Problem 1 [10/10 pts]
Sketch the asymptotes of the bode plot magnitude and phase for following transfer function. Embellish the asymptote plots with a rough estimate of the transitions for each break point. After completing the hand sketches, verify your result with the Matlab commands tf and bode or the GUI for asymptotic bode plots (linked on course website). Turn in your hand sketches and the Matlab results on the same scales.

\[ G(s) = \frac{s+2}{s(s+10)(s^2+2s+2)} \]

First, we rewrite our transfer function:
\[ G(s) = (s + 2) \left( \frac{1}{s} \right) \left( \frac{10}{s+10} \right) \left( \frac{1}{2} \right) \left( \frac{2}{s^2+2s+2} \right) \]

Then, we rewrite our transfer function into standard forms:
\[ G(s) = 2 \left( \frac{s}{2} + 1 \right) \left( \frac{1}{s} \right) \left( \frac{1}{10+1} \right) \left( \frac{1}{2} \right) \left( \frac{1}{\left( \frac{s}{\sqrt{2}} \right)^2 + s+1} \right) \]

Or, with a single gain term (moved out front):
\[ G(s) = \left( \frac{1}{10} \right) \left( \frac{s}{2} + 1 \right) \left( \frac{1}{s} \right) \left( \frac{1}{10+1} \right) \left( \frac{1}{\left( \frac{s}{\sqrt{2}} \right)^2 + s+1} \right) \]

Each of which correspond to a standard form found in the text.

We will plot each transfer function individually:

\[ G(s) = \left( \frac{1}{10} \right) \left( \frac{s}{2} + 1 \right) \left( \frac{1}{s} \right) \left( \frac{1}{10+1} \right) \left( \frac{1}{\left( \frac{s}{\sqrt{2}} \right)^2 + s+1} \right) \]

For \( G_1(s) = \left( \frac{1}{10} \right) \):
\[ |G_1(s)| = 20 \log_{10} \left( \frac{1}{10} \right) \approx -20 \text{ dB} \]
\[ \angle G_1(s) = 0^\circ \]
For $G_2(s) = \left( \frac{s}{2} + 1 \right)$:

with $\omega^z = 2$

Case 1: $\omega << \omega^z$

$G_2(i\omega) \approx 1$

$|G_2(s)| \approx 0$ dB

$\angle G_2(s) \approx 0^\circ$

Case 2: $\omega >> \omega^z$

$G_2(i\omega) \approx i\frac{\omega}{2}$

This corresponds, in decibels, to a slope of the magnitude plot of 20 dB/decade. (Notice in the above that when $\omega$ increases by a factor of 10, the magnitude of $G_2(i\omega)$ also increases by a factor of 10.)

Also, we have a phase angle in this case:

$\angle G_2(i\omega) \approx 90^\circ$

Case 3: $G_2(s)$: $\omega = \omega^z$

$G_2(i\omega^z) = i + 1$
\(|G_2(i\omega^2)| = 20 \log_{10}(\sqrt{2}) \approx 3 \text{ dB}\)

\(\angle G_2(s) = 45^\circ\)

Fig. 2. Bode plot of \(G_2(s) = (\frac{s}{2} + 1)\). [1 pt]

For \(G_3(s) = \left( \frac{1}{s} \right)\):

\[G_3(i\omega) = \left( \frac{i}{i\omega} \right) = \left( \frac{1}{\omega} \right)\]

This corresponds, in decibels, to a slope of the magnitude plot of -20 dB/decade. (Notice in the above that when \(\omega\) increases by a factor of 10, the magnitude of \(G_3(i\omega)\) decreases by a factor of 10.)

Also, because it is a simply a negative imaginary number we have a phase angle:

\(\angle G_3(i\omega) = -90^\circ\)

Lastly for \(G_3(s)\): \(\omega = 1\)

\[G_3(i\omega^2) = |\frac{-i}{1}| = 1 = 0 \text{ dB}\]
For $G_4(s) = \left( \frac{1}{10+1} \right)$:

$$G_4(i\omega) = \left( \frac{1}{1+i\omega/10} \right)$$

Case 1: $\omega \ll \omega^p$

$$G_4(i\omega) \approx 1$$

$$|G_4(i\omega)| = 20\log_{10}(1) \approx 0 \text{ dB}$$

$$\angle G_4(i\omega) = 0^\circ$$

Case 2: $\omega \gg \omega^p$

$$G_4(i\omega) \approx \frac{1}{10} = -\frac{i\omega}{10}\omega$$

This corresponds, in decibels, to a slope of the magnitude plot of -20 dB/decade. (Notice in the above that when $\omega$ increases by a factor of 10, the magnitude of $G_4(i\omega)$ decreases by a factor of 10.)

Also, because it is a simply a negative imaginary number we have a phase angle:

$$\angle G_4(i\omega) = -90^\circ$$

Case 3: $G_4(s)$: $\omega = \omega^p$
\[ G_4(i\omega^p) = \frac{1}{1+i} = \frac{1-i}{2} \]

\[ |G_4(i\omega^p)| = 20\log_{10}\left(\frac{\sqrt{2}}{2}\right) \approx -3 \text{ dB} \]

\[ \angle G_4(s) = -45^\circ \]

Fig. 4. Bode plot of \( G_4(s) = \left(\frac{1}{s+1}\right) \). [1 pt]

Next, we focus on \( G_5(s) \):

\[ G_5(s) = \left(\frac{2}{s^2+2s+2}\right) = \left(\frac{1}{\left(\frac{s}{\sqrt{2}}\right)^2+s+1}\right) \]

From pg. 412, we see that:

\[ \omega_n = \sqrt{2} \quad \text{ and } \quad \zeta = \frac{\sqrt{2}}{2} \]

We can rewrite \( G_5(s) \) as:

\[ G_5(i\omega) = \frac{1}{1-(\frac{i\omega}{\sqrt{2^2}})^2+i\omega} \]

Case 1: \( \omega << \omega_n \)

\[ G_5(i\omega) \approx 1 \]

\[ |G_5(i\omega)|_{dB} = 0 \text{ dB} \quad \text{ and } \quad \angle G_5(i\omega) = 0^\circ \]
Case 2: $\omega \gg \omega_n$

$$G_5(i\omega) \approx -\left(\frac{\sqrt{2}}{\omega}\right)^2$$

This corresponds, in decibels, to a slope of the magnitude plot of -40 dB/decade. (Notice in the above that when $\omega$ increases by a factor of 10, the magnitude of $G_5(i\omega)$ decreases by a factor of 100.)

Also, we have a phase angle in this case:

$$\angle G_5(i\omega) \approx -180^\circ$$

Case 3: $\omega = \omega_n$

$$G_5(i\omega_n) = \frac{1}{i\sqrt{2}} = -\frac{i\sqrt{2}}{2} \quad \text{and} \quad \angle G_5(i\omega_n) = -90^\circ$$

For an exact plotting we will need:

$$|G_5(i\omega_n)|_{dB} = 20\log\left(\frac{1}{\sqrt{2}}\right) \approx -3 \text{ dB}$$

Note: Although this component is a second-order underdamped, complex conjugate pole, which often has a resonance peak, the damping ratio is very large and thus the peak is non-existent. See also Table 9.2 on pg. 414 for the peak magnitude in terms of $\zeta$ values.

Fig. 5. Bode plot of $G_5(s) = \frac{\frac{1}{s^2 + 2s + 2}}{2s^2 + 2s + 2} = \frac{(\frac{1}{\sqrt{2}s})^2 + 3 + 1}{\sqrt{2}s + 2 + 1}$. [1 pt]
Now add each plot together [2 pts]:

Fig. 6. Bode plot of $G(s) = \left( \frac{1}{10} \right) \left( \frac{s + 2}{s} \right) \left( \frac{1}{s + 10} \right) \left( \frac{2}{s^2 + 2s + 2} \right)$.

Check with MATLAB [3 pts]:
Fig. 7. MATLAB Bode plot of \( G(s) = \left( \frac{1}{10} \right) \left( \frac{1}{s+1} \right) \left( \frac{1}{s^2+1} \right) \left( \frac{1}{s+10} \right) \left( \frac{1}{s^{\sqrt{2}}+1} \right) \).

1 A = zpk(-2,[0,-10, roots([1 2 2])'],1)
2 opts = bodeoptions;
3 opts.Xlabel.FontSize = 15;
4 opts.Ylabel.FontSize = 15;
5 opts.TickLabel.FontSize = 15;
6 opts.Title.FontSize = 15;
7 h = bodeplot(A,opts);
Problem 2 [15/15 pts]

(a) The differential equation is given:

\[ J \ddot{\theta}_m(t) + b \dot{\theta}_m(t) = k t i_a(t). \]  

(1)

By Applying Laplace transform on both sides,

\[ L \{ J \ddot{\theta}_m(t) \} + L \{ b \dot{\theta}_m(t) \} = L \{ k t i_a(t) \} ; \]

\[ J \left( s^2 \Theta_m(s) - s \theta_m(0) - \dot{\theta}_m(0) \right) + b (s \Theta_m(s) - \theta_m(0)) = k t I_a(s). \]

assume zero initial conditions, i.e., \( \dot{\theta}_m(0) = \theta_m(0) = 0 \),

\[ J s^2 \Theta_m(s) + b s \Theta_m(s) = k t I_a(s). \]

Rearrange,

\[ \Theta_m(s) \left( J s^2 + b s \right) = k t I_a(s); \]

\[ \frac{\Theta_m(s)}{I_a(s)} = \frac{k_t}{(J s^2 + b s)} = \frac{k_t}{s (J s + b)} \]  [2 pts]

(b) Let the closed-loop transfer function be

\[ H(s) = \frac{\Theta_m(s)}{R(s)} , \]

then apply the block diagram algebraic manipulation law for feedback loop in Table 8.3,

\[ H(s) = \frac{K(s) \Theta_m(s)}{1 + K(s) \Theta_m(s)} \]

From part (a), we know:

\[ \frac{\Theta_m(s)}{I_a(s)} = \frac{k_t}{(J s^2 + b s)} = \frac{k_t}{s (J s + b)} . \]

And

\[ K(s) = k_p. \]

By substituting into Eqn. (2),

\[ H(s) = \frac{k_p \frac{k_t}{s}}{1 + k_p \frac{k_t}{s}} \]

Rearrange,

\[ H(s) = \frac{\Theta_m(s)}{R(s)} = \frac{k_p k_t}{J s^2 + b s + k_p k_t} \]  [2 pts]

(c) By substituting \( J = 2, b = 4, k_t = 5, k_p = 10 \) into

\[ H(s) = \frac{k_p k_t}{J s^2 + b s + k_p k_t} , \]
The transfer function becomes

\[ H(s) = \frac{50}{2s^2 + 4s + 50}. \]

Rearrange,

\[ H(s) = \frac{1}{(\frac{s}{5})^2 + \frac{2}{25}s + 1}. \]

\[ \Rightarrow H(j\omega) = \frac{1}{(\frac{j\omega}{5})^2 + \frac{2}{25}j\omega + 1} = \frac{1}{(1 - \omega^2) + \frac{2}{25}j\omega} \]

So we get

\[ \omega_n = 5 \text{ rad/sec}; \]
\[ \frac{2}{25} = \frac{2\zeta}{\omega_n} \]
\[ \Rightarrow \zeta = \frac{1}{5}. \]

1) when \( \omega << \omega_n \),

\[ H(j\omega) = 1 \]
\[ \Rightarrow |H(j\omega)|_dB = 0 \text{ dB}; \]
\[ \angle H(j\omega) = 0^\circ. \]

2) when \( \omega >> \omega_n \),

\[ H(j\omega) = -\frac{25}{\omega^2} \]
\[ \Rightarrow \text{slope} = -40 \text{ dB/dec} \]
\[ \angle H(j\omega) = -180^\circ. \]

3) when \( \omega = \omega_n = 5 \) rad/sec,

\[ H(j\omega_n) = -\frac{5}{2}j \]
\[ \Rightarrow |H(j\omega)|_dB \approx 7.96 \text{ dB}. \]
\[ \angle H(j\omega) = -90^\circ. \]

Accordingly, we can get [3 pts]:
We can also use Matlab for verification:
The Matlab code is also attached for reference:

```matlab
clear all
den = [2 4 50];
sys = tf(num, den);
w = logspace(-2, 3, 1000);
[mag, phase] = bode(sys, w);
magdb = 20 * log10(mag);

figure(1)
subplot(2, 1, 1)
semilogx(w, squeeze(magdb), 'LineWidth', 1.5), grid;
ylabel('Magnitude (dB)', 'FontSize', 12)
xlabel('\omega (rad/sec)', 'FontSize', 12)
subplot(2, 1, 2)
semilogx(w, squeeze(phase), 'LineWidth', 1.5), grid;
ylabel('Phase (deg)', 'FontSize', 12)
xlabel('\omega (rad/sec)', 'FontSize', 12)
```

(d) In this problem,

\[ r(t) = \frac{2}{u_1(t)} + \sin 10t \]

Since this is a linear system, we can consider the output \( \theta_m(s) \) to be the superposition of output induced by \( u_1(t) \) and \( u_2(t) \) respectively.

For input that looks like \( u(t) = A \sin(\omega t) \), we know that

\[ y_{ss}(t) = A |H(j\omega)| \sin(\omega t + \angle H(j\omega)) \]

Similarly, for inputs like \( u(t) = A \cos(\omega t) \), \( y_{ss}(t) = A |H(j\omega)| \cos(\omega t + \angle H(j\omega)) \).

1. Consider \( u_1(t) = 2 \), which can be written as

\[ u_1(t) = 2 \cos(0t + 0) \]

has a frequency \( \omega_1 = 0 \) rad/s. According to bode plot in Part (c), at \( \omega_1 = 0 \) rad/s, the magnitude of transfer function is

\[ 20 \log(|H(0)|) = 0 \text{ dB}, \]

\( i.e., |H(0)| = 1 \).

The phase angle of transfer function at this frequency is

\[ \angle H(0) = 0^\circ = 0 \text{ rad}. \]

So the output corresponding to \( u_1(t) \) is:

\[ y_{ss1}(t) = |H(0)| \cdot (2 \cos(0t + \angle H(0))) = (1) \cdot (2 \cos(0t + 0)) = 2. \]
Similarly, 
\[ u_2(t) = \sin(10t) \]
has a frequency \( \omega_2 = 10 \text{ rad/s} \). According to bode plot, at \( \omega_2 = 10 \text{ rad/s} \), the magnitude of transfer function is

\[
|H(10j)| = \left| \frac{50}{50 - 2\omega_2^2 + 4\omega_2 j} \right|
\]
\[
= \left( \frac{50}{\sqrt{(50-200)^2 + 40^2}} \right)
\]
\[
= 0.32.
\]

The phase angle of transfer function at \( \omega_2 = 10 \text{ rad/s} \) is

\[
\angle H(10j) = 180 + \tan^{-1}\left( \frac{4\omega_2}{50 - 2\omega_2^2} \right)
\]
\[
= 195^\circ
\]
\[
= -165^\circ
\]
\[
= -2.88 \text{ rad}
\]
\(|H(10j)|\) and \(\angle H(10j)\) can also be approximated using the bode plot, where

\[
20 \log |H(10j)| = -10 \text{ dB}
\]
\[\Rightarrow |H(10j)| = 10^{-\frac{10}{20}} = 0.32
\]

\(\angle H(10j) = -165^\circ = -2.88 \text{ rad}\)

So the output corresponding to the input \(u_2(t)\) is

\[
y_{ss2}(t) = |H(10j)| \sin(10t + \angle H(10j))
\]
\[= 0.32 \sin(10t - 2.88).
\]

So the steady state output is

\[
\theta_{ss}(t) = y_{ss1}(t) + y_{ss2}(t) = 2 + 0.32 \sin(10t - 2.88) \quad [3 \text{ pts}].
\]

(e) According to the definition of bandwidth, we need to find the frequency at which the magnitude of transfer function is 3 dB lower than the DC gain. This is shown in the bode plot as follows,

![Bode plot](image)

By estimating using our bode plot in Part (c), we can see the bandwidth frequency

\[
w_{bw} = 7.5 \text{ rad/sec} \quad [2 \text{ pts}].
\]

We can also verify this using Matlab with command `bandwidth(sys)`.

```matlab
clear all
```
(f) According to the statement,

\[ \Omega_m(s) = \mathcal{L}\{\Omega_m(t)\} \]
\[ = \mathcal{L}\{\dot{\Theta}_m(t)\} \]
\[ = s\Theta_m(s) - \theta_m(0) \]

So that

\[ \frac{\Omega_m(s)}{R(s)} = s \frac{\Theta_m(s)}{R(s)} \]

So that the bode plot of \( \frac{\Omega_m(s)}{R(s)} \) is the bode plot of \( \frac{\Theta_m(s)}{R(s)} \) superimposed by the bode plot of \( s \).

Bode plot of \( \frac{\Omega_m(s)}{R(s)} \),
By adding the two together [3 pts],

We can also verify this through Matlab,
Matlab code is also attached for reference,

clear all
num1=[1 0];
den1=[1];
sys1=tf(num1,den1);
w=logspace(-2,3,1000);
[mag1,phase1]=bode(sys1,w);
magdb1=20*log10(mag1);

num2=[50];
den2=[2 4 50];
sys2=tf(num2,den2);
[mag2,phase2]=bode(sys2,w);
magdb2=20*log10(mag2);

num3=[50 0];
den3=[2 4 50];
sys3=tf(num3,den3);
[mag3,phase3]=bode(sys3,w);
magdb3=20*log10(mag3);

figure(4)
subplot(2,1,1)
semilogx(w,squeeze(magdb1),'--b',w,squeeze(magdb2),'--g',w,squeeze(magdb3)...
    ,'-r','LineWidth',1.5),grid;
ylabel('Magnitude (dB)','Fontsize',12)
xlabel('
omega (rad/sec)','Fontsize',12)
legend('s','\Theta_m(s)/R(s)', 's\Theta_m(s)/R(s)')

subplot(2,1,2)
semilogx(w,squeeze(phase1),'--b',w,squeeze(phase2),'--g',w,squeeze(phase3),'-r','LineWidth',1.5),grid;
ylabel('Phase (deg)','Fontsize',12)
xlabel('
omega (rad/sec)','Fontsize',12)
legend('s','\Theta_m(s)/R(s)', 's\Theta_m(s)/R(s)')