Problem 1. [8 pts]
Suppose you are given the plant:

\[ P(s) = \frac{1}{s^2 + (1 + \alpha)s + (1 - \alpha)} \]

where \( \alpha \) is a system parameter that is subject to variations; \( 0 < \alpha < \infty \). The plant is in a typical unity feedback loop and controlled by P-control where \( k_p = 1 \).

(1) Use the root-locus method to determine what variations in \( \alpha \) can be tolerated before instability occurs. Show your hand-drawn sketch and a verification by the Matlab function rlocus for full credit.

[Equation]

\[
\frac{Y}{R} = \frac{k(s)p(s)}{1 + k(s)p(s)} = \frac{1}{s^2 + (1 + \alpha)s + 2} = \frac{1}{s^2 + (1 + \alpha)s + 2 - \alpha}
\]

Not in our standard form

\[
\frac{Y}{R} = \frac{N(s)}{1 + kG(s)}
\]

Divide numerator and denominator by all terms that do not have the varied parameter \( \alpha \) in them.

\[
\frac{Y}{R} = \frac{1}{s^2 + (1 + \alpha)s + (2 - \alpha)} \cdot \frac{\frac{1}{s^2 + (1 + \alpha)s + 2}}{\frac{1}{s^2 + (1 + \alpha)s + 2}} = \frac{1}{1 + \alpha(s-1)}
\]

\[ G(s) = \frac{s - 1}{s^2 + s + 2} \] [2 pts]

(1) Plot the poles and zeros of \( G(s) \) where \( n_z = 1, z_i = 1, n_p = 2, p_i = -\frac{1}{2} \pm \frac{1}{2}\sqrt{7}i \)

(2) The root locus exists to the left of an odd number of poles and zeros on the real axis

(3) Asymptotes \( \Theta = \frac{180^\circ + n360^\circ}{(n_z - n_p)} \rightarrow \Theta_0 = \frac{180^\circ}{-1} = -180^\circ \)

pole at \(-\frac{1}{2} \pm \frac{1}{2}\sqrt{7}i \)

(4) Departure angles
\[
\angle(s - p_j) = \sum_{i=1}^{n_x} \angle(p_j - z_i) - \sum_{i=1, i \neq j}^{n_p} \angle(p_j - p_i) - 180 = \tan^{-1}\left(\frac{\frac{1}{2} \sqrt{7}}{-\frac{3}{2}}\right) - 90 - 180 = -131
\]

RL is symmetric about real axis, pole at \(-\frac{1}{2} - \frac{1}{2} \sqrt{7}i \Rightarrow \angle(s - p_j) = 130\)

5) Break-in point \(N(s) = s - 1, D(s) = s^2 + s + 2\).

\[
\frac{d}{ds} \left(\frac{1}{G(s)}\right) = \frac{dD(s)}{ds} N(s) - D(s) \frac{dN(s)}{ds} = 0
\]
\[
(2s + 1)(s - 1) - (s^2 + s + 2) = 0
\]
\[
2s^2 - 2s + s - 1 - s^2 - s - 2 = 0
\]
\[
s^2 - 2s - 3 = 0
\]
\[
(s - 3)(s + 1) = 0
\]
\[
s = -1 \text{ (only valid root), } 3
\]

Gain:

1 pole crosses into RHP at point (0,0).

\[
1 + \alpha G(s) = 0 \rightarrow \alpha = \frac{1}{|G(s)|} = \frac{|s^2 + s + 2|}{|s - 1|}
\]
\[
= \frac{|s - (-\frac{1}{2} + \frac{1}{2} \sqrt{7})||s - (-\frac{1}{2} - \frac{1}{2} \sqrt{7})|}{|s - 1|}
\]
\[
= \sqrt{(1/2)^2 + (1/2 \cdot \sqrt{7})^2} \sqrt{(1/2)^2 + (1/2 \cdot \sqrt{7})^2}
\]
\[
= \frac{1}{\sqrt{1/4 + 7/4 \sqrt{1/4 + 7/4}}} = 2
\]

Since an increasing \(\alpha\) drives pole into the RHP, \(\alpha\) should be less than 2 for stability. [2 pts]
1 % Create transfer function
2 G=tf([1 -1],[1 1 2]);
3 % Plot root locus
4 rlocus(G)
Root Locus

System: G
Gain: 2
Pole: 0.00147
Damping: −1
Overshoot (%): 0
Frequency (rad/s): 0.00147

\[ 2 \text{ pts} \]
Problem 2. [17 pts]

Part 1 Write the transfer function between the plant input \( T(s) \) and plant output \( \Theta_1(s) \) in terms of \( J_1, J_2, k, \) and \( b \) only. Write the transfer function \( \frac{\Theta_1(s)}{T(s)} \) in its simplest form.

\[
\sum \dot{M} = J\ddot{\Theta}
\]

\[\Rightarrow J_1\ddot{\Theta}_1 = T(t) - k(\Theta_1 - \Theta_2) - b(\dot{\Theta}_1 - \dot{\Theta}_2)\]
\[J_1\ddot{\Theta}_1 + b\dot{\Theta}_1 - b\dot{\Theta}_2 + k\Theta_1 - k\Theta_2 = T(t)\]

\[\Rightarrow J_2\ddot{\Theta}_2 = 0 - k(\Theta_2 - \Theta_1) - b(\dot{\Theta}_2 - \dot{\Theta}_1)\]
\[J_2\ddot{\Theta}_2 + b\dot{\Theta}_2 - b\dot{\Theta}_1 + k\Theta_2 - k\Theta_1 = 0\]

Laplace transform both equations assuming zero initial conditions.

\[
J_1s^2\Theta_1(s) + bs\Theta_1(s) - bs\Theta_2(s) + k\Theta_1(s) - k\Theta_2(s) = T(s)
\]
\[J_2s^2\Theta_2(s) + bs\Theta_2(s) - bs\Theta_1(s) + k\Theta_2(s) - k\Theta_1(s) = 0\]
\[\Theta_1(s)(J_1s^2 + bs + k) - \Theta_2(s)(bs + k) = T(s)\]
\[\Theta_2(s)(J_2s^2 + bs + k) - \Theta_1(s)(bs + k) = 0\]
\[\Theta_2(J_2s^2 + bs + k) = \Theta_1(bs + k)\]
\[\Theta_2(s) = \Theta_1(s)\frac{bs + k}{J_2s^2 + bs + k}\]

\[\Rightarrow \Theta_1(s)(J_1s^2 + bs + k) - \Theta_1(s)\frac{bs + k}{J_2s^2 + bs + k}(bs + k) = T(s)\]
\[\Theta_1(s)[(J_1s^2 + bs + k) - \frac{(bs + k)^2}{J_2s^2 + bs + k}] = T(s)\]
\[\Theta_1(s)[\frac{(J_1s^2 + bs + k)(J_2s^2 + bs + k) - (bs + k)^2}{J_2s^2 + bs + k}] = T(s)\]
\[
\frac{\Theta_1(s)}{T(s)} = \frac{J_2s^2 + bs + k}{(J_1s^2 + bs + k)(J_2s^2 + bs + k) - (bs + k)^2}
\]
\[
\frac{\Theta_1(s)}{T(s)} = \frac{J_2s^2 + bs + k}{J_1J_2s^4 + (J_1 + J_2)bs^3 + (J_1 + J_2)ks^2}
\]

[3 pts]

Part 2 \( J_1 = 5, J_2 = 1, b = 0.2, \) and \( k = 36. \) Given a controller design:

\[K(s) = K\frac{s + 1}{s + 12}\]

design \( K \) such that the closed loop system has an optimal response. You may use the Matlab function \( \text{rlocus} \) instead of plotting by hand. By optimal response possible I mean the fastest rise-time and lowest overshoot. Show your root locus plot and design value \( K. \)
\[
G(s) = K(s)P(s)
\]
\[
K(s) = K \frac{s + 1}{s + 12}
\]
\[
\Theta_1(s) = \frac{1 \cdot s^2 + 0.2s + 36}{s^2 + 0.2s + 36}
\]
\[
T(s) = \frac{s^2 + 0.2s + 36}{s^2(5s^2 + 1.2s + 216)}
\]
\[
\Theta_1(s) = \frac{1 \cdot s^2 + 0.2s + 36}{5 \cdot s^4 + (5 + 1) \cdot 0.2s^3 + (5 + 1)36s^2}
\]
\[
\Theta_1(s) = \frac{s^2 + 0.2s + 36}{s^2(5s^2 + 1.2s + 216)}
\]
\[
Y(s) = \frac{K(s)}{1 + K(s)} \frac{\Theta_1(s)}{T(s)}
\]
\[
= \frac{K \frac{s + 1}{s + 12}}{1 + K \frac{s + 1}{s + 12}} \frac{s^2 + 0.2s + 36}{s^2(5s^2 + 1.2s + 216)}
\]
\[
= \frac{KG(s)}{1 + KG(s)}
\]

1. % Create transfer function
   C=tf([1 1],[1 12]);
2. G=tf([1 .2 36],conv([1 0 0],[5 1.2 216]));
3. % Plot root locus
   rltool(C*G)

Root locus plot for optimal response is shown below. \( K = 411 \) for optimal response. [1 pt]
Figure: Root locus Plot for optimal response. The diagonal lines represent overshoot. [1 pt]

Part 3 Use your root locus plot to design a $K$ such that the response is sub-optimal.

Root locus plot for optimal response is shown below. $K = 49.5$ for sub-optimal response. [1 pt]
Part 4 Use simulink to compare your optimal design to the sub-optimal design. Simulate the output when the reference signal is a step, a ramp, and the smoothed step given in the attached data file. Does an optimal design in terms of step-response metrics (rise time and overshoot) yield a better response for the ramp and smooth step response?

The response is better for a ramp response.

```matlab
1 % AME 30315 Homework 10, Problem 2
2 close all
3
4 % Load smoothed_step.mat file to put time and ref into the workspace
5 simin=[time',ref'];
6```
sim('problem2')

% Step Response

% Plot Reference
figure(1)
plot(tout,optStep(:,1),'--','LineWidth',1.5)
hold on

% Plot optimal step response
plot(tout,optStep(:,2),'g','LineWidth',1.5)

% Plot suboptimal step response
plot(tout,suboptStep(:,2),'r','LineWidth',1.5)

legend('Reference','Optimal','Suboptimal')
axis([0 15 0 1.5])
xlabel('t')
ylabel('$\theta(t)$')

% Ramp Response

% Plot Reference
figure(2)
plot(tout,optRamp(:,1),'--','LineWidth',1.5)
hold on

% Plot optimal ramp response
plot(tout,optRamp(:,2),'g','LineWidth',1.5)

% Plot suboptimal step response
plot(tout,suboptRamp(:,2),'r','LineWidth',1.5)

legend('Reference','Optimal','Suboptimal')
axis([0 5 0 6])
xlabel('t')
ylabel('$\theta(t)$')

% Smoothed Step Response

% Plot Reference
figure(3)
plot(tout,optSmooth(:,1),'--','LineWidth',1.5)
hold on

% Plot optimal step response
plot(tout,optSmooth(:,2),'g','LineWidth',1.5)

% Plot suboptimal step response
plot(tout,suboptSmooth(:,2),'r','LineWidth',1.5)

legend('Reference','Optimal','Suboptimal')
axis([0 15 0 1.5])
xlabel('t')
ylabel(\theta(t))
Step response. [2 pts]

Ramp response. [2 pts]
Part 5 The reference signal for this example is a desired satellite attitude in time. Why is a step change in position (step reference) a stupid task to ask a satellite to perform? Why is a smoothed step reference position a better idea?

A satellite cannot instantaneously change its position, so a smoothed step is better because it’s more realistic for how a satellite can actually change position. [2 pts]

Part 6 In your test of different values of $K$, you may have noticed that you get really good performance when you increase $K$ to a very large number. The closed-loop poles coming from the complex conjugate poles move towards the complex conjugate zeros and practically cancel second order, underdamped modes. Discuss possible negative effects of increasing $K$ to a really large value.

A large $K$ improves the system performance, but the thrusters are limited in how much torque they can provide. For example, if the gain is increased more to $K = 2000$ then the predicted smooth step response more closely matches the reference signal than for the lower gain values as shown in the figure below. However, one negative effect of increasing $K$ is that more torque is required to drive the system. There is a limit to how much torque the thrusters can actually generate so the best simulation of system performance may not be achievable in reality. The required input torque corresponding to different $K$ values is shown below.
Figure: Smoothed Step Response