Problems:

- Review problems for Midterm 1 and 2
- 9.20
- 9.21
- 9.38 (just first bullet)
- 10.10.1–3
- 10.14.1–4
- 10.15.1–6

- Consider the type 1 system in Fig. 1. We would like to design the controller $K(s)$ to meet the following requirements: 1) The steady state change of $y(t)$ due to a constant unit disturbance $w(t) = 1(t)$ should be less than $\frac{4}{5}$, and 2) the damping ratio $\zeta = 0.7$. Using the root-locus techniques:
  
  (a) Show that proportional control is not adequate.
  
  (b) Show that proportional derivative control will work.
  
  (c) Find values of the gains $k_p$ and $k_d$ for $K(s) = k_p + k_ds$ that meet the design specifications.

![Fig. 1. Feedback loop with a step disturbance.](image-url)

- Design a velocity control for a tape-drive servomechanism. The transfer function from
current $I(s)$ to tape velocity $\Omega(s)$ (units are mm per msec per ampere) is:

$$\frac{\Omega(s)}{I(s)} = \frac{15(s^2 + 0.9s + 0.8)}{(s + 1)(s^2 + 1.1s + 1)}$$

We wish to design a type 1 feedback system so that the response to a step reference satisfies

$$t_r \leq 4 \text{ msec}, \ t_s \leq 15 \text{ msec}, \ O \leq 5\%$$

Assume a proportional-integral controller of the form $k_p(s + \alpha)/s$.

(a) Sketch out the region of the complex plane where you want to place the closed loop poles. Pick a pole location well within your designed region.

(b) Design controller zero location $\alpha$ such that the root locus passes through the chosen pole location.

(c) Sketch the root locus plot.

(d) Choose $k_p$ such that the closed loop poles are in the desired regions.

- In homework you investigated the typical unity feedback loop for different types of reference signals and came to the conclusion that the $\lim_{t \to \infty} e(t) = e_{ss}$ of the closed-loop system under unity feedback is dependent on system type. We parameterized the open-loop system:

$$K(s)P(s) = \frac{B(s)}{s^pA(s)}$$

where $A(s)$ and $B(s)$ are Type 0 polynomials and $p$ is the system type. If $\frac{E(s)}{R(s)} = \frac{1}{1+K(s)P(s)}$ is stable, then $e_{ss}$ is given by the table below.

The fact that $e_{ss} = 0$ for a step reference applied to a Type 1 system and that $e_{ss} = 0$ for a ramp reference applied to a Type 2 system is the result of a more general principle termed
the Internal Model Principle.

To paraphrase, the Internal Model Principle states that given a stable closed-loop system and a reference signal with a Laplace Transform $R(s)$, $e_{ss} = 0$ if the open-loop transfer function $K(s)P(s)$ contains $R(s)$.

**Test the Internal Model Principle with a sinusoidal reference signal.** Demonstrate that if $R(s) = \frac{1}{s^2+1}$, $e_{ss} = 0$ if $K(s)P(s) = \frac{1}{s^2+1} \frac{B(s)}{A(s)}$.

- Consider the feedback loop in Fig. 2. $W$ is the Laplace Transform of a sensor noise signal.

We wish to design a controlled system such that the overshoot to a step reference is less than 15% and the rise time is less than 0.9 sec.

![Feedback control system with sensor noise.](image)

(a) Assume $W(s) = 0$ for simplicity. First investigate a proportional controller $K(s) = k_p$.

Draw the root locus for the system AND indicate whether or not a proportional controller can achieve the stated design specifications.
For full credit:

(i) Show your root locus plot calculations for the asymptotes, asymptote intersection point, and break-away point.

(ii) Lightly shade in the region of the complex plane where the design specifications are satisfied.

(iii) Determine whether or not proportional control is adequate; reference your root locus plot and shaded regions.

(b) Again, assume $W(s) = 0$ for simplicity. Use the root locus method to design a lead-controller to meet the specifications.

• Your boss has tasked you with the development of a feedback control system for a physical plant that can be described by the transfer function:

$$P(s) = \frac{s + 4}{s(s + 3 + 2i)(s + 3 - 2i)}.$$ 

Your boss wants the closed-loop system to have a rise time of less than 0.9 sec and an overshoot of less than 25%. As a first attempt, you decide to design a proportional controller with a proportional gain $k_p$.

(a) (20 pts) In Fig. 3, sketch the values of $s$ such that $1 + k_pP(s) = 0$ as $k_p$ ranges from 0 to $\infty$ (also known as the root locus). For full credit, clearly show and/or calculate the:

(i) open-loop poles and zeros

(ii) asymptotes of unbounded poles and the associated intersection point

(iii) departure angles of the complex conjugate poles (tables in the backmatter are available to help with the trigonometry, round to the nearest table entry).
(iv) root locus region on the real axis

(b) (5 pts) Also in Fig. 3, sketch the region of the complex plane in which the step response of a second-order underdamped system will have the rise time and overshoot specified by your boss. Notes: lines of constant \( \zeta \) are given by the diagonal lines in Fig. 3.

(c) (15 pts) There IS a range of \( k_p \) such that all closed-loop poles lie within the specified region. Find this range. Estimated values from your plot are allowable and you may leave your answer in terms of radicals and \( x^y \) terms.

(d) (25 pts) You’ve decided that your proportional control design will not provide you with the closed-loop bandwidth you desire. Scrap your proportional controller and use the root locus method to design a lead-controller such that the closed-loop system has \( \omega_n > 6 \) rad/sec and \( \zeta > 0.50 \). The open-loop bode plot of \( K(s)P(s) \) for \( K(s) = 1 \) is given in Fig. 4.

(i) (5pts) Calculate the uncompensated \( g_m \) and \( \phi_m \).

(ii) (13pts) Calculate the gain and pole and zero locations of a lead-controller that meets the design specifications. Use nice round numbers to make it easier on yourself and the graders; the graders are looking for your thought process, not absolute accuracy.
Fig. 3. Blank root locus paper for Problem 1. Diagonal lines are lines of constant $\zeta$. 
- Your often wrong, cheapskate colleague is back to their old tricks again. This time their bad ideas may be very dangerous. Your team is trying to balance an inverted pendulum (Fig. 5a) where the input to the system is the force $f(t)$ applied to the pendulum base and the output is the pendulum angle $\theta(t)$. The equations of motion for the pendulum are:

$$(M + m) \ddot{x} - ml \ddot{\theta} \cos \theta + ml^2 \dot{\theta}^2 \sin \theta = f(t)$$

$$l \ddot{\theta} - g \sin \theta = \ddot{x} \cos \theta$$

which can be linearized and written as a Transfer Function for the plant:

$$P(s) = \frac{\Theta(s)}{F(s)} = \frac{1}{s^2 Ml - (M + m) g}.$$
Instead of purchasing a sensor \( (H) \) to measure \( \theta(t) \) to complete a feedback loop (Fig. 5b), your colleague wants to use a microprocessor to compute a differential equation corresponding to the inverse of the plant dynamics and use this in the configuration in Fig. 5c. His or her rationale is that this system will provide perfect dynamics \( (\Theta(s) = R(s)) \),

\[
\Theta(s) = \frac{1}{s^2 Ml - (M + m) g} \frac{s^2 Ml - (M + m) g}{1} R(s) = R(s),
\]

while a feedback loop will not yield perfect dynamics. Write a letter to your colleague explaining why a feedback loop is the only way to go for this system and why their idea will not work and may even be dangerous.

Fig. 5. a) Inverted pendulum on a moving base. \( g \) is gravitational acceleration. b) Standard feedback loop to balance an inverted pendulum. c) Your colleague’s proposed scheme.

- You are designing a controlled system to accurately position the angle of a motor shaft.

The plant (motor) has the transfer function

\[
\frac{\Theta_m(s)}{I_a(s)} = \frac{k_\tau}{s \left( s + \frac{3}{2} \right)}
\]

where \( k_\tau \) is an unknown scalar parameter. The motor is in the feedback loop given by Fig. 6.
(A) (20pts) As a preliminary test you examined the response of the closed-loop system to a unit step reference signal \( r(t) = 1(t) \); the measured response for motor position is given in Fig. 7. Given that you were using a proportional controller with \( K(s) = 1 \), estimate the value of \( k_{\tau} \) from the step response.

(B) (10pts) Next, you are tasked with designing the closed-loop system to have a rise time of 0.6 sec and an overshoot of 15%. Sketch in the complex plane given in Fig. 8 the region at which you should place the closed-loop poles given these performance metrics. A map of overshoot as a function of \( \zeta \) is given in the backmatter.

(C) (30pts) Now, design a new controller, \( K(s) \), that achieves both metrics. Use \( k_{\tau} = 10 \) if you are not confident in your answer in Part A. Assume that the closed-loop system approximates a second-order underdamped system, even if your chosen design adds an extra pole or zero to the closed-loop transfer function.
Fig. 7.  Step response of the closed-loop system in Fig. 6 for an unknown $k_r$. 

Fig. 8.  Complex plane. Diagonal lines are level sets of a fixed $\zeta$. Arcs are level sets of a fixed $\omega_n$. 
• $G(s)$ is defined by the transfer function:

$$G(s) = \frac{(s + 0.619)}{s^2(s + 3.21 + 4.77i)(s + 3.21 - 4.77i)}.$$  

Fig. 9 is a plot of the values of $s$ such that:

$$1 + kG(s) = 0$$

as $k$ ranges from 0 to $\infty$; also called a root locus plot. Each x marks a pole of $G(s)$ and each o marks a zero of $G(s)$. There is a double pole at the origin marked by a hatch mark.

(a) (15 pts) For what range of values of $k$ are all the roots of $1 + kG(s) = 0$ in the left-half plane (LHP)? Your answer should be in terms of a subset of the dimensions (letters) in Fig. 9 and nothing else; NOT all dimensions will be used. Only consider positive values of $k$.

(b) (5 pts) $\hat{G}(s) = 5G(s)$. In words, how would the shape of the root locus plot in Fig. 9 change if instead you plotted the root locus for $\hat{G}(s)$?

(c) (5pts) Consider the root locus plot for $\hat{G}(s)$. For what range of values of $k$ are all the roots of $1 + k\hat{G}(s) = 0$ in the LHP? Only consider positive values of $k$. 
Fig. 9. Root Locus plot of $G(s)$. 