \in \mathcal{F}$$

and

$$A \in (\mathcal{F} + \mathcal{G}) + \mathcal{H} \iff \{m : \{n : A - n \in \mathcal{H}\} - m \in \mathcal{G}\} \in \mathcal{F}$$

and that $\{n : A - n \in \mathcal{H}\} - m = \{n : A - n - m \in \mathcal{H}\}$; indeed

$$\begin{aligned} x \in \{n : A - n \in \mathcal{H}\} - m &\iff x + m \in \{n : A - n \in \mathcal{H}\} \\ &\iff A - x - m \in \mathcal{H} \\ &\iff x \in \{n : A - n - m \in \mathcal{H}\}. \end{aligned}$$

$\square$

Having put a topological and algebraic structure on $\beta\mathbb{N}$, we now connect the two.

Lemma 3.4 *For fixed $\mathcal{G}$, the map $+_{\mathcal{G}} : \beta\mathbb{N} \to \beta\mathbb{N}$ given by $+_{\mathcal{G}}(\mathcal{F}) = \mathcal{F} + \mathcal{G}$ is continuous.*

Proof: We show that the inverse image of a basic open set is open. Indeed,

$$\begin{aligned} +_{\mathcal{G}}^{-1}(A^\star) &= \{\mathcal{F} : \mathcal{F} + \mathcal{G} \in A^\star\} \\ &= \{\mathcal{F} : A \in \mathcal{F} + \mathcal{G}\} \\ &= \{\mathcal{F} : \{n : A - n \in \mathcal{G}\} \in \mathcal{F}\} \\ &= \{n : A - n \in \mathcal{G}\}^\star. \end{aligned}$$

□

The significance of continuity of addition is that it allows us to conclude that $\beta\mathbb{N}$ has an idempotent (an element $\mathcal{F}$ satisfying $\mathcal{F} + \mathcal{F} = \mathcal{F}$). The proof works for any compact semigroup with one-sided continuous addition.

Lemma 3.5 *(Idempotent lemma) There is $\mathcal{F} \in \beta\mathbb{N}$ with $\mathcal{F} + \mathcal{F} = \mathcal{F}$.*

Proof: Let $\mathcal{A}$ be the set of compact semigroups that are contained in $\beta\mathbb{N}$. Because $\beta\mathbb{N} \in \mathcal{A}$ it is non-empty. It is partially ordered by inclusion. Every chain $\mathcal{C}$ has $\cap_{C\in\mathcal{C}} C$ as a non-empty lower bound (it is non-empty and compact since all the C's are compact, and is easily seen to be a semigroup). By Zorn's lemma, there is a minimal compact semigroup A. We claim that any $\mathcal{F} \in A$ is idempotent.

We first observe that $A + \mathcal{F}$ is a compact (by left continuity of addition) semigroup (if $\mathcal{F}_1 + \mathcal{F}$ and $\mathcal{F}_2 + \mathcal{F}$ are elements of $A + \mathcal{F}$ then so is $(\mathcal{F}_1 + \mathcal{F}) + (\mathcal{F}_2 + \mathcal{F}) = (\mathcal{F}_1 + \mathcal{F} + \mathcal{F}_2) + \mathcal{G})$. Since $A + \mathcal{F} \subseteq A$, we have that $A + \mathcal{F} = A$ by minimality.

Now set $B = \{\mathcal{G} \in A : \mathcal{G} + \mathcal{F} = \mathcal{F}\}$. Because $A = A + \mathcal{F}$, B is non-empty. By continuity it is compact. It is also a semigroup: $\mathcal{G}_1 + \mathcal{F} = \mathcal{F}$ and $\mathcal{G}_2 + \mathcal{F} = \mathcal{F}$ imply $(\mathcal{G}_1 + \mathcal{G}_2) + \mathcal{F} = \mathcal{F}$. Since $B \subseteq A$, by minimality of A in fact $B = A$. So $\mathcal{F} \in B$ and $\mathcal{F} + \mathcal{F} = \mathcal{F}$. □

4 An application to analysis — the Banach limit of a sequence

Let $\{x_i\}$ be a bounded sequence of reals. We want to define a limit for this sequence, which we will denote $\lim^\star x_i$, satisfying three properties:

- Agreement (**A**): If $\lim x_i$ exists in the usual sense, then $\lim^\star x_i = \lim x_i$,
- Linearity (**L**): If $\{y_i\}$ is another bounded sequence and c_x, c_y are reals, then $\lim^\star(c_x x_i + c_y y_i) = c_x \lim^\star x_i + c_y \lim^\star y_i$, and
- Boundedness (**B**): If $|x_i| \leq A$ for all i then $|\lim^\star x_i| \leq A$.

In the presence of **A** and **L**, the boundedness condition is equivalent to the statement that for a sequence satisfying $x_i \geq 0$ for all i, $\lim^\star x_i \geq 0$.

The usual definition of limit of a bounded sequence $\{x_i\}$ of reals may be stated as follows:

$$\lim x_i = \ell \iff \forall \varepsilon > 0,\ |\mathbb{N} \setminus \{i : |x_i - \ell| < \varepsilon\}| < \infty$$

(the limit is ℓ if for all $\varepsilon > 0$ all but finitely many of the x_i are within ε of ℓ). Since every non-principle ultrafilter contains all cofinite sets (sets whose complements are finite), that suggests the following approach to defining $\lim^\star x_i$. Fix a non-principle ultrafilter $\mathcal{F}$. For each bounded $\{x_i\}$, $\ell \in \mathbb{R}$ and $\varepsilon > 0$, set

$$U_x(\ell, \varepsilon) = \{i : |x_i - \ell| < \varepsilon\}$$

(note that this depends on $\mathcal{F}$, but we suppress this in the notation).

Definition 4.1 *(Banach limit)*

$$\lim{}^\star x_i = \ell \Longleftrightarrow \forall \varepsilon > 0,\ U_x(\ell, \varepsilon) \in \mathcal{F}.$$

We must show that this is a good definition; that for every bounded sequence of reals there is a unique ℓ with $\lim^\star x_i = \ell$. Uniqueness is easy. For $\ell \neq \ell'$ and any $\varepsilon < (|\ell - \ell'|)/2$ the sets $U_x(\ell, \varepsilon)$ and $U_x(\ell', \varepsilon)$ are disjoint and so at most one of them can be in $\mathcal{F}$.

For existence, let $L = \sup |x_i|$ and set $I_0 = [-L, L]$. Divide I_0 into two equal disjoint intervals $I_0' = [-L, 0)$ and $I_0'' = [0, L]$. Exactly one of $\{i : x_i \in I_0'\}$, $\{i : x_i \in I_0'\}$ must belong to $\mathcal{F}$; let I_1 be the interval with that property. Inductively, we construct a nested sequence of intervals $I_0, I_1, \ldots$ with the properties that for each n the length of I_n is $2L/2^n$ and $\{i : x_i \in I_n\} \in \mathcal{F}$. The first of these properties implies that $\cap_n \bar{I}_n = \{\ell\}$ for some real ℓ, and the second implies that for all $\varepsilon > 0$ the set $U_x(\ell, \varepsilon)$ is in $\mathcal{F}$.

This shows that $\lim^\star x_i$ is well defined. That it satisfies **A** and **B** is clear. For **L**, suppose that $\lim^\star x_i = \ell_x$ and $\lim^\star y_i = \ell_y$ and fix $\varepsilon > 0$. The sets $U_x(\ell_x, \varepsilon/2c_x)$ and $U_y(\ell_y, \varepsilon/2c_y)$ are both in $\mathcal{F}$, and so their intersection is too. For i in the intersection we have $|x_i - \ell_x| < \varepsilon/2c_x$ and $|y_i - \ell_y| < \varepsilon/2c_y$ and so, by the triangle inequality,

$$|(c_x x_i + c_y y_i) - (c_x \ell_x + c_y \ell_y)| \leq \varepsilon.$$

It follows that $U_{c_x x + c_y y}(c_x \ell_x + c_y \ell_y, \varepsilon) \in \mathcal{F}$. This gives **L** assuming neither of $c_x, c_y = 0$; if either or both are 0 the argument is easily modified.

One issue with the definition of $\lim^\star x_i$ is that it is sensitive to translation (changes in indexing). For example, if $\mathcal{F}$ includes the set $\{1, 3, 5, \ldots\}$ and $\{x_i\} = \{0, 1, 0, 1, \ldots\}$, $\{y_i\} = \{1, 0, 1, 0 \ldots\}$ then $\lim^\star x_i = 0$ whereas $\lim^\star y_i = 1$. There is a fix for this: set

$$\lim{}^{\star\star} x_i = \lim{}^\star z_i$$

where $z_i = (x_1 + \ldots + x_i)/i$. One can check that $\lim^{\star\star} x_i$ satisfies all of **A**, **L** and **B**, as well as translation invariance: if $y_i = x_{i+1}$ then $\lim^{\star\star} x_i = \lim^{\star\star} y_i$.

Remark 4.2 *Our approach to defining a generalized limit is taken from [9], where it is suggested that it is folklore. The more standard approach is the one taken originally by Banach [2]. The set of bounded sequences of reals forms a normed vector space $\mathcal{B}$ over $\mathbb{R}$ with the usual addition and scalar multiplication, and the norm given by $||\{x_i\}|| = \sup_i |x_i|$. The set of convergent sequences forms a subspace $\mathcal{C}$. The map $f : \mathcal{C} \to \mathbb{R}$ that takes a* *convergent sequence to its limit is a linear function with operator norm* 1*. By the Hahn-Banach theorem this map extends to a linear map* $f' : \mathcal{C} \to \mathbb{R}$ *that also has operator norm* 1*. For* $\{x_i\} \in \mathcal{B}$*,* $f'(\{x_i\})$ *is referred to as the* Banach limit *of* $\{x_i\}$*. It satisfies the analog of* **A** *because it extends* f*. It satisfies the analog of* **L** *because it is linear, and it satisfies the analog of* **B** *because it has operator norm* 1*.*

5 An application to voting — Arrow's theorem

Let X be a set of voters (not necessarily finite) and let $C = \{c_1, \ldots, c_n\}$ be a finite set of candidates. Each $x \in X$ provides a permutation π_x of C (thought of as x's preference ranking of the candidates). A *social welfare function* (SWF) is a function $f : S_c^{|X|} \to S_C$ (where S_C is the set of permutations of C), which we think of as a way of aggregating the individual rankings into a societal ranking.

Here are two properties that we might except a "fair" SWF to satisfy.

- Unanimity (**U**): If all individuals present the same permutation, then f produces that permutation.
- Irrelevant alternatives (**IA**): The relative ranking of two candidates c_i, c_j in the output of f depends only on the relative rankings of c_i, c_j in each individual input.

Together these two properties imply

- Local unanimity (**LU**): If all individuals present a permutation in which c_i is ranked above c_j, then f produces a permutation in which c_i is ranked above c_j.

Indeed, if we rearrange all the input permutations so that they are identical, with c_i ranked first by everyone and c_j second, then by **U** f ranks c_i above c_j, and by **IA** this is the relative ranking of c_i and c_j originally.

We extend the idea of **U** in the following definition.

Definition 5.1 *A set* $F \subseteq X$ *is* decisive *if whenever all* $x \in F$ *present the same permutation,* f *outputs that permutation.*

Note that **U** is the assertion that X is decisive. Are there other decisive sets? The following, the main result of this section, answers this question. It is taken from [8], via [9].

Theorem 5.2 *Suppose* $n \geq 3$*. Let* f *be a SWF that satisfies* **U** *and* **IA***. Then* $\mathcal{F} = \{F \subseteq X\}$ *is an ultrafilter.*

Proof: Say that F is *block decisive* if whenever all $x \in F$ present the same permutation, and all $x \in F^c$ present the same permutation (possibly different from that presented by those $x \in \mathcal{F}$), then the permutation presented by those $x \in F$ is the outcome.

Lemma 5.3 *If* F *is block decisive, then it is decisive.*

Proof: Suppose F is block decisive but not decisive, and consider an input in which all $x \in F$ present a permutation π in which c_i is ranked above c_j, the permutations presented by all $x \in V_1$ have c_i above c_j, the permutations presented by all $x \in V_2$ have c_j above c_i (where (F, V_1, V_2) is a partition of X), and the output ranks c_j above c_i. Pick a c_k different from c_i, c_j. Modify the input permutations so that c_k is between c_i and c_j for all $x \in F$, and below c_i and c_j for all other x. By **IA**, for this new input the output still ranks c_j above c_i. All permutations in the new input ranks c_i above c_k, and so the output must too by **LU**). Now modify the input again, by having all $x \in V_1 \cup V_2$ present the same permutation in which c_j is ranked above c_k. By **IA**, the output for this new input ranks c_j above c_k, contradicting the block-decisiveness of F. □

Clearly, if F is decisive it is block-decisive, so we have $\mathcal{F} = \{\text{block decisive } F \subseteq X\}$. By **U** we have $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. We next show that if $F \notin \mathcal{F}$, then F^c is.

Lemma 5.4 *If F is not block decisive, then F^c is. Equivalently, if $|X| = 2$ there is a decisive element.*

Proof: We show that F^c is *ij-decisive* for all $i \neq j$: the relative ranking of c_i and c_j in the output agrees with the relative ranking in the permutation presented by all $x \in F^c$. To see this, we begin by noting that since $F \notin \mathcal{F}$ there is some pair $i \neq j$ and a pair of permutations π, τ with π ranking c_i above c_j and τ ranking c_j above c_i such that if all $x \in \mathcal{F}$ present π and all $x \in F^c$ present τ, the outcome ranks c_j above c_i. By **IA**, whenever all $x \in F$ rank c_i above c_j and all $x \in F^c$ rank c_j above c_i, the outcome ranks c_j above c_i. Now pick $k \neq i, j$. We present a list of seven inputs to the SWF, only mentioning the relative rankings of (some subset of) c_i, c_j and c_k for $x \in F$, $x \in F^c$ and for the output, with the output for the first input being the conclusion we have just drawn, and the other outputs being easily deduced from **LU** and **IA**. The interpretation of, for example, ijk, is that the particular permutation under consideration ranks c_i above c_j above c_k.

Column	1	2	3	4	5	6	7
$x \in F$	ij	ikj	ik	ijk	jk	jik	ji
$x \in F^c$	ji	kji	ki	kij	kj	ikj	ij
Outcome	ji	kji	ki	kij	kj	ikj	ij

The conclusion is that whenever all $x \in F^c$ rank c_i above c_j, so does the outcome. It follows that F^c is ij-decisive for this particular i, j. But, by column three above and the same argument, it is also ik-decisive, and by column five above and the same argument F^c is jk-decisive. Since k was arbitrary, we can pick any $\ell \neq k, i, j$ and repeat the argument to show that F^c is $k\ell$-decisive and so decisive. □

To complete the proof that $\mathcal{F}$ is an ultrafilter, we need to show that it is closed under taking intersections. We'll show that if F_1 and F_2 are any sets, then at least one of $(F_1 \cup F_2)^c, F_2 \setminus F_1, F_1 \setminus F_2$ and $F_1 \cap F_2$ are (block) decisive (and so exactly one, since we can't have disjoint block decisive sets). If both F_1 and F_2 are decisive, then the only possibility for a decisive set from among our list of four is $F_1 \cap F_2$, as required.

Lemma 5.5 *One of* $(F_1 \cup F_2)^c, F_2 \setminus F_1, F_1 \setminus F_2$ *and* $F_1 \cap F_2$ *is block decisive. Equivalently, if* $|X| \leq 4$ *then there is a decisive element.*

Proof: We have already dealt with $|X| = 2$. Now we deal with $X = \{x_1, x_2, x_3, x_4\}$ (the same argument will do for $|X| = 3$). If x_1 and x_2 present permutations as a block, and x_3 and x_4 also, then one of two is block decisive and so decisive; for definiteness say that it is x_1 and x_2. Now if we fix a permutation π to be presented by x_3 and x_4, we get an SWF (depending on π), in which one of x_1, x_2 is decisive; for definiteness say that it is x_1. We claim that x_1 is decisive in the original SWF. If not, there is $i \neq j$ and inputs in which x_1 ranks c_i above c_j and x_2 ranks c_j above c_i, as does the output (if x_2 ranks c_i above c_j too, then so does the output, by decisiveness of x_1 and x_2 as a block). If neither c_i nor c_j are at the bottom of π, then we can do the following. Writing c_k for the last element of π, modify x_1 so that c_k is between c_i and c_j; modify x_2 so that c_k is below c_i; and modify both x_3 and x_4 so that c_k is at the end of their rankings. By **LU** and **IA** the output has c_j above c_i above c_k. Now if we replace the inputs of x_3 and x_4 with π, the output (by **IA**) still ranks c_j above c_k even though x_1 ranks c_k above c_j, contradicting the decisiveness of x_1 when x_3 and x_4 present π. If one of c_i, c_j is at the bottom of π but the other is not at the top, the same argument can be repeated in a symmetric manner with "bottom" replaced by "top". If c_i, c_j occupy the top and the bottom of π, then we can use the "replacement" process described in the $|X| = 2$ argument to find an $\ell \neq i, j$ and inputs in which x_1 ranks c_ℓ above c_j and x_2 ranks c_j above c_ℓ, as does the output (just do the first two steps of the seven step process). Since it is not the case that c_ℓ, c_j occupy the top and the bottom of π, we can proceed with the above-described argument. □

This completes the verification that the set of decisive sets of voters forms an ultrafilter. □

What is the significance of this theorem? Well, a property that we would not expect a fair SWF to satisfy is that if have a *dictator*, or an individual x with the property that the outcome of f equals the input of x (and does not depend on the other inputs). A dictator is exactly a decisive set of size 1. But if X is finite, then the ultrafilter $\mathcal{F}$ of decisive sets must be of the form $\mathcal{F}(x)$ for some $x \in X$ (a principle ultrafilter), and so contains $\{x\}$. We have arrived at a celebrated result of economist Kenneth Arrow.

Theorem 5.6 *(Arrow's Theorem) If* $|X| < \infty$ *and* $n \geq 3$*, the only SWF's that satisfy* **U** *and* **IA** *have a dictator.* □

Arrow did not use ultrafilters in his original proof [1]; if he had, he would have been able to draw a comforting conclusion about voting in infinite societies. In the proof of our main result in this section we took an SWF satisfying **U** and **IA** and corresponded to it an ultrafilter on X. The correspondence goes the other way, too. Let $\mathcal{F}$ be an ultrafilter on X and define an SWF f by declaring the output to be that unique permutation π with the property that $\{x \in X : \pi_x = \pi\} \in \mathcal{F}$. Because $X \in \mathcal{F}$, this SWF satisfies **U**. Because $\mathcal{F}$ cannot contain a pair of disjoint sets, it satisfies **IA**. Indeed, fix two candidates c_i and c_j and two inputs to f, in both of which all $x \in A$ rank c_i above c_j and all $x \in A^c$ rank c_j above c_i. If the first input results in c_i being ranked above c_j, then we have $A \in \mathcal{F}$, and so, since we then can't have $A^c \in \mathcal{F}$, it must be the case that the first input results in c_i being ranked above c_j.

If the ultrafilter $\mathcal{F}$ is non-principle (and so necessarily $|X| = \infty$) then the corresponding SWF cannot be a dictatorship, since for every $x \in X$ there must be some $A \in \mathcal{F}$ with $x \notin A$ (consider $X \setminus \{x\}$: if this is not in $\mathcal{F}$ then x is and $\mathcal{F} = \mathcal{F}(x)$ is principle). We have shown the following.

Theorem 5.7 *(Arrow's Theorem') Fix $n \geq 3$. There is a one-to-one correspondence between ultrafilters on X and SWF's from $S_n^{|X|}$ to S_n that satisfy* **U** *and* **IA***. The non-dictatorship SWF's are those corresponding to non-principle ultrafilters. In particular, Arrow's Theorem is equivalent to the assertion that all ultrafilters on a finite set are principle.*

Remark 5.8 *The construction of an SWF that we have described picks out a "large" set of inputs that are identical (the notion of "large" being determined by an ultrafilter), and declares that to be the output. An obvious choice for a non-dictatorship SWF with $|X| < \infty$ is to take as output the most commonly occurring input. This clearly satisfies* **U***, but not* **IA***: consider the case where $X = \{x_1, \ldots, x_7\}$ and $C = \{c_1, c_2, c_3\}$. If each of x_1, x_2, x_3 input the ranking c_1 above c_2 above c_3, each of x_4, x_5 input the ranking c_2 above c_3 above c_1, and each of x_6, x_7 input the ranking c_3 above c_2 above c_1, then the output is c_1 above c_2 above c_3 and so in particular c_1 is above c_2. But if x_4, x_5 change their ranking to c_3 above c_2 above c_1 (not changing the relative ranking of c_2 and c_1) then the output switches to c_3 above c_2 above c_1 and so in particular c_2 is above c_1.*

6 An application to combinatorics — Ramsey theory

Because of the correspondence with measures, an element of an ultrafilter may be viewed as a "large" subset of X. Since Ramsey theory is concerned with finding classes in a partition which are "large" in the sense that they contain homogeneous substructures, it is not unreasonable that there is a connection. We begin with the classical proof of the classical Ramsey's Theorem [10].

Theorem 6.1 *(Ramsey's Theorem) Whenever the edges of an infinite complete graph are coloured with finitely many colours, there is an infinite complete monochromatic subgraph.*

Proof: Pick x_1 arbitrarily. At least one colour must leave x_1 infinitely often; choose one such, c_1. From the vertices that are joined to x_1 by an edge of colour c_1, choose x_2 arbitrarily. At least one colour must leave x_2 to an unchosen vertex infinitely often; choose one such, c_2. From the unchosen vertices that are joined to x_2 by an edge of colour c_2, choose x_3 arbitrarily. Repeating, we get a sequence of vertices $x_1, x_2, \ldots$, and a sequence of colours $c_1, c_2, \ldots$ with the property that the colour of $x_i x_j$ $(i < j)$ is c_i. At least one colour must occur infinitely often in the list of c_i's; choose one such, c say. The set $\{x_i : c_i = c\}$ is the vertex set of an infinite monochromatic graph. $\square$

We now re-proof Ramsey's theorem using ultrafilters. The proof (taken from [4]) is "constructive" in the sense that we will use a non-principle ultrafilter to find a colour which occurs "frequently", and find an infinite monochromatic subset of that colour.

Proof of Ramsey's theorem using ultrafilters: Fix a non-principle ultrafilter $\mathcal{F}$ on X, the (infinite) vertex set of the graph. Fix $x \in X$. Foe each colour i set

$$A_x^i = \{\{x, y\} \in X^{(2)} : \chi(\{x, y\}) = i\}$$

(where χ is the colouring of the edges). The sets A_x^i are disjoint and have $X \setminus \{x\}$ as their union, so the union is in $\mathcal{F}$ ($\mathcal{F}$ is non-principle). It follows that exactly there is exactly one i with A_x^i in $\mathcal{F}$. By the same reasoning there is exactly one i with the property that

$$B := \{x : A_x^i \in \mathcal{F}\} \in \mathcal{F}.$$

We find an infinite monochromatic subset of colour i by the following inductive construction. Choose $a_1 \in B$ arbitrarily. Having chosen $a_1, \ldots, a_n$ with the property that $\chi(\{a_s, a_t\}) = i$ for all $1 \leq s \neq t \leq n$, set

$$S = B \cap \cap_{s=1}^n \{y : \chi(\{a_s, y\}) = i\}.$$

S is a finite intersection of sets in $\mathcal{F}$ so is in $\mathcal{F}$. Choose $a_{n+1} \in S$ distinct from $\{a_1, \ldots, a_n\}$ (possible since $\mathcal{F}$ is non-principle and so S is infinite). $\square$

An observation of Hindman (presented in [4]) draws a rather strong connection between Ramsey Theory and ultrafilters.

Lemma 6.2 *Let $\mathcal{G}$ be a family of non-empty subsets of X. Then the following are equivalent.*

1. *Whenever X is finitely coloured, there is a monochromatic $G \in \mathcal{G}$.*

2. *There is an ultrafilter $\mathcal{F}$ on X with the property that for each $A \in \mathcal{F}$ there is a $G \in \mathcal{G}$ with $G \subseteq A$.*

Proof: Suppose we have such an ultrafilter. In any partition of X into finitely many pieces, there is one such piece, A say, in $\mathcal{F}$. Any $G \in \mathcal{G}$ with $G \subseteq A$ is monochromatic.

Suppose, on the other hand, that we have the Ramsey property. Set

$$\mathcal{B} = \{A \subseteq X : A \cap G \neq \emptyset \text{ for all } G \in \mathcal{G}\}.$$

and let $\mathcal{B}^+$ be the set of all finite intersections of elements of $\mathcal{B}$. We claim that $\mathcal{B}^+$ is an intersecting family. It is clear that if $A, B \in \mathcal{B}^+$ then $A \cap B \in \mathcal{B}^+$ and $X \in \mathcal{B}^+$, so it remains to show that $\emptyset \notin \mathcal{B}^+$. So, let $A_1, \ldots, A_k$ be elements of B. Partition X into 2^k pieces $\{C_S\}_{S \subset \{1, \ldots, k\}}$ via

$$x \in C_S \iff x \in \cap_{i \in S} A_i \cap \cap_{i \notin S} A_i^c.$$

By the Ramsey property, there's $S \subseteq \{1, \dots, k\}$ and $G \in \mathcal{G}$ with $G \subseteq \cap_{i \in S} A_i \cap \cap_{i \notin S} A_i^c$. But since $A_i \cap G \neq \emptyset$ for all i, it must be the case that $G \subseteq \cap_{i=1}^k A_i$ and so in particular $\cap_i A_i \neq \emptyset$.

Since $\mathcal{B}^+$ is intersecting there is an ultrafilter $\mathcal{F} \supseteq \mathcal{B}^+$ on X. Fix $A \in \mathcal{F}$. We have $A^c \notin \mathcal{F}$ and so $A^c \notin \mathcal{B}^+$ and $A^c \notin \mathcal{B}$. It follows that there is $G \in \mathcal{G}$ with $G \cap A^c = \emptyset$ and so $G \subseteq A$. □

Example 6.3 *The pigeon principle (whenever $\mathbb{N}$ is partitioned into finitely many pieces, one of the pieces is infinite) corresponds to the situation where $X = \mathbb{N}$ and $\mathcal{G} = \{A \subseteq \mathbb{N} : A \text{ infinite}\}$. Any non-principle ultrafilter works for the corresponding $\mathcal{F}$.*

We now present Schur's theorem [12], another gem of Ramsey theory, which can easily be deduced from Ramsey's theorem (and was first proved before Ramsey's theorem).

Theorem 6.4 *(Schur's Theorem) Whenever the natural numbers are partitioned into finitely many classes, it is possible to find x and y such that x, y and $x+y$ all belong to the same class (and so in particular it is possible to solve $x + y = z$ within a single partition class).*

Proof: Let χ be the colouring (partitioning) of $\mathbb{N}$. This induces a colouring on pairs of natural numbers: the colour of the pair xy is $\chi(|x-y|)$. By Ramsey's theorem there is an infinite set $\{x_1, \dots\}$ with the property that $\chi(|x_i - x_j|) = c$ for some c for all $i \neq j$. Set $x = x_3 - x_2$ and $y = x_2 - x_1$ (where without loss of generality we assume $x_3 > x_2 > x_1$). Then x, y and $x + y = x_3 - x_1$ are all coloured c. □

For an infinite set $A = \{x_1, \dots\} \subseteq \mathbb{N}$, write

$$FS(A) = \left\{ \sum_{i \in X} x_i : X \subseteq \mathbb{N},\ |X| < \infty \right\}$$

for the set of finite sums of A. The following result of Hindman [6] vastly generalizes Schur's Theorem. The original proof was combinatorial and quite involved. Not long after its publication, Glazer and Galvin (no relation) used ultrafilters to give a startlingly simple proof, not much more than a corollary of the result that $\beta\mathbb{N}$ has an idempotent element. The ultrafilter proof was first presented in [3]; for our presentation we have followed [4].

Theorem 6.5 *(Hindman's Theorem) Whenever the natural numbers are partitioned into finitely many classes, it is possible to find an infinite set A with the property that $FS(A)$ lies entirely inside one partition class.*

Proof: Fix a colouring χ. We'll inductively construct sequences $A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$ and (distinct) $a_1, a_2, \dots$ with the properties that $a_i \in A_{i-1}$, $A_i \in \mathcal{F}$ and $a_{i+1} + A_{i+1} \subseteq A_i$, and with χ constant on A_0. This will give the result; for consider any finite sum from among the a_i's, say

$$a_7 + a_4 + a_3.$$

We have $a_7 \in A_6 \subseteq A_5 \subseteq A_4$, so $a_7 + a_4 \subseteq A_3$, so $a_7 + a_4 + a_3 \subseteq A_2 \subseteq A_1 \subseteq A_0$.

Fix an idempotent ultrafilter $\mathcal{F} \in \beta\mathbb{N}$. There is a unique colour i with $A_0 := \{n \in \mathbb{N} : \chi(n) = i\}$. Now for any $B \subseteq \mathbb{N}$ set $B' = \{n : B - n \in \mathcal{F}\}$. If $B \in \mathcal{F}$ then, since $\mathcal{F}$ is idempotent, $B' \in \mathcal{F}$ and so also $B \cap B' \in \mathcal{F}$.

Select $a_1 \in A_0 \cap A_0'$ and set $A_1 = A_0 \cap (A_0 - a_1) - \{a_1\}$, so $A_1 \subseteq A_0$, $a_1 + A_1 \subseteq A_0$ and $A_1 \in \mathcal{F}$ (removing one element from a set in $\mathcal{F}$ does not take it out of $\mathcal{F}$, since $\mathcal{F}$ is non-principle). Having defined A_n, select $a_{n+1} \in A_n \cap A_n'$ and set $A_{n+1} = A_n \cap (A_n - a_{n+1}) - \{a_{n+1}\}$, so $A_{n+1} \subseteq A_n$, $a_{n+1} + A_{n+1} \subseteq A_n$ and $A_{n+1} \in \mathcal{F}$. $\square$

Remark 6.6 *An easy corollary of Hindman's theorem is that whenever the natural numbers are partitioned into finitely many classes, it is possible to find an infinite set A with the property that $FP(A)$ lies entirely inside one partition class, where*

$$FP(A) = \left\{ \prod_{i \in X} x_i : X \subseteq \mathbb{N},\ |X| < \infty \right\}$$

(the given partition of $\{2^n : n \in \mathbb{N}\}$ induces a partition of $\mathbb{N}$ in a natural way; apply the finite sums theorem to that partition). That raises a natural question: is it the case that whenever the natural numbers are partitioned into finitely many classes, it is possible to find an infinite set A with the property that $FS(A) \cup FP(A)$ lies entirely inside one partition class? This is an extremely hard problem, as it combines the additive and multiplicative structure of $\mathbb{N}$. Even the following seemingly simple case is open.

Conjecture 6.7 *Whenever the natural numbers are partitioned into finitely many classes, it is possible to find two numbers a and b such that $a, b, a + b$ and ab all lie in one partition class.*

This is only known (see [5]) when the number of classes is two! Even more astonishingly, the conjecture remains open if we ignore the class of a and b (see [7]).

Conjecture 6.8 *Whenever the natural numbers are partitioned into finitely many classes, it is possible to find two numbers a and b such that $a + b$ and ab both lie in one partition class.*

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