(5) Recall that

\[ \frac{\Theta(s)}{V_{in}(s)} = \frac{1}{6s^2 + s} = G(s). \]  

(1)

with proportional control, \( C(s) = k \)

\[ \frac{\Theta(s)}{\Theta_d(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{kG(s)}{1 + kG(s)} \]

\[ = \frac{\frac{k}{6s^2 + s}}{1 + \frac{k}{6s^2 + s}} = \frac{kG(s)}{1 + kG(s)} \]

where \( G(s) = \frac{1}{6s^2 + s} = \frac{1/6}{s^{1/6}} \)

Asymptote angles: [1 pt]

\( n_z = 0, \ n_p = 2 \)

\[ \Theta_n = \frac{180 + n360}{n_z - n_p}, \quad n = 0, 1, ..., n_p - n_z - 1 \]

\[ \Theta_0 = \frac{180}{2} = -90^\circ \]

\[ \Theta_1 = \frac{180 + 360}{-2} = \frac{540}{-2} = -270^\circ = 90^\circ \]

Asymptotes intersect the real axis at

\[ S_{int} = \frac{\sum_{i=1}^{n_z} z_i - \sum_{i=1}^{n_p} p_i}{n_z - n_p} = \frac{0 - (0 - 1/6)}{-2} = -\frac{1}{12} \]

On real axis, RL is to the left of an odd number of poles and zeros.
RL is in region from \( s \in (-\frac{1}{6}, 0) \).
RL breaks away from real axis at point, where

\[ \frac{dD(s)}{ds}N(s) - D(s)\frac{dN(s)}{ds} = 0 \]

\[ N(s) = 1, D(s) = s^2 + \frac{1}{6}s \]

\[ (2s + \frac{1}{6})1 - (s^2 + \frac{1}{6}s) \cdot 0 = 0 \]

Break-away point: \( s = -\frac{1}{12} \) [2 pts]

Poles with a damping ratio of 0.5 lie align a line 30° from the imaginary axis, from section 9.5.
RL and a line 30° from imaginary axis intersect at the point \( s = (-\frac{1}{12}, \frac{\sqrt{2}}{12}) \).

\[ k = \frac{1}{|G(s)|} \]
where

\[ |G(s)| = \frac{k \prod_{i=1}^{n_z} |s - z_i|}{\prod_{i=1}^{n_p} |s - p_i|} \]

\[ k = 1 \frac{\prod_{i=1}^{n_p} |s - p_i|}{\prod_{i=1}^{n_z} |s - z_i|} \]

\[ = 1 \frac{1/6 \cdot 1/6}{1} = \frac{1}{6} \]

at \( k = \frac{1}{6} \), the system has a damping ratio of \( \zeta = 0.5 \) [3 pts]

(6)

\[ V_{in} = k[(\Theta_d - \Theta) + \frac{1}{2}(s\Theta_d - s\Theta)] \]

\[ = k\frac{1}{2}(s + 2)(\Theta_d - \Theta) \]

\[ \frac{V_{in}(s)}{\Theta_d - \Theta} = k\frac{1}{2}(s + 2) = k(s) \]

\[ \Theta = \frac{k(s)p(s)}{1 + k(s)p(s)} = \frac{k\frac{s(s+2)}{(s+1/6)}(s+2)}{1 + k\frac{s(s+2)}{(s+1/6)}} \]

\[ G(s) = \frac{1}{12} \frac{s + 2}{s(s + 1/6)} \] [1pt]

\( n_z = 1, z_i = -2, n_p = 2, p_i = 0, -\frac{1}{6} \)

Derivative term adds a zero to the system.

\[ \Theta_0 = \frac{180}{-1} = -180 \]

\[ S_{int} = \frac{-2 - (0 - 1/6)}{-1} = -\frac{11}{6} \]

RL lies to the left of odd number of pole and zeros. \( s \in (-\infty, -2), (-\frac{1}{6}, 0) \) [1 pt]

Break-away, break-in points

\( N(s) = s + 2, D(s) = s^2 + \frac{1}{6}s \)

\[ (2s + 1/6)(s + 2) - (s^2 + 1/6)s)1 = 0 \]

\[ s^2 + 3s + \frac{1}{3} = 0 \]

\[ s = -2.88, -0.116 \] [1pt]

Initially the poles lie on the real axis, which means that there will be no oscillations in the system response. As \( K \) increases, the poles gain an imaginary component. The percent overshoot will increase, the setting time will decrease (since the poles move towards the left) and the rise time will decrease (since moving away from the origin). This will happen until the poles hit the top (and bottom) of the circle. After this point, the percent overshoot will decrease, with rise time and setting time still decreasing. Then with large enough \( K \) values, the poles will return to being strictly real. [2 pts]
Figure 1: Root locus Plot for Problem 9.12.5 and 9.12.6. [2 pts for each plot]

9.18 [7 pts] Consider

\[ G_p(s) = \frac{1}{(s-1)(s+1)}, \quad G_c(s) = \frac{s + 2}{s + 3}, \quad G_s(s) = 1 \]

in the block diagram illustrated in Figure 2. Sketch the root locus plot for the system and by referring to the plot, determine the range of \( k \) values for which the system is stable.

\[ G(s) = \frac{s + 2}{(s - 1)(s + 1)(s + 3)} \]

\( n_z = 1, \ z_i = -2, \ n_p = 3, \ p_i = 1, -1, -3 \)
2 poles go unbounded, 1 pole goes to the zero.

\[ \Theta_0 = \frac{180}{-2} = -90 \]
\[ \Theta_1 = \frac{180 + 360}{-2} = 90 \]
\[ s_{int} = \frac{-2 - (1 - 1 - 3)}{-2} = -\frac{1}{2} \quad [1pt] \]

break-away point:

\[ N(s) = s + 2 \]
\[ D(s) = (s - 1)(s + 1)(s + 3) = (s^2 - 1)(s + 3) = s^3 + 3s^2 - s - 3 \]
\[ \frac{dD(s)}{ds} N(s) - D(s) \frac{dN(s)}{ds} = (3s^2 + 6s - 1)(s + 2) - (s^3 + 3s^2 - s - 3) \cdot 1 \]
\[ = 3s^3 + 6s^2 - s + 6s^2 + 12s - 2 - (s^3 + 3s^2 - s - 3) = 0 \]
\[ 2s^3 + 4s^2 + 12s + 1 = 0 \]
\[ s = -0.0812 (\text{only valid root}), -2.2054 \pm 0.8619i \quad [2pts] \]

Open-loop pole at \( s = \pm 1 \) crosses into LHP at point \((0,0)\).

\[ k = \frac{1}{G(s)} = \frac{1}{k} \prod_{i=1}^{n_p} |s - p_i|, \quad \text{where} \quad s = 0 + 0i \]
\[ k = \frac{3 \cdot 1 \cdot 1}{2} = \frac{3}{2} \quad [2pts] \]

|s - p_i| and |s - z_i| values are the distances from the OL poles and zeros to the point \((0,0)\).

Root locus plot is shown in Fig. 3 [2 pts]

---

**Problem 3 [8 pts]**

Suppose you are given the plant:

\[ P(s) = \frac{1}{s^2 + (1 + \alpha)s + (1 - \alpha)} \]

where \( \alpha \) is a system parameter that is subject to variations; \( 0 < \alpha < \infty \). The plant is in a typical unity feedback loop and controlled by \( P \)-control where \( k_p = 1 \).

(1) Use the root-locus method to determine what variations in \( \alpha \) can be tolerated before instability occurs. Show your hand-drawn sketch and a verification by the Matlab function \texttt{rlocus} for full credit.

\[ \frac{Y}{R} = \frac{k(s)p(s)}{1 + k(s)p(s)} = \frac{\frac{1}{s^2 + (1 + \alpha)s + 1 - \alpha}}{1 + \frac{1}{s^2 + (1 + \alpha)s + 1 - \alpha}} = \frac{1}{s^2 + (1 + \alpha)s + 2 - \alpha} \]
Not in our standard form

\[ \frac{Y}{R} = \frac{kG(s)}{1+kG(s)} \]

Divide numerator and denominator by all terms that do not have the varied parameter \( \alpha \) in them.

\[ \frac{Y}{R} = \frac{1}{s^2 + (1+\alpha)s + (2-\alpha)} \cdot \frac{\frac{1}{s^2+1}}{\frac{1}{s^2+\alpha(s-1)}} = \frac{\frac{1}{s^2+1}}{\frac{1}{s^2+\alpha(s-1)}} \]

\[ G(s) = \frac{s-1}{s^2+s+2} \]

[2 pts]

\( n_z = 1, \ z_i = 1, \ n_p = 2, \ p_i = -\frac{1}{2} + \frac{1}{2}\sqrt{7}i \)

Asymptotes \( \Theta_0 = \frac{180}{7} = -180^\circ \)

pole at \( -\frac{1}{2} + \frac{1}{2}\sqrt{7}i \)

Departure angles [1 pt]

\[ \angle(s - p_i) = n_z \angle(p_j - z_i) - \sum_{i=1,i\neq j}^{n_p} \angle(p_j - z_i) - 180 = \tan^{-1}\left(\frac{\frac{1}{2}\sqrt{7}}{\frac{1}{2}}\right) - 90 - 180 = -131 \]

RL is symmetric about real axis, pole at \( -\frac{1}{2} - \frac{1}{2}\sqrt{7}i \Rightarrow \angle(s - p_j) = 130 \)

Break-in point \( N(s) = s - 1, \ D(s) = s^2 + 2 + 2. \)

\[ \frac{dD(s)}{ds}N(s) - D(s)\frac{dN(s)}{ds} = 0 \]

\[ (2s + 1)(s - 1) - (s^2 + s + 2)1 = 0 \]

\[ 2s^2 - 2s + s - 1 - s^2 - s - 2 = 0 \]

\[ s^2 - 2s - 3 = 0 \]

\[ (s - 3)(s + 1) = 0 \]

\[ s = -1 \text{ (only valid root)}, 3 \]

[1 pt]

Gain:

1 pole crosses into RHP at point \( (0,0) \).

\[ k = \sqrt{(1/2)^2 + (1/2 \cdot \sqrt{7})^2} \]

\[ = \sqrt{1/4 + 7/4} = \frac{\sqrt{1/4 + 7/4}}{1} = 2 \]

[1 pt]

Since an increasing \( k \) drives pole into the RHP, \( k \) should be less than 2.

(2) Verify your answer using the Routh Array for the closed-loop system.

\[ \frac{Y}{R} = \frac{1}{s^2 + (1+\alpha)s + (2-\alpha)} \]

\[ s^2 \quad 1 \quad 2 - \alpha \]

\[ s^1 \quad 1 + \alpha \quad 0 \]

\[ s^0 \quad 2 - \alpha \]

Therefore, \(-1 < \alpha < 2\) for 1st column all positive. Note root locus only shows \( \alpha > 0 \). [1 pt]
Figure 3: Root locus Plot for 9.18 and Problem 3. [2 pts for each plot]