Problem 1 [20 pts]

(1)[5 pts] Write the transfer function between the plant input \( T(s) \) and plant output \( \Theta_1(s) \) in terms of \( J_1, J_2, k, \) and \( b \) only. Write the transfer function \( \frac{\Theta_1(s)}{T(s)} \) in its simplest form.

\[
\sum M = J \dot{\Theta} \\
\Rightarrow J_1 \dot{\Theta} = T(t) - k(\Theta_1 - \Theta_2) - b(\dot{\Theta}_1 - \dot{\Theta}_2) \\
J_1 \dot{\Theta}_1 + b \dot{\Theta}_1 - b \dot{\Theta}_2 + k \Theta_1 - k \Theta_2 = T(t) \\
\Rightarrow J_2 \dot{\Theta}_2 = 0 - k(\Theta_2 - \Theta_1) - b(\dot{\Theta}_2 - \dot{\Theta}_1) \\
J_2 \dot{\Theta}_2 + b \dot{\Theta}_2 - b \dot{\Theta}_1 + k \Theta_2 - k \Theta_1 = 0
\]

Laplace transform both equations assuming zero initial conditions.

\[
J_1 s^2 \Theta_1(s) + bs \Theta_1(s) - bs \Theta_2(s) + k \Theta_1(s) - k \Theta_2(s) = T(s) \\
J_2 s^2 \Theta_2(s) + bs \Theta_2(s) - bs \Theta_1(s) + k \Theta_2(s) - k \Theta_1(s) = 0 \\
\Theta_1(s)(J_1 s^2 + bs + k) - \Theta_2(s)(bs + k) = T(s) \\
\Theta_2(s)(J_2 s^2 + bs + k) - \Theta_1(s)(bs + k) = 0 \\
\Theta_2(s)(J_2 s^2 + bs + k) = \Theta_1(s)(bs + k) \\
\Theta_2(s) = \Theta_1(s) \frac{bs + k}{J_2 s^2 + bs + k}
\]

\[
\Rightarrow \Theta_1(s)(J_1 s^2 + bs + k) - \Theta_1(s) \frac{bs + k}{J_2 s^2 + bs + k} = T(s) \\
\Theta_1(s)[(J_1 s^2 + bs + k) - \frac{(bs + k)^2}{J_2 s^2 + bs + k}] = T(s) \\
\Theta_1(s)(J_1 s^2 + bs + k)(J_2 s^2 + bs + k) - (bs + k)^2 \\
\Theta_1(s) = \frac{J_1 s^2 + bs + k}{T(s)} \\
\Theta_1(s) = \frac{J_1 s^2 + bs + k}{J_1 J_2 s^4 + (J_1 + J_2)bs^3 + (J_1 + J_2)ks^2}
\]

(2)[5 pts] \( J_1 = 5, J_2 = 1, b = 0.2, \) and \( k = 36 \). Given a controller design:

\[
K(s) = K \frac{s + 1}{s + 12}
\]

design \( K \) such that the closed loop system has an optimal response. You may use the Matlab function \( \text{rlocus} \) instead of plotting by hand. By optimal response possible I mean the fastest rise-time and lowest overshoot. Show your root locus plot and design value \( K \).
\[
Y(s) = \frac{K(s) \Theta_1}{T(s)}
\]

\[
= \frac{\frac{s+1}{s+12}}{1 + \frac{s^2 + 0.2s + 36}{s^2 + 0.2s + 36}}
\]

\[
= \frac{s^2 + 0.2s + 36}{s^2 + 0.2s + 36}
\]

\[
= \frac{1 + KG(s)}{1 + KG(s)}
\]

Root locus plot for optimal response is shown in Fig. 1. \(K = 403\) for optimal response.

Figure 1: Root locus Plot for optimal response.

(3) [2 pts] Use your root locus plot to design a \(K\) such that the response is sub-optimal.

Root locus plot for optimal response is shown in Fig. 2. \(K = 57.8\) for sub-optimal response.
(4) [8 pts] Use simulink to compare your optimal design to the sub-optimal design. Simulate the output when the reference signal is a step, a ramp, and a sinusoid. Does an optimal design in terms of step-response metrics (rise time and overshoot) yield a better response for the ramp and sinusoid response?

The response is better for a ramp response.
Figure 4: Step response.

Figure 5: Ramp response.
Figure 6: Sin response.