Problem 1. [6 pts]

\[ K(s)P(s) = \frac{B(s)}{s^pA(s)} \]

\[ E(s) = R(s) - Y(s) = \left(1 - \frac{Y(s)}{R(s)}\right)R(s) \]

\[ = 1 - \frac{K(s)P(s)}{1 + K(s)P(s)}R(s) \]

\[ = \frac{1 + K(s)P(s) - K(s)P(s)}{1 + K(s)P(s)}R(s) \]

\[ = \frac{1}{1 + K(s)P(s)}R(s) \]

\[ = \frac{1}{1 + \frac{B(s)}{s^pA(s)}}R(s) = \frac{1}{\frac{s^pA(s) + B(s)}{s^pA(s)}}R(s) = \frac{s^pA(s)}{s^pA(s) + B(s)}R(s) \]

\[ e_{ss}(t) = \lim_{t \to \infty} e(t) \]

Assume that poles of \( E(s) \) are in the left-half plane and apply final value theorem

\[ \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \frac{s^pA(s)}{s^pA(s) + B(s)}R(s) \]

For \( R(s) = \frac{1}{s} \) [2 pts]

\[ e_{ss}(t) = \lim_{s \to 0} s \frac{s^pA(s)}{s^pA(s) + B(s)} \frac{1}{s} = \lim_{s \to 0} \frac{s^pA(s)}{s} \frac{s^pA(s)}{s^pA(s) + B(s)} \]

\[ \rightarrow \text{ when } p = 0 \]

\[ e_{ss}(t) = \lim_{s \to 0} \frac{A(s)}{A(s) + B(s)} = \frac{A(0)}{A(0) + B(0)} \]

\[ \rightarrow \text{ when } p = 1 \]

\[ e_{ss}(t) = \lim_{s \to 0} \frac{SA(s)}{SA(s) + B(s)} = \frac{0}{B(0)} = 0 \]
→ when $p = 2$

$$
e_{ss} (t) = \lim_{s \to 0} \frac{s^2 A (s)}{s^2 A (s) + B (s)} = \frac{0}{B (0)} = 0
$$

For $R (s) = \frac{1}{s^2}$ [2 pts]

$$
e_{ss} (t) = \lim_{s \to 0} s \frac{s^p A (s)}{s^p A (s) + B (s)} \frac{1}{s^2} = \lim_{s \to 0} \frac{s^{p-2} A (s)}{s^{p-1} A (s) + B (s)}$$

→ when $p = 0$

$$
e_{ss} (t) = \lim_{s \to 0} s \frac{A (s)}{A (s) + B (s)} = \frac{A (0)}{0 A (0) + B (0)} = \infty
$$

→ when $p = 1$

$$
e_{ss} (t) = \lim_{s \to 0} s \frac{s A (s)}{s A (s) + B (s)} = \lim_{s \to 0} \frac{A (s)}{s A (s) + B (s)} = \frac{A (0)}{B (0)}
$$

→ when $p = 2$

$$
e_{ss} (t) = \lim_{s \to 0} s \frac{s^2 A (s)}{s^2 A (s) + B (s)} = \lim_{s \to 0} s \frac{A (s)}{s A (s) + B (s)} = 0
$$

For $R (s) = \frac{2}{s^3}$ [2 pts]

$$
e_{ss} (t) = \lim_{s \to 0} s \frac{s^p A (s)}{s^p A (s) + B (s)} \frac{2}{s^3} = \lim_{s \to 0} \frac{2 s^{p-3} A (s)}{s^{p-2} s^2 A (s) + B (s)}$$

→ when $p = 0$

$$
e_{ss} (t) = \lim_{s \to 0} s \frac{2 A (s)}{2 s^2 A (s) + B (s)} = \infty
$$

→ when $p = 1$

$$
e_{ss} (t) = \lim_{s \to 0} s \frac{s A (s)}{s A (s) + B (s)} = \lim_{s \to 0} \frac{2 A (s)}{s A (s) + B (s)} = \infty
$$

→ when $p = 2$

$$
e_{ss} (t) = \lim_{s \to 0} s \frac{s^2 A (s)}{s^2 s^2 A (s) + B (s)} = \lim_{s \to 0} \frac{2 A (s)}{s^2 A (s) + B (s)} = \frac{2A (0)}{B (0)}
Problem 2. [12 pts]
First system

1) From pole-zero map \( n_z = 0 \) and \( n_p = 2 \). The root locus starts at the poles of \( G(s) \) and ends at the zeros.
\[
p_{1,2} \approx -0.25 \pm 1.75i
\]

2) There are no poles or zeros on the real axis, so the root locus does not exist along the real axis. This means that there are also no break in or break-out points.

3) The asymptotic angles are
\[
\theta_n = \frac{(180^\circ + n \times 360^\circ)}{n_z - n_p}
\]
\[
\theta_0 = \frac{180^\circ}{-2} = -90^\circ
\]
\[
\theta_1 = \frac{180^\circ + 360^\circ}{-2} = -270^\circ
\]
Because the root locus never crosses the imaginary axis, the system is stable for all values of $K$. [1 pt]

From Nyquist plot
→ $Z =$ number of unstable closed-loop poles
→ $P =$ number of unstable open-loop poles
→ $N =$ number of clockwise encirclement of -1

$$Z = N + P = 0 + 0 \rightarrow Z = 0$$

Based on the Nyquist criterion there are no closed-loop poles in the right-half plane. Also, for any value of $K$, the Nyquist plot will never expand to encircle $s = -1$. [2 pts]

Second system

$$G(s) = \frac{\hat{k}(s + 1)}{s(s - 10)}$$

where the exact value of $\hat{k}$ is unknown based on the given information.

1) From the pole-zero map $n_z = 1$ and $n_p = 2$, and the root locus starts at the poles of $G(s)$ and ends at the zeros, $z_1 = -1$, $p_1 = 0$, $p_2 = 10$.

2) The root locus exists to the left of an odd number of poles and zeros along the real axis.

3) The asymptote angles are

$$\theta_0 = \frac{180^\circ}{-1} = -180^\circ$$

4) No complex conjugate pairs of poles or zeros of $G(s)$.

5) 

$$G(s) = \frac{N(s)}{D(s)} = \frac{s + 1}{s(s - 10)}$$
Compute the break-in and break-out points

\[
\frac{d}{ds} \left( \frac{1}{G(s)} \right) = 0 \rightarrow N(s) \frac{dD(s)}{ds} - D(s) \frac{dN(s)}{ds} = 0
\]

\[
\rightarrow (2s - 10)(s + 1) - (s^2 - 10s)(1) = 0
\]

\[
\rightarrow 2s^2 - 8s - 10 - s^2 + 10s = 0
\]

\[
\rightarrow s^2 + 2s - 10 = 0 \rightarrow s \approx 2.3 \text{ (breakout point)} \& \ s \approx 4.3 \text{ (break-in point)}
\]

The root locus crosses the imaginary axis from the right-half plane into the left-half plane at \( K = 10 \) so for a sufficiently high \( K \) the system is stable. [1 pt]
From Nyquist plot

\[ N = 1 \]
\[ P = 1 \]
\[ Z = N + P + 1 + 1 = 2 \text{ when } K = 1 \]

For \( K = 1 \), the closed-loop system is unstable, but for a sufficiently large \( K \) value there will be a counterclockwise encirclement of -1. This means that \( N = -1 \) and \( Z = -1 + 1 = 0 \), so the system will be stable. [2 pts]

For example, if \( \hat{k} \) is approximately 9.5 and \( K \) is increased from 1 to 3, this changes the Nyquist plot such that there is a counterclockwise encirclement of -1.

Third system

\[ G(s) \approx \frac{(\hat{k})(s^2 + 2s + 2)}{s^2(s + 6)(s + 4)(s + 1)} \]

From the pole-zero map \( n_z = 2 \) and \( n_p = 5 \) (there is a double pole at the origin). The root locus starts at the poles of \( G(s) \) and ends at the zeros.

\( p_1 = -6, \ p_2 = -4, \ p_3 = -1, \ p_{4,5} = 0 \)
\( z_1 = -1 + i, \ z_2 = -1 - i \)
The root locus exists to the left of an odd number of poles and zeros along the real axis.

The asymptote angles are

\[ \theta_n = \frac{180^\circ + n 360^\circ}{n_z - n_p} \]

\[ \theta_0 = \frac{180^\circ}{-3} = -60^\circ \]

\[ \theta_1 = \frac{180^\circ + 360^\circ}{-3} = -180^\circ \]

\[ \theta_2 = \frac{180^\circ + 2(360^\circ)}{-3} = -300^\circ \]

Intersection

\[ s_{\text{int}} = \frac{\sum_{i=1}^{n_z} z_i - \sum_{i=1}^{n_p} p_i}{n_z - n_p} = \frac{(-1 + i) + (-1 - i) - (6) - (-4) - (1)}{-3} = -3 \]

Calculate arrival angles to complex zeros.

\[ \angle (s - z_j) = 180^\circ - \sum_{i=1, i \neq j}^{n_z} \angle (z_j - z_i) + \sum_{i=1, i \neq j}^{n_p} \angle (z_j - p_i) \]

\[ \angle (s - z_1) = 180^\circ - \angle (z_1 - z_2) + \angle (z_1 - p_1) + \angle (z_1 - p_2) + \angle (z_1 - p_3) + \angle (z_1 - p_4) + \angle (z_1 - p_5) \]

\[ = 180^\circ - \arctan 2 (2, 0) + \arctan 2 (1, 5) + \arctan 2 (1, 3) + \arctan 2 (1, 0) + \arctan 2 (1, -1) + \arctan 2 (1, -1) \]

\[ = 180^\circ - 90^\circ + 11.3^\circ + 18.4^\circ + 90^\circ + 135^\circ + 135^\circ \]

\[ = -360^\circ + 479.7^\circ = 119.7^\circ \]

The root locus is symmetric about the real axis so \( \angle (s - z_2) \approx -119.7^\circ \)

Find the break-out point from the real axis where \( G(s) = \frac{N(s)}{D(s)} = \frac{\dot{k}}{s^5 + 11s^4 + 34s^3 + 24s^2} \)

\[ \frac{d}{ds} \left( \frac{1}{G(s)} \right) = 0 \rightarrow N(s) \frac{dD(s)}{ds} - D(s) \frac{dN(s)}{ds} = 0 \]

\[ \rightarrow \dot{k} \left( s^2 + 2s + 2 \right) \left( 5s^4 + 44s^3 + 102s^2 + 48s \right) - \dot{k} \left( s^5 + 11s^4 + 34s^3 + 24s^2 \right) \left( 2s + 2 \right) = 0 \]

valid root is \( s = -2.1 \) because we’re looking for a root between -2 and -4 to be on the root locus.
From the root locus plot, for low $K$ values the system is unstable, for moderate values the system is stable, and for large $K$ the system is unstable. [1 pt]

From Nyquist plot [2 pts for analysis] The Nyquist plot given is for $K = 1$ which corresponds to within the stable portion on the root locus. This is because $\hat{k}$ is relatively large, but we do not know the exact value.

For example, if $\hat{k} = 100$, then the stable range of $K$ is about $0.26 < K < 1.34$ as determined from the root locus.

$$P = 0$$

$$N = -1 + 1 = 0$$

$$Z = N + P = 0 \rightarrow$$ This verifies stability for intermediate $K$ values
If $0 < K < 0.26$

\[
\begin{align*}
P &= 0 \\
N &= 1 \\
Z &= 1 + 0 = 1 
\end{align*}
\]

$\rightarrow$ This verifies instability for low $K$ values

Corresponding to the example $\hat{k} = 100$, for a very low $K$ value (like 0.1) causes a shift in the Nyquist plot such that there is only 1 clockwise encirclement of -1.

If $K > 1.34$

\[
\begin{align*}
P &= 0 \\
N &= 2 \\
Z &= 2 + 0 = 2 
\end{align*}
\]

$\rightarrow$ This verifies instability for large $K$ values

Corresponding to the example $\hat{k} = 100$, for a larger $K$ value, Nyquist plot is changed such that it encircles -1 twice as shown in the figure below for $K = 2$. 
Problem 3. [12 pts]

1) For X axis use MATLAB to obtain the Bode plot of the uncompensated system.

```matlab
s = tf('s');
num = 400*(s+0.2)*(s+2)*(s+100)*(s^2 + 7.108*s + 1263);
den = s*(s+1)*(s+3)*(s+4)*(s+43)*(s^2 + 12*s + 1600);
sys = num/den;
figure,margin(sys);
```

The uncompensated system has a gain margin of $-2.87 \text{dB}$ and a phase margin of $-3.27^\circ$.

For a lead compensator of the form $G_c = \frac{s + 1}{s + z}$ let $\alpha = \frac{p}{z}$.

To make the phase margin greater that $50^\circ$, choose $\alpha = 15$. From Fig. 10.30 in the text we can increase the phase margin by approximately $60^\circ$.

Substitute $\alpha = 15$ into the right-hand side of Eqn. (10.5) in the text.

$$|G(i\sqrt{zp})|_{dB} = -11.76$$

Therefore we need the frequency in the uncompensated Bode plot where the uncompensated gain is approximately $-11.76$ which is $\sqrt{zp} = 55$.

Solving $\alpha = \frac{p}{z} = 15$ and $\sqrt{zp} = 55$ for $p$ and $z$, gives $p = 213.01$, $z = 14.2$.

Hence

$$G_{cx}(s) = \frac{s + 1}{14.2s + 1}$$

Verify and adjust our design using MATLAB:
s = tf('s');
num = 400*(s+0.2)*(s+2)*(s+100)*(s^2 + 7.108*s + 1263);
den = s*(s+1)*(s+3)*(s+4)*(s+43)*(s^2 + 12*s + 1600);
sys = num/den;
a = 15;
wm = 55;
p = sqrt(a)*wm;
z = p/a;
comp = (s/z + 1)/(s/p + 1);
figure;
margin(comp*sys);
figure;
bode(feedback(comp*sys,1));
grid on;
title('Closed-loop Bode Diagram');
From the figures, our controller satisfies all the specifications so we don’t need to adjust it.

For Y axis, use MATLAB to obtain the Bode plot of the uncompensated system.

```matlab
s = tf('s');
num = 400*(s+0.2)*(s+100)*(s^2 + 10.66*s + 1263);
den = s*(s+1)*(s+4)*(s+43)*(s^2 + 12.68*s + 1787);
sys = num/den;
figure,margin(sys);
```
The uncompensated system has a gain margin of 8.63 dB and a phase margin of 2.22°.

We choose $\alpha = 10$ which can increase the phase margin by approximately 55°.

Thus

$$|G(i\sqrt{zp})|_{dB} = -10 \rightarrow \sqrt{zp} = 53.$$  

Solving $\alpha = \frac{p}{z} = 10$ and $\sqrt{zp} = 53$ for $p$ and $z$, gives $p = 167.6$, $z = 16.76$.

Hence

$$G_{cy}(s) = \frac{s}{167.6} + \frac{1}{s}$$

Verify and adjust our design using MATLAB:

```matlab
1 s = tf('s');
2 num = 400*(s+0.2)*(s+100)*(s^2 + 10.66*s + 1263);
3 den = s*(s+1)*(s+4)*(s+43)*(s^2 + 12.68*s + 1787);
4 sys = num/den;
5
6 a = 10;
7 wm = 53;
8 p = sqrt(a)*wm;
9 z = p/a;
10 comp = (s/z + 1)/(s/p + 1);
11
12 figure;
13 margin(comp*sys);
```
figure;
bode(feedback(comp*sys,1));
grid on;
title('Closed-loop Bode Diagram');
From the figures, our controller satisfies all the specifications so we don’t need to adjust it.

2) Using Simulink and MATLAB to get the step response.

X axis:
Using rule of thumb, $\zeta \sim \frac{P_o}{100} = 0.545$

$\Rightarrow$ Percent overshoot $P.O. = 100\% \cdot e^{\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)} = 13\%$

From the figure below, the simulated overshoot matches our expectation well.
From the figure, the input to the plant exceeds +10 V for a step reference, so we need to modify our controller. By adding a gain factor and changing $\sqrt{z_p}$, we get a new controller:

$$G_{cx} (s) = 0.6 \frac{8}{103.3} + 1 \frac{8}{154.9} + 1$$

Check phase margin and gain margins:
Check closed-loop bandwidth:

Check step response:
Y axis:

\[ \zeta \sim \frac{P_m}{100} = 0.5 \]

→ Percent overshoot \( P.O. = 100\% \cdot e^{\left(\frac{-\zeta \pi}{\sqrt{1 - \zeta^2}}\right)} = 16.3\% \)

From the figure below, the simulated overshoot matches our expectation well.
From the figure, the maximum input to the plant is exactly +10 V. For safety, we can add a gain factor of 0.9 to the controller:

\[ G_{cy}(s) = 0.9 \frac{s}{16.76} + 1 \]

Check phase margin and gain margins:
Check closed-loop bandwidth:

Check step response:
3) See part (2) 'check closed-loop bandwidth'.

4) Build up a Simulink model:

Simulation results:
Generally, the controller give good tracking.

Grading instructions:

Note: There exists variety of controllers working for this system. The controllers designed in the solution is just one case.

- [5 pts] for designing controllers by hand.
- [4 pts] for verifying and adjusting controllers using a software package.
- [3 pts] for tracking simulation.