Problem 1 [10/10 pts]

\[ G(s) = \Theta(s)/I(s) \] and we are interested in the closed-loop pole locations as the parameter \( k \) is varied.

\[
\frac{\Theta(s)}{I(s)} = \bar{k} \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s - \omega_n^2}
\]

We will use \( \bar{k}, \zeta, \) and \( \omega_n \) values for our system found from the hanging configuration so that the transfer function is:

\[
\frac{\Theta(s)}{I(s)} = 0.248 \frac{(6.23)^2}{s^2 + 2(0.039)(6.23)s - (6.23)^2}
\]

Our pole locations are:

\[ s_{1,2} = 5.99, -6.48 \]

Rule 9.1 - The root locus starts at the open-loop poles \( (k = 0) \).

By Rule 9.2 we know for our case (no zeros), as \( k \to \infty \), the root locus will grow unbounded for both poles.
Rule 9.3 gives the asymptotes along which the root locus will grow unbounded.

\[
\theta_n = \frac{180^\circ + n \cdot 360^\circ}{n_z - n_p} \\
\theta_0 = \frac{180^\circ + (0) \cdot 360^\circ}{-2} = -90^\circ \\
\theta_1 = \frac{180^\circ + (1) \cdot 360^\circ}{-2} = -270^\circ = 90^\circ
\]

Next, Rule 9.4 gives where the asymptotes intersect the real axis.

\[
s_{int} = \frac{\sum_{i=1}^{n_z} z_i - \sum_{i=1}^{n_p} p_i}{n_z - n_p} \\
s_{int} = \frac{0 - (5.99 - 6.48)}{-2} = -0.245
\]

By Rule 9.5 we recognize that on the real axis the root locus exists to the left of an odd number of poles and zeros.

Lastly, Rule 9.6 gives the angle at which a branch of the root locus leaves a pole \(p_j\)

\[
\angle(s - p_j) = \sum_{i=1}^{n_z} \angle(p_j - z_i) - \sum_{i=1, i \neq j}^{n_p} \angle(p_j - p_i) - 180^\circ \\
p_{-6.48} = 0 - [180^\circ + 0] - 180^\circ = 0^\circ \\
p_{5.99} = 0 - [0^\circ] - 180^\circ = -180^\circ = 180^\circ
\]
We can make a complete root locus sketch [3 pts, must show steps]

Thus, we can see that a proportional controller will be able to attain a stable closed-loop system.

We will use the magnitude criterion to find the minimum $k$ which will give us a stable closed loop system:

\[
    k = \frac{1}{|G(s)|}
\]

\[
    |G(s)|_{s=0} = \hat{k} \prod_{i=1}^{n_z} \frac{|s - z_i|}{\prod_{i=1}^{n_p} |s - p_i|}
\]

\[
    |G(s)|_{s=0} = (0.248)(6.23^2) \frac{1}{|6.48| |5.99|} = 0.248
\]

Therefore:

\[
    k = \frac{1}{|G(s)|} = \frac{1}{0.248} = 4.0325
\]

For any $k > 4.0325$ we will have a stable closed loop system [3 pts].

To determine if p-control is sufficient to meet our design criteria of $\%OS < 25\%$ and $t_r < 1.5$ sec, we want to map this region in the complex plane.

We use the rise time approximation formula:
\[
t_r \approx \frac{1.8}{\omega_n} \rightarrow \omega_n > 1.2
\]

And using Fig. 9.24 on p. 361 of Goodwine, we find that the damping ratio must be:

\[\zeta > 0.4.\]

Plotting, in Matlab [2 pts]

We can see from the trajectories of the root locus branches that there does not exist a value of \(k\) such that all the closed-loop poles fall within the design criteria region simultaneously (unshaded region) [2 pts].

```matlab
G = zpk([], [5.99, -6.48], 0.248*6.23^2)
rltool
```
Problem 2 [25/25 pts]

We choose a point within the specified region which satisfies the design criteria, knowing that:

\[ s_{desired} = -\zeta \omega_n \pm j \omega_n \sqrt{1 - \zeta^2} = -\zeta \omega_n \pm j \omega_d \]

We choose \( \zeta = 0.7 \) and \( \omega_n = 2 \), so that our desired point is: \( s \approx -1.4 + 1.43j \) which meets our design criteria [2 pts].

We want to design a lead controller to make the root locus go through the desired point in the complex plane. A lead controller has the form:

\[ K_{lead}(s) = k \frac{s + z}{s + p}. \]

Then for the root locus analysis, we define:

\[ G(s) = \frac{s + z}{s + p} \frac{0.248(6.23)^2}{s^2 + 2(0.039)(6.23)s - (6.23)^2} \]

And design \( k \) such that \( k \in (0+, \infty) \).

We choose a fixed \( z \)-value, and we need to adjust \( p \) to make the root locus go through our desired point. Some rules of thumb include that \( z \) can be placed at \( z = \omega_n \); however, this places the zero in between the poles of our plant. We chose \( z = 8 \) to the left of the stable plant pole.

\[ \angle G(s) = \sum_{i=1}^{n_z} \angle (s - z_i) - \sum_{i=1}^{n_p} \angle (s - p_i) = 180^\circ + 360^\circ (n - 1) \]

We find the angles referring to Fig. 1.
Then,
\[ \theta_2 - (\theta_1 + \theta_3 + \theta_4) = 180^\circ + 360^\circ(n - 1) \]
where \( \theta_1 \) is an unknown function of \( p \), so we solve for \( \theta_1 \):

\[
\theta_1 = \theta_2 - \theta_3 - \theta_4 - 180^\circ - 360^\circ(n - 1) \\
= \tan^{-1}\left(\frac{\omega_d}{z - \zeta \omega_n}\right) - \tan^{-1}\left(\frac{\omega_d}{6.48 - \zeta \omega_n}\right) - \left(180 + \tan^{-1}\left(\frac{\omega_d}{-(\zeta \omega_n + 5.99)}\right)\right) - 180 - 360(n - 1)
\]
So

\[ \theta_1 \approx 7.45^\circ. \]

From trig,
\[
\tan \theta_1 = \frac{\omega_d}{p - \zeta \omega_n} \\
p = 12.33
\]

Our controller has the form

\[ K_{lead}(s) = k \frac{s + 8}{s + 12.33} \]

[10 pts].

where \( k \) is still an unknown parameter.

Before finding \( k \), we can plot the root locus.

By Rule 9.1, the root locus starts at the open-loop poles (\( k=0 \)).
Rule 9.2, \( n_z = 1 \) and \( n_p = 3 \), therefore, the root locus will have one bounded branch and 2 unbounded branches.

Rule 9.3, calculate the asymptotes

\[
\theta_n = \frac{180^\circ + n360^\circ}{n_z - n_p}
\]

to find that \( \theta_0 = -90^\circ \) and \( \theta_1 = +90^\circ \)

Rule 9.4, Asymptote intersection with real axis:

\[
s_{\text{int}} = \frac{\sum_{i=1}^{n_z} z_i - \sum_{i=1}^{n_p} P_i}{n_z - n_p}
\]

\[
s_{\text{int}} = \frac{-z - (-p - 6.48 + 5.99)}{-2} = -2.41
\]

Rule 9.5, the root locus exists on the real axis to the left of an odd number of poles and zeros.

Rule 9.6, we do not need because all of the open loop poles and zeros are on the real axis.

Rule 9.7, Break away points from the real axis can be calculated where we let \( G(s) = N(s)/D(s) \)

\[
\frac{dD(s)}{ds}N(s) - \frac{dN(s)}{ds}D(s) = 0
\]  \( \text{(1)} \)

The components of this equation are:

\[
N(s) = 0.248(6.23^2)(s + 8)
\]

\[
\frac{dN(s)}{ds} = 0.248(6.23^2)
\]

\[
D(s) = (s + 12.33)(s + 6.48)(s - 5.99)
\]

\[
\frac{dD(s)}{ds} = 3s^2 + 12.82(2)s - 32.7735
\]

So that we find the roots of Eq. 1 using Mathematica:

\[
s = -8.52279 \pm 2.58162i , -1.364
\]

Then, \( s = -1.364 \) is the only breakaway point that makes sense.

We can plot our root locus by hand [5 pts, must show steps]:
Solve for $k$ such that the root locus passes through a given location. We will use the magnitude criterion to find the minimum $k$ which will give us a stable closed loop system:

$$k = \frac{1}{|G(s)|}$$

where

$$|G(s)|_{s=0} = \hat{k} \prod_{i=1}^{n_z} \frac{|s - z_i|}{|s - p_i|}$$

$$G(s) \approx 0.1483 \rightarrow k \approx 6.73 \quad [5 \text{ pts}]$$

Then, the controller by design that will achieve the desired design specifications is:

$$K_{lead}(s) = 6.73 \frac{s + 8}{s + 12.33}$$

We verify our sketch using Matlab `rltool` [3 pts],
We want to evaluate whether the design criteria is actually satisfied, first by evaluating our linear system.

When we plot the step response of our system with the controller we designed, we see that:
Fig. 2. $k=6.73$, $z=8$, $p=12.33$ as designed following the root locus specifications.

The overshoot is 4% and the rise time is 1.5 seconds which meets our design criteria. The steady-state value is off; for a step reference input, the output should fall to 1, but we did not set a design criteria for that parameter.
Problem 3 [10/10 pts]

We now want to study the full nonlinear system for the inverted pendulum.

Based on homework 9, we can determine the constant parameters for the plant in the provided simulink model:

\[
\begin{align*}
\frac{g}{l} &= \omega_n^2 = 6.23^2 \\
\frac{b}{ml^2} &= 2\zeta\omega_n = 2(0.039)(6.23) = 0.4859 \\
\frac{k_r}{ml^2} &= \bar{k}\omega_n^2 = 9.625
\end{align*}
\]

These are what we plug into the nonlinear plant diagram.

Using our design parameters we get: [5 pts]

This has an overshoot of: \( OS = \frac{110 - 80}{80} \rightarrow \% OS = 38\% \) and a rise time of 0.25 seconds. This does not meet our overshoot design criteria [5 pts].

To bring the response closer to our desired gain values, we increase the gain and shift the compensator pole and zero. We tried gain \( k = 15 \), \( z = 7 \), and \( p = 20 \).
This decreases the overshoot so that $\%OS = 17\%$ and did not effect the rise time. Our design criteria is satisfied.

```matlab
1 close all
2 clear all
3
4 % Construct the system transfer function
5 G=tf([0.248*6.23^2],[1 2*0.039*6.23 -6.23^2])
6
7 % Assign value for the compensator pole and zero
8 K = 6.73;
9 z = 8; % zero
10 p = 12.33; % pole
```
% Construct the lead compensator transfer function
K_lead=tf([1 z],[1 p]);
rltool
sim('Problem3sim')

% Plot system response from simulink data
figure(2)
plot(Time,Reference,'-b',Time,System_Response,'-r','LineWidth',1.5)
legend('Reference','System Response')
xlabel('Time [sec]')
ylabel('Theta [deg]')