EFFECTIVELY BOUNDED IDEMPOTENT GENERATION OF CERTAIN 2 × 2 SINGULAR MATRICES BY IDEMPOTENT MATRICES OVER REAL QUADRATIC NUMBER RINGS

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1. Introduction

Let \( A \) be an integral domain, and let \( M_n(A) \) denote the set of all \( n \times n \) matrices with entries in \( A \). A matrix \( M \in M_n(A) \) is singular if the determinant of \( M \) is zero in \( A \). A matrix \( M \in M_n(A) \) is called an idempotent matrix if \( M^2 = M \). Note that the \( n \times n \) identity matrix \( 1_n \) is idempotent. It is obvious that every idempotent matrix \( M \neq 1_n \) is singular. So it is natural to ask whether or not every singular matrix can be written as a product of idempotent matrices, which can be viewed as an analogue of finite generation in group theory. Several works have been devoted to this problem. For a field \( A \), Erdos [5] proved that every singular matrix with entries in \( A \) is a product of idempotent matrices. For \( A \) being a division ring or a Euclidean ring, Laffey [7] showed that every singular matrix over \( A \) is a product of idempotents over \( A \). Very recently Cossu and Zanardo [3] considered a similar problem for a certain set of \( 2 \times 2 \) singular matrices over real quadratic number rings; more precisely Cossu and Zanardo proved that if \( A \) is a real quadratic number ring, then every matrix over \( A \) of the form

\[
\begin{pmatrix}
x & y \\ 0 & 0
\end{pmatrix}
\]

for arbitrary elements \( x, y \in A \) can be written as a product of idempotents.

In this paper, we consider the following analogue of bounded generation from group theory in the setting of singular matrices.

Problem 1.1. Let \( A \) be an integral domain. Describe the largest set of singular matrices over \( A \), each of whose members can be written as a product of a bounded number of idempotent matrices over \( A \). (For more precise definition of bounded generation in the set of singular matrices, see Definition 2.9.)

In contrast to the problem of finite generation of singular matrices by idempotent matrices, there are not many works in literature devoted to studying the above problem. In [5] and [6], Erdos and Howie showed that every \( 2 \times 2 \) singular matrix with entries from \( \mathbb{Q} \) can be written as a product of two idempotent matrices over \( \mathbb{Q} \). The conclusion no longer holds for \( 2 \times 2 \) singular matrices with entries in \( \mathbb{Z} \). For \( n \geq 3 \), Laffey [8] proved that every \( n \times n \) singular matrix with entries in \( \mathbb{Z} \) can be written as a product of \( 36n + 217 \) idempotent matrices with entries in \( \mathbb{Z} \). Lenders and Xue [9] improved Laffey’s result which shows that every \( n \times n \) singular matrix with entries in \( \mathbb{Z} \) can be written as a product of \( 2n + 1 \) idempotent matrices with entries in \( \mathbb{Z} \) for each \( n \geq 3 \).

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In this paper, we consider Problem 1.1 for a certain set of $2 \times 2$ singular matrices over quadratic number rings which can be viewed as a natural generalization of the results by Cossu and Zanardo [3]. More precisely we prove the following result.

**Theorem 1.2.** (see Theorem 3.8) Let $O_k$ be the ring of integers of a real quadratic number field $k = \mathbb{Q}(\sqrt{\alpha})$, where $\alpha$ is a positive square-free integer. Then every matrix of the form $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$ for elements $x, y \in O_k$ can be written as a product of at most 15 idempotent matrices with entries in $O_k$.

**Remark 1.3.** In [2, Theorem 6.1], Cohn proved that if $O$ is the ring of integers of the imaginary quadratic number field $\mathbb{Q}(\sqrt{-\alpha})$, where $\alpha$ is a positive square-free integer such that $\alpha \neq 1, 2, 3, 7, 11$, then there exists an invertible $2 \times 2$ matrix with entries in $O$ that cannot be written as a product of elementary matrices with entries in $O$. For such a domain $O$, Cossu and Zanardo (see [4, Proposition 3.4]) showed that there exists a singular $2 \times 2$ matrix with entries in $O$ that cannot be written as a product of idempotent matrices. In view of this, we only study Problem 1.1 the rings of integers of real quadratic number fields $\mathbb{Q}(\sqrt{\alpha})$, where $\alpha$ is a positive square-free integer.

Theorem 1.2 is a simplified version of Theorem 3.8 proved in Section 3 which implies that for every matrix of the form $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$ for elements $x, y \in O_k$, there are 19 invertible linear transformations induced by elements in $SL_2(O_k)$ that are needed to convert the original matrix into a product of at most 15 idempotent matrices with entries in $O_k$. Thus Theorem 3.8 also contains an effective algorithm how to convert a matrix of the form $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$ for elements $x, y \in O_k$ into a product of a bounded number of idempotent matrices.

The proof of Theorem 1.2 follows the strategy of that of the main theorem of Cossu and Zanardo (see [3, Theorem 3.2], but there is one key difference between these two proofs. In the proof of Theorem 3.2 in [3], Cossu and Zanardo exploited the Euclidean algorithm for $\mathbb{Z}$ to deduce the fact that for each matrix $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$ for elements $x, y \in O_k$, there exist an integer $h \in \mathbb{Z}$ and an element $\beta \in O_k$ such that if $\begin{pmatrix} h & \beta \\ 0 & 0 \end{pmatrix}$ is a product of idempotent matrices, so does $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$. Since applying the Euclidean algorithm for a couple of integers can generate an arbitrarily long sequence of divisions, one can not obtain a bounded number of transformations needed to convert $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$ into $\begin{pmatrix} h & \beta \\ 0 & 0 \end{pmatrix}$, which results in a weaker conclusion than Theorem 1.2 in our paper. In order to bound the number of transformations used to convert $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$ for $x, y \in O_k$, into $\begin{pmatrix} h & \beta \\ 0 & 0 \end{pmatrix}$ for some $h \in \mathbb{Z}$ and $\beta \in O_k$, we introduce a new approach in which Dirichlet’s theorem on primes in arithmetic progressions will be exploited (see Lemmas 3.5 and 3.6.)

The structure of our paper is as follows. In Section 2, we introduce some basic notions and notation that will be used throughout the paper. In Section 3, we prove Theorem 1.2 (see Theorem 3.8)–our main theorem.

### 2. Basic notions and notation

In this section, we introduce some basic notions and notation which will be used throughout this paper. Throughout this subsection, let $\mathcal{A}$ denote an integral domain, and let $M_2(\mathcal{A})$ be the set of all $2 \times 2$ matrices with entries in $\mathcal{A}$, and let $SL_2(\mathcal{A})$ be the set of all $2 \times 2$ matrices with entries in $\mathcal{A}$ of determinant 1.

For $x, y \in \mathcal{A}$, denote by $[x \ y]$ the matrix $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$.
For an element $a \in A$, write
\[
a_{1,1} = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}, \\
a_{1,2} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \\
a_{2,1} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \\
a_{2,2} = \begin{pmatrix} 0 & -1 \\ -1 & a \end{pmatrix}.
\]

**Definition 2.1.** (transformation $\rightarrow a_{1,1}$)
Let $A, B$ be $2 \times 2$ matrices in $\mathcal{M}_2(A)$. We write
\[
A \rightarrow a_{1,1} B
\]
for some element $a \in A$ if and only if
\[
B = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}^{-1} A \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}.
\]
That is, $\begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}$ conjugates $A$ to $B$.
We also use the notation
\[
A^{a_{1,1}} = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}^{-1} A \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}.
\]
It is obvious that $A \rightarrow a_{1,1} B$ if and only if
\[
B = A^{a_{1,1}} = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix} A \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}.
\]

**Definition 2.2.** (transformation $\rightarrow a_{1,2}$)
Let $A, B$ be $2 \times 2$ matrices in $\mathcal{M}_2(A)$. We write
\[
A \rightarrow a_{1,2} B
\]
for some element $a \in A$ if and only if
\[
B = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{-1} A \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.
\]
That is, $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ conjugates $A$ to $B$.
We also use the notation
\[
A^{a_{1,2}} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{-1} A \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.
\]
It is obvious that $A \rightarrow a_{1,2} B$ if and only if
\[
B = A^{a_{1,2}} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.
\]

**Definition 2.3.** (transformation $\rightarrow a_{2,1}$)
Let $A, B$ be $2 \times 2$ matrices in $\mathcal{M}_2(A)$. We write
\[
A \rightarrow a_{2,1} B
\]
for some element $a \in A$ if and only if
\[
B = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}^{-1} A \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}.
\]
That is, \( \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \) conjugates \( A \) to \( B \).

We also use the notation

\[
A^{a,1} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}^{-1} A \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}.
\]

It is obvious that \( A \rightarrow_{a,1} B \) if and only if

\[
B = A^{a,1} = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}.
\]

**Definition 2.4.** (transformation \( \rightarrow_{a,2} \))

Let \( A, B \) be \( 2 \times 2 \) matrices in \( M_2(A) \). We write

\[
A \rightarrow_{a,2} B
\]

for some element \( a \in A \) if and only if

\[
B = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}^{-1} A \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}.
\]

That is, \( \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix} \) conjugates \( A \) to \( B \).

We also use the notation

\[
A^{a,2} = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}^{-1} A \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}.
\]

It is obvious that \( A \rightarrow_{a,2} B \) if and only if

\[
B = A^{a,2} = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}.
\]

The next results are obvious.

**Lemma 2.5.** Let \( x, y \) be elements in an integral domain \( A \). Then

\[
[x \ y] \rightarrow_{0,2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} [-y \ x].
\]

**Lemma 2.6.** Let \( a, b, u \) be elements in an integral domain \( A \). Then

\[
\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}^{u,1} = \begin{pmatrix} 0 & -1 \\ 1 & u \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} u & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -bu & -b \\ u(a + bu) & a + bu \end{pmatrix}.
\]

In particular,

\[
\begin{pmatrix} a & 0 \\ 1 & 0 \end{pmatrix}^{(-a),1} = [a \ -1],
\]

and

\[
\begin{pmatrix} a & 0 \\ -1 & 0 \end{pmatrix}^{(a),1} = [-a \ 1].
\]

**Definition 2.7.** Let \( M \) and \( A_1, A_2, \ldots \) be \( 2 \times 2 \) matrices in \( M_2(A) \). For each \( n \geq 1 \), we inductively define the \( 2 \times 2 \) matrix \( M^{A_1 A_2 \cdots A_n} \) as follows. For \( n = 1 \), set \( M_1 = M^{A_1} = A_1^{-1} M A_1 \), and for each \( n \geq 2 \),

\[
M_n = M^{A_1 A_2 \cdots A_n} = M_{n-1}^{A_n} = A_n^{-1} M_{n-1} A_n.
\]

The following result is obvious.

**Lemma 2.8.** Let \( A \) be an integral domain. Let \( M_1, \ldots, M_n \) be \( 2 \times 2 \) matrices with entries in \( A \), and let \( A_1, \ldots, A_k \) be \( 2 \times 2 \) matrices in \( SL_2(A) \). Then

\[
(M_1 \cdots M_n)^{A_1 \cdots A_k} = \left( M_1^{A_1 \cdots A_k} \right) \left( M_2^{A_1 \cdots A_k} \right) \cdots \left( M_n^{A_1 \cdots A_k} \right).
\]
The main aim in this paper is to study the following notion for the set of $2 \times 2$ matrices over quadratic number fields.

**Definition 2.9.** (effectively bounded idempotent generation)

Let $\mathcal{A}$ be an integral domain. A collection of $2 \times 2$ matrices $(M_i)_{i \in I}$ in $\mathcal{M}_2(\mathcal{A})$ is said to admit an **effectively bounded idempotent generation** over $\mathcal{A}$ if there exist positive integers $m$ and a nonnegative integer $n$ such that for every $i \in I$, there exist matrices $E_1, \ldots, E_{s_i}$ in $\mathcal{SL}_2(\mathcal{A})$ with $s_i \leq n$ and there exist idempotent matrices $A_1, \ldots, A_{r_i}$ in $\mathcal{M}_2(\mathcal{A})$ with $r_i \leq m$ for which

$$M_i^{E_1 \cdots E_{s_i}} = A_1 A_2 \cdots A_{r_i}.$$  

Let $\text{EBIG}^n_m(\mathcal{A})$ denote the largest subset of $\mathcal{M}_2(\mathcal{A})$ whose members satisfy the above condition. Then we can write $(M_i)_{i \in I} \subset \text{EBIG}^n_m(\mathcal{A})$.

**Remark 2.10.**

(i) Note that each idempotent matrix with entries in $\mathcal{A}$ is an element in $\text{EBIG}^0_1(\mathcal{A})$.

(ii) By Lemma 2.8, if $(M_i)_{i \in I} \subset \text{EBIG}^n_m(\mathcal{A})$, then it follows from (1) that for each $i \in I$,

$$M_i = (A_1 \cdots A_{r_i})^{E_{s_1} \cdots E_{s_i}^{-1}}$$

$$= \begin{pmatrix} E_{s_1}^{-1} \cdots E_{s_i}^{-1} \\ A_1^{E_{s_1}^{-1} \cdots E_{s_i}^{-1}} \\ A_2^{E_{s_1}^{-1} \cdots E_{s_i}^{-1}} \cdots \ A_{r_i}^{E_{s_1}^{-1} \cdots E_{s_i}^{-1}} \end{pmatrix}.$$  

Since each $A_j^{E_{s_1}^{-1} \cdots E_{s_i}^{-1}}$ is idempotent and $r_i \leq m$, every matrix $M_i$ in the sequence $(M_i)_{i \in I}$ can be written as a product of at most $m$ idempotent matrices.

(iii) Definition 2.9 signifies that after applying at most $n$ invertible linear transformations induced by elements in $\mathcal{SL}_2(\mathcal{A})$, every matrix $M_i$ in the sequence $(M_i)_{i \in I}$ can be converted into a product of at most $n$ idempotent matrices, where $m$ is independent of the $M_i$.

The next two lemmas are obvious.

**Lemma 2.11.** Let $\mathcal{A}$ be an integral domain. Let $M \in \text{EBIG}^n_m(\mathcal{A})$ for some positive integers $n, m$, and let $A_1, \ldots, A_{r_i}$ be $2 \times 2$ matrices in $\mathcal{SL}_2(\mathcal{A})$. Then

$$M^{A_1 \cdots A_{r_i}} \in \text{EBIG}^{n+\ell}_m(\mathcal{A}).$$

**Lemma 2.12.** Let $\mathcal{A}$ be an integral domain, and let $M \in \text{EBIG}^n_m(\mathcal{A})$ for some positive integer $m$ and some nonnegative integer $n$. Then $M \in \text{EBIG}^{n+r}_m(\mathcal{A})$ for any nonnegative integers $r, s$.

**Proof.** Note that the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is an idempotent matrix, and also an element in $\mathcal{SL}_2(\mathcal{A})$. Hence the lemma follows immediately.

**Lemma 2.13.** Let $\mathcal{A}$ be an integral domain. Let $M \in \text{EBIG}^n_{m_1}(\mathcal{A})$ and $N \in \text{EBIG}^n_{m_2}(\mathcal{A})$, where $m_1, m_2$ are positive integers, and $n_1, n_2$ are nonnegative integers. Then $MN \in \text{EBIG}^{\min(n_1,n_2)}_{m_1+m_2}(\mathcal{A})$.

**Proof.** By assumption, there exist elements $E_1, \ldots, E_{s_1}, E_1, \ldots, E_{s_2}$, $r_1, r_2$ integers with $0 \leq s_1 \leq n_1$ and $1 \leq r_1 \leq m_1$ such that

$$M^{E_1 \cdots E_{s_1}} = A_1 \cdots A_{r_1},$$

$$N^{E_{s_1} \cdots E_{s_2}^{-1}} = A'_{1} A'_{2} \cdots A'_{r_2}.$$  

Multiplying both sides of the above equation by the product $E_{s_1}^{-1} \cdots E_1^{-1}$, we deduce from Lemma 2.8 that

$$M = \begin{pmatrix} E_{s_1}^{-1} \cdots E_1^{-1} \\ A_1^{E_{s_1}^{-1} \cdots E_1^{-1}} \\ A_2^{E_{s_1}^{-1} \cdots E_1^{-1}} \cdots \ A_{r_i}^{E_{s_1}^{-1} \cdots E_1^{-1}} \end{pmatrix}$$

$$= A'_1 A'_2 \cdots A'_{r_2},$$

where

$$A'_i = A_i^{E_{s_1}^{-1} \cdots E_1^{-1}}.$$
for each \(1 \leq i \leq r_1\). Note that each \(A_i'\) is an idempotent matrix.

On the other hand, since \(N \in \text{EBIG}_{n_2}^{m_2}(A)\), there exist elements \(F_1, \ldots, F_{s_2} \in \text{SL}_2(A)\) and idempotent matrices \(B_1, \ldots, B_{r_2}\), where \(s_2, r_2\) are integers with \(0 \leq s_2 \leq n_2\) and \(1 \leq r_2 \leq m_2\) such that

\[
N^{F_1 \cdots F_{s_2}} = B_1 \cdots B_{r_2}.
\]

By (2) and (3), we deduce from Lemma 2.8 that

\[
(MN)^{F_1 \cdots F_{s_2}} = M^{F_1 \cdots F_{s_2}} N^{F_1 \cdots F_{s_2}} = \left(A_1^{F_1 \cdots F_{s_2}} \right) \cdots \left(A_{r_1}^{F_1 \cdots F_{s_2}} \right) B_1 \cdots B_{r_2}.
\]

Since the \(A_i^{F_1 \cdots F_{s_2}}\) are idempotent, and \(r_1 + r_2 \leq m_1 + m_2\), \(s_2 \leq n_2\), the above equation implies that \(MN \in \text{EBIG}_{m_1 + m_2}^{n_1}(A)\). Exchanging the roles of \(M, N\), we also obtain that \(MN \in \text{EBIG}_{m_1 + m_2}^{n_1}(A)\), and thus \(MN \in \text{EBIG}_{\text{min}(n_1, n_2)}^{m_1 + m_2}(A)\).

\[\square\]

3. Effectively bounded idempotent generation

In this section, we prove our main theorem (see Theorem 3.8). We begin by proving several results that will be needed in the proof of our main theorem.

**Lemma 3.1.** Let \(A\) be an integral domain such that there exists a positive integer \(n_0\) for which every matrix in \(\text{SL}_2(A)\) is a product of at most \(n_0\) elementary matrices. Set

\[
S = \{ (x, y) \in A^2 \mid \text{there exist } z, w \in A \text{ such that } \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \text{SL}_2(A) \}
\]

and

\[
I = \{ [x \ y] \mid (x, y) \in S \}
\]

Then \(I \subset \text{EBIG}_{2n_0+2}^{2n_0}(A)\).

**Proof.** We will use a similar argument as in the proofs of Lemmas 2.2 and 2.3 in Cossu and Zanardo [3].

Take any \([x \ y] \in I\). By assumption, there exists \(z, w \in A\) such that \(\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \text{SL}_2(A)\), and thus \(xw - yz = 1\). Hence \(\begin{pmatrix} x & z \\ y & w \end{pmatrix} \in \text{SL}_2(A)\). By assumption, there exist elements \(q_0, q_1, \ldots, q_{2n_0-1}\) in \(A\) such that

\[
\begin{pmatrix} x & z \\ y & w \end{pmatrix} = \begin{pmatrix} 1 & q_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & q_{2n_0-2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_{2n_0-1} & 1 \end{pmatrix}.
\]

For each \(1 \leq i \leq 2n_0 - 2\), define

\[
r_{i+2} = r_i - r_{i+1}q_{i+1},
\]

where we set

\[
\begin{align*}
r_1 &= x - yq_0, \\
r_2 &= y - r_1q_1.
\end{align*}
\]

On setting \(r_{-1} = x\), \(r_0 = y\), and \(r_{2n_0} = 0\), and following the same arguments as in Lemma 2.2 in [3], we deduce that

\[
r_i = q_{i+1}r_{i+1} + r_{i+2}
\]

for each integer \(-1 \leq i \leq 2n_0 - 2\).

By the above equation, one can verify that for each \(-1 \leq k \leq 2n_0 - 3\),

\[
[r_k \ r_{k+1}]^{-q_{k+1}}_{2 \times 1} = \begin{pmatrix} 1 & 0 \\ q_{k+1} & 0 \end{pmatrix} [r_{k+2} \ r_{k+1}],
\]

and thus
and
\[
[r_{k+2} \ r_{k+1}]_{1,2}^{(-q_{k+2})} = [r_{k+2} \ r_{k+3}].
\]

Thus for each \(-1 \leq k \leq 2n_0 - 3\), we deduce from Lemma 2.13 that
\begin{equation}
[\begin{array}{c}
[r_k \ r_{k+1}]_{1,2}^{(-q_{k+2})}
\end{array}]
\end{equation}

Applying (5) \(n_0\) times repeatedly for odd integers \(k = -1, 1, 3, \ldots, 2n_0 - 3\), and using Lemma 2.13, we deduce that
\[
[x \ y]_{1,2}^{(-q_0)_{1,2} \ldots (-q_{2n_0 - 2})_{1,2} (-q_{2n_0 - 1})_{1,2}} = [r_{-1} \ r_0]_{1,2}^{(-q_0)_{1,2} \ldots (-q_{2n_0 - 2})_{1,2} (-q_{2n_0 - 1})_{1,2}}
\]
\begin{equation}
= \prod_{0 \leq h \leq n_0 - 1} \begin{pmatrix} 1 & 0 \\ q_{2h} & 0 \end{pmatrix} [r_{2n_0 - 1} \ r_{2n_0}].
\end{equation}

(Here the notation \(\prod_{0 \leq h \leq j_0} \alpha_h\) represents the product \(\alpha_{i_0}\alpha_{i_0 + 1} \ldots \alpha_{j_0}\) in exactly this ordering of terms \(\alpha_h\) appearing in the product.)

Note that
\[
[r_{2n_0 - 1} \ r_{2n_0}] = [r_{2n_0 - 1} \ 0] = \begin{pmatrix} 1 & -1 \\ 0 & 1 - r_{2n_0 - 1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Since \(\begin{pmatrix} 1 & -1 \\ 0 & 1 - r_{2n_0 - 1} \end{pmatrix}\), \(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\), and \(\begin{pmatrix} 1 & 0 \\ q_{2h} & 0 \end{pmatrix}^{(-q_{2h + 1})_{1,2} \ldots (-q_{2n_0 - 2})_{1,2} (-q_{2n_0 - 1})_{1,2}}\) are idempotent for each \(0 \leq h \leq n_0 - 1\), we deduce from (6) that \([x \ y] \in \text{EBIG}_{n_0 + 2}(\mathcal{A})\), which verifies the lemma.

\[\square\]

**Corollary 3.2.** Let \(k = \mathbb{Q}(\sqrt{\alpha})\), where \(\alpha\) is a positive square-free integer, and let \(\mathcal{O}_k\) be its ring of integers. Set
\[
\mathcal{S} = \{(x, y) \in \mathcal{O}_k^2 \mid \text{there exist } z, w \in \mathcal{O}_k \text{ such that } (x \ y) \in \text{SL}_2(\mathcal{O}_k)\},
\]
and
\[
\mathcal{I} = \{(x \ y) \mid (x, y) \in \mathcal{S}\}
\]

Then \(\mathcal{I} \subseteq \text{EBIG}_{10}(\mathcal{O}_k)\).

**Proof.** By Theorem 1.1 in Morgan-Rapinchuk-Sury [10], every matrix in \(\text{SL}_2(\mathcal{O}_k)\) is a product of at most 9 elementary matrices. Hence using Lemma 3.1 with \(n_0 = 9\), we deduce that \(\mathcal{I} \subseteq \text{EBIG}_{18}(\mathcal{O}_k)\).

\[\square\]

**Corollary 3.3.** Let \(\mathcal{A}\) be an integral domain such that there exists a positive integer \(n_0\) for which every matrix in \(\text{SL}_2(\mathcal{A})\) is a product of at most \(n_0\) elementary matrices. Set
\[
\mathcal{I} = \left\{(x \ y) \mid x \in \mathcal{A}\right\}
\]

Then \(\mathcal{I} \subseteq \text{EBIG}_{n_0 + 2}(\mathcal{A})\).

**Proof.** Take an element of the form \(\begin{pmatrix} x & 0 \\ 1 & 0 \end{pmatrix}\) in \(\mathcal{I}\). By Lemma 2.6, we see that
\[
[\begin{pmatrix} x & 0 \\ 1 & 0 \end{pmatrix}]_{-1,1} = [x - 1].
\]

By Lemma 3.1, and since \(\begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} \in \text{SL}_2(\mathcal{A})\), \([x - 1]\) belongs in \(\text{EBIG}_{n_0 + 2}(\mathcal{A})\), and thus \(\begin{pmatrix} x & 0 \\ 1 & 0 \end{pmatrix} \in \text{EBIG}_{n_0 + 2}^{2n_0 + 1}\).

\[\square\]
Similarly one can prove that \( \begin{pmatrix} x & 0 \\ -1 & 0 \end{pmatrix} \in \text{EBIG}_{m+2}^{2n_0+1} \), which implies the lemma immediately. 

\[ \square \]

From Corollaries 3.2 and 3.3, the following result is obvious.

**Corollary 3.4.** Let \( k = \mathbb{Q}(\sqrt{\alpha}) \), where \( \alpha \) is a positive square-free integer, and let \( \mathcal{O}_k \) be its ring of integers. Set

\[ \mathcal{I} = \left\{ \begin{pmatrix} x & 0 \\ \pm 1 & 0 \end{pmatrix} \mid x \in \mathcal{O}_k \right\}. \]

Then \( \mathcal{I} \subset \text{EBIG}_{11}^{19}(\mathcal{O}_k) \).

**Lemma 3.5.** Let \( k = \mathbb{Q}(\sqrt{\alpha}) \), where \( \alpha \) is a positive square-free integer such that \( \alpha \equiv 2, 3 \pmod{4} \), and let \( \mathcal{O}_k \) be its ring of integers. Let \( x, y \) be elements in \( \mathcal{O}_k \). Then there exist an integer \( h \in \mathbb{Z} \) and an element \( \beta \in \mathcal{O}_k \) such that if \( [h, \beta] \in \text{EBIG}_{m}^{n}(\mathcal{O}_k) \), then \( [x \ y] \) is an element in \( \text{EBIG}_{m+24}^{n+3}(\mathcal{O}_k) \).

**Proof.** Let \( x = x_1 + x_2\sqrt{\alpha} \), and \( y = y_1 + y_2\sqrt{\alpha} \), where \( x_1, x_2, y_1, y_2 \) are integers.

Suppose that \( x_2 = 0 \). Letting \( h = x = x_1 \in \mathbb{Z} \) and \( \beta = y \in \mathcal{O}_k \), we deduce that if \( [h, \beta] \in \text{EBIG}_{m}^{n}(\mathcal{O}_k) \), then \( [x \ y] \) is an element in \( \text{EBIG}_{m+24}^{n+3}(\mathcal{O}_k) \), and thus by Lemma 2.12, \( [x \ y] \) is an element in \( \text{EBIG}_{m+24}^{n+3}(\mathcal{O}_k) \).

Suppose that \( y_2 = 0 \). We see that

\[ [x \ y] = [x \ y_1] \rightarrow_{0,2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} [-y_1 \ x]. \]

Letting \( h = -y_1 \in \mathbb{Z} \) and \( \beta = x \in \mathcal{O}_k \), we deduce from the above equation that if \( [h, \beta] \in \text{EBIG}_{m}^{n}(\mathcal{O}_k) \), then \( [x \ y] \) is an element in \( \text{EBIG}_{m+24}^{n+3}(\mathcal{O}_k) \), and thus by Lemma 2.12, \( [x \ y] \) is an element in \( \text{EBIG}_{m+24}^{n+3}(\mathcal{O}_k) \).

Suppose that both \( x_1, y_1 \) are zero. Then

\[ [x \ y] = \begin{pmatrix} x_2\sqrt{\alpha} & y_2\sqrt{\alpha} \\ \sqrt{\alpha} & 0 \end{pmatrix} \]
\[ = \begin{pmatrix} 1 & -1 \\ 0 & \sqrt{\alpha} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & \sqrt{\alpha} \end{pmatrix} \begin{pmatrix} x_2 & y_2 \end{pmatrix} \]

Letting \( h = x_2 \in \mathbb{Z} \) and \( \beta = y_2 \in \mathcal{O}_k \), and since \( \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 - \sqrt{\alpha} & 0 \end{pmatrix} \) are idempotent matrices, we deduce that if \( [h, \beta] \in \text{EBIG}_{m}^{n}(\mathcal{O}_k) \), then \( [x \ y] \) is an element in \( \text{EBIG}_{m+24}^{n+3}(\mathcal{O}_k) \), and thus by Lemma 2.12, \( [x \ y] \) is an element in \( \text{EBIG}_{m+24}^{n+3}(\mathcal{O}_k) \).

For the rest of the proof, without loss of generality, we can assume that the following assumptions are true:

(i) both \( x_2, y_2 \) are nonzero;
(ii) at least one of \( x_1, y_1 \) is nonzero.

We consider the following cases.

\( \star \) **Case 1.** \( \gcd(x_1, x_2) = 1. \)

Since \( \gcd(x_1, x_2) = 1 \), there exist integers \( a_0, b_0 \) such that

\[ a_0x_1 + b_0x_2 = 1. \]

Thus

\[ ax_1 + bx_2 = y_2, \]

where \( a = -a_0y_2 \in \mathbb{Z} \) and \( b = -b_0y_2 \in \mathbb{Z} \).
We see that
\[
[x_1 + x_2 \sqrt{\alpha}, y_1 + y_2 \sqrt{\alpha}] \rightarrow_{\overrightarrow{b+a \sqrt{\alpha}}_{1,2}} [x_1 + x_2 \sqrt{\alpha}, (x_1 + x_2 \sqrt{\alpha}) (b + a \sqrt{\alpha}) + y_1 + y_2 \sqrt{\alpha}]
\]
\[
= [x_1 + x_2 \sqrt{\alpha}, (bx_1 + ax_2 \alpha + y_1) + (ax_1 + bx_2 + y_2) \sqrt{\alpha}]
\]
\[
= [x_1 + x_2 \sqrt{\alpha}, bx_1 + ax_2 \alpha + y_1] \text{ (see (7))}
\]
\[
\rightarrow_{0,2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} [-bx_1 + ax_2 \alpha + y_1, x_1 + x_2 \sqrt{\alpha}].
\]

By Corollary 3.4, \((0 \quad 0) \in \text{EBIG}_{11}^{19}(\mathcal{O}_k)\). Since \(h = -(bx_1 + ax_2 \alpha + y_1) \in \mathbb{Z}\) and \(\beta = x_1 + x_2 \sqrt{\alpha}\) is an element in \(\mathcal{O}_k\), we deduce from Lemma 2.13 that if \([h, \beta] \in \text{EBIG}_{m}^{n}(\mathcal{O}_k)\), then \([x \quad y] = [x_1 + x_2 \sqrt{\alpha}, y_1 + y_2 \sqrt{\alpha}] \in \text{EBIG}_{m+1}^{\min(19, n)}(\mathcal{O}_k)\).

*Case 2.* \(\gcd(y_1, y_2) = 1\).

We see that
\[
[x_1 + x_2 \sqrt{\alpha}, y_1 + y_2 \sqrt{\alpha}] \rightarrow_{0,2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} [-y_1 - y_2 \sqrt{\alpha}, x_1 + x_2 \sqrt{\alpha}].
\]

Since \(\gcd(-y_1, -y_2) = 1\), using the result in Case 1 with \(x_1 + x_2 \sqrt{\alpha}\) replaced by \(-y_1 - y_2 \sqrt{\alpha}\), we deduce that there exists an integer \(h \in \mathbb{Z}\) and an element \(\beta \in \mathcal{O}_k\) such that if \([h, \beta] \in \text{EBIG}_{m}^{n}(\mathcal{O}_k)\), then \([-y_1 - y_2 \sqrt{\alpha}, x_1 + x_2 \sqrt{\alpha}] \in \text{EBIG}_{m+1}^{\min(19, n)+2}(\mathcal{O}_k)\). By Corollary 3.4, \((0 \quad 0) \in \text{EBIG}_{m+1}^{19}(\mathcal{O}_k)\). Thus it follows from the above equation and Lemma 2.13 that there exists an integer \(h \in \mathbb{Z}\) and an element \(\beta \in \mathcal{O}_k\) such that if \([h, \beta] \in \text{EBIG}_{m}^{n}(\mathcal{O}_k)\), then \([x \quad y]\) is an element in \(\text{EBIG}_{m+22}^{\min(19, \min(19, n)+2)}(\mathcal{O}_k)\) which is equivalent to \([x \quad y] \in \text{EBIG}_{m+22}^{\min(18, \min(19, n)+2)}(\mathcal{O}_k)\).

*Case 3.* \(s = \gcd(x_1, x_2) > 1\) and \(r = \gcd(y_1, y_2) > 1\).

*Subcase 3A.* \(\gcd(s, r) = 1\).

Set \(\lambda = \gcd(x_1, y_1)\), and \(\epsilon = \gcd(x_2, y_2)\). Since \(x_2, y_2\) are nonzero, and at least one of \(x_1, y_1\) is nonzero, \(\lambda, \epsilon\) are positive integers.

Write \(x_1 = \lambda z_1, y_1 = \lambda w_1, x_2 = \epsilon z_2, y_2 = \epsilon w_2\), where \(z_1, z_2, w_1, w_2\) are all integers such that \(\gcd(z_1, w_1) = \gcd(z_2, w_2) = 1\).

Assume first that \(z_1 w_2 - z_2 w_1 = 0\).

We see that \(z_1 w_2 = z_2 w_1\). Since \(x_2, y_2\) are nonzero and at least one of \(x_1, y_1\) is nonzero, the last identity implies that all of \(z_1, z_2, w_1, w_2\) are nonzero.

Since \(\gcd(z_1, w_1) = 1\) and \(z_1\) divides \(z_2 w_1\), we deduce that \(z_1\) divides \(z_2\). On the other hand, since \(\gcd(z_2, w_2) = 1\) and \(z_2\) divides \(z_1 w_2\), it follows that \(z_2\) divides \(z_1\). Thus \(z_1 = \delta z_2\), where \(\delta \in \{\pm 1\}\). Thus \(w_1 = \delta w_2\). Therefore
\[
[x \quad y] = [\lambda z_1 + \epsilon z_2 \sqrt{\alpha}, \lambda w_1 + \epsilon w_2 \sqrt{\alpha}]
\]
\[
= [z_2 (\lambda \delta + \epsilon \sqrt{\alpha}) \quad w_2 (\lambda \delta + \epsilon \sqrt{\alpha})]
\]
\[
= [\lambda \delta + \epsilon \sqrt{\alpha} \quad 0] [z_2 \quad w_2].
\]

We see that \(s = \gcd(x_1, x_2) = \gcd(\lambda \delta z_2, \epsilon z_2) = |z_2| \gcd(\lambda \delta, \epsilon)\) and \(r = \gcd(y_1, y_2) = \gcd(\lambda \delta w_2, \epsilon w_2) = |w_2| \gcd(\lambda \delta, \epsilon)\). Since \(\gcd(s, r) = 1\) and \(\gcd(z_2, w_2) = 1\), we deduce that \(\gcd(\lambda \delta, \epsilon) = 1\).

Since \(\gcd(z_2, w_2) = 1\), Corollary 3.2 implies that \([z_2 \quad w_2]\) is in \(\text{EBIG}_{11}^{19}(\mathcal{O}_k)\). By the result in Case 1 with \(x_1, x_2\) replaced by \(\lambda \delta, \epsilon\), respectively, we deduce that there exist an integer \(h \in \mathbb{Z}\) and an element \(\beta \in \mathcal{O}_k\) such that if \([h, \beta] \in \text{EBIG}_{m}^{n}(\mathcal{O}_k)\), then \(\lambda \delta + \epsilon \sqrt{\alpha} \quad 0 \in \text{EBIG}_{m+11}^{\min(19, n)+2}(\mathcal{O}_k)\). Using Lemma 2.13, we deduce that there exist an integer \(h \in \mathbb{Z}\) and an element \(\beta \in \mathcal{O}_k\) such that if \([h, \beta] \in \text{EBIG}_{m}^{n}(\mathcal{O}_k)\), then \([x \quad y] \in \text{EBIG}_{m+22}^{\min(18, \min(19, n)+2)}(\mathcal{O}_k)\) which is equivalent to \([x \quad y] \in \text{EBIG}_{m+22}^{\min(18, n+2)}(\mathcal{O}_k)\).

Assume now that \(z_1 w_2 - z_2 w_1 \neq 0\).

Set \(I = \{\text{primes } \ell \text{ such that } z_1 w_2 - z_2 w_1 \equiv 0 \pmod{\ell}\}\).

Note that \(I\) is a finite nonempty set.
Note that the assumption implies that \( \lambda, \epsilon \) are relatively prime, and thus at least one of them is odd. Suppose first that \( \lambda \) is odd, i.e., every prime factor of \( \lambda \) is odd.

Set

\[
J_\lambda = \{ \text{primes } \ell \text{ such that } \lambda \equiv 0 \pmod{\ell} \text{ and } z_2 \not\equiv 0 \pmod{\ell} \}. 
\]

Note that all primes in \( J_\lambda \), if any, are odd. If \( J_\lambda \neq \emptyset \), write \( J_\lambda = \mathcal{X}_\lambda \cup \mathcal{Y}_\lambda \), where

\[
\mathcal{X}_\lambda = \{ \text{primes } \ell \in J_\lambda \text{ such that } z_1 \equiv 0 \pmod{\ell} \},
\]

and

\[
\mathcal{Y}_\lambda = \{ \text{primes } \ell \in J_\lambda \text{ such that } z_1 \not\equiv 0 \pmod{\ell} \}.
\]

Note that \( \mathcal{X}_\lambda \cap \mathcal{Y}_\lambda = \emptyset \). Since \( \ell \) is odd for every prime \( \ell \) in \( \mathcal{Y}_\lambda \), one can choose, for each prime \( \ell \in \mathcal{Y}_\lambda \), an integer \( b_\ell \) such that \( b_\ell \not\equiv -z_1^{-1}w_1 \pmod{\ell} \) and \( b_\ell \not\equiv -z_2^{-1}w_2 \pmod{\ell} \). For each prime \( \ell \in \mathcal{X}_\lambda \), choose an integer \( a_\ell \) such that \( a_\ell \not\equiv -z_2^{-1}w_2 \pmod{\ell} \). By the Chinese Remainder Theorem, there exists an integer \( u_\lambda \) such that

\[
\begin{align*}
\begin{cases}
u_\lambda \equiv a_\ell \pmod{\ell} & \text{for every prime } \ell \in \mathcal{X}_\lambda, \\
u_\lambda \equiv b_\ell \pmod{\ell} & \text{for every prime } \ell \in \mathcal{Y}_\lambda.
\end{cases}
\end{align*}
\]

We contend that \( \gcd(z_1P_\lambda, z_1u_\lambda + w_1) = 1 \). Indeed, we first prove that \( \gcd(P_\lambda, z_1u_\lambda + w_1) = 1 \), and thus

\[
\gcd(z_1P_\lambda, z_1u_\lambda + w_1) = 1.
\]

By Dirichlet’s theorem on primes in arithmetic progressions, there exist infinitely many integers \( f \) such that \( p = (z_1P_\lambda)f + z_1u_\lambda + w_1 \) is a prime for which \( p \not\in \mathcal{I} \) and \( \gcd(p, \epsilon) = 1 \). Take such an integer \( f \), and set \( p = (z_1P_\lambda)f + z_1u_\lambda + w_1 = z_1e + w_1 \), where \( e = P_\lambda f + u_\lambda \).

We see that

\[
\begin{align*}
[x_1 + x_2\sqrt{\alpha} \ y_1 + y_2\sqrt{\alpha}] &\rightarrow_{e_1, 2}[x_1 + x_2\sqrt{\alpha} \ y_1 + y_2\sqrt{\alpha}] \\
&= [x_1 + x_2\sqrt{\alpha} \ \lambda(z_1e + w_1) + \epsilon(z_2e + w_2)\sqrt{\alpha}] \\
&= [x_1 + x_2\sqrt{\alpha} \ p\lambda + \epsilon(z_2e + w_2)\sqrt{\alpha}] \\
&= [x_1 + x_2\sqrt{\alpha} \ y_1' + y_2'\sqrt{\alpha}],
\end{align*}
\]

where

\[
\begin{align*}
y_1' &= p\lambda, \\
y_2' &= \epsilon(z_2e + w_2).
\end{align*}
\]

We contend that \( \gcd(y_1', y_2') = 1 \). Indeed, we first prove that \( p \) does not divide \( z_2e + w_2 \). Assume the contrary, i.e., \( p = z_1e + w_1 \equiv 0 \pmod{p} \) and \( z_2e + w_2 \equiv 0 \pmod{p} \). Thus \( z_1w_2 - z_2w_1 \equiv 0 \pmod{p} \), which implies that \( p \in \mathcal{I} \), a contradiction to the choice of \( p \). Thus \( \gcd(p, z_2e + w_2) = 1 \). By the choice of \( p \), \( \gcd(p, \epsilon) = 1 \), and thus

\[
\gcd(p, \epsilon(z_2e + w_2)) = 1.
\]

We now prove that \( \gcd(\lambda, z_2e + w_2) = 1 \). Assume the contrary, i.e., there exists a prime factor \( \ell \) of \( \lambda \) such that \( z_2e + w_2 \equiv 0 \pmod{\ell} \). If \( z_2 \equiv 0 \pmod{\ell} \), then it follows that \( w_2 \equiv 0 \pmod{\ell} \), which is a contradiction since \( \gcd(z_2, w_2) = 1 \). Hence \( z_2 \not\equiv 0 \pmod{\ell} \), and thus \( \ell \in J_\lambda \).
Recall that $e = P_\lambda f + u_\lambda$ and since $\ell$ divides $P_\lambda$, we deduce that
\[
2z_2e + w_2 = 2z_2u_\lambda + w_2 \equiv 0 \pmod{\ell}.
\]
Thus $u_\lambda \equiv -z_2^{-1}w_2 \pmod{\ell}$, which is a contradiction to the choice of $u_\lambda$. Thus $\gcd(\lambda, 2z_2e + w_2) = 1$. Since $\gcd(\lambda, e) = 1$, we deduce that
\[
\gcd(\lambda, e(2z_2e + w_2)) = 1.
\]

By (10), (11), we deduce that $\gcd(y'_1, y'_2) = \gcd(p\lambda, e(2z_2e + w_2)) = 1$. From (9), we can use the result in Case 2 for $(y'_1, y'_2)$ in place of $(y_1, y_2)$ to deduce that there exists an integer $h \in \mathbb{Z}$ and an element $\beta \in \mathcal{O}_k$ such that if $[h, \beta] \in \mathbf{EBIG}^n_m(\mathcal{O}_k)$, then $[x_1 + x_2\sqrt{\alpha}, y'_1 + y'_2\sqrt{\alpha}]$ is an element in $\mathbf{EBIG}^{\min(19, n+2)}_m(\mathcal{O}_k)$. It follows from (9) that that there exists an integer $h \in \mathbb{Z}$ and an element $\beta \in \mathcal{O}_k$ such that if $[h, \beta] \in \mathbf{EBIG}^n_m(\mathcal{O}_k)$, then $[x_1 + x_2\sqrt{\alpha}, y_1 + y_2\sqrt{\alpha}]$ is an element in $\mathbf{EBIG}^{\min(20, n+3)}_m(\mathcal{O}_k)$.

Suppose now that $\epsilon$ is odd, i.e., every prime factor of $\epsilon$ is odd. We use a similar argument as above with $\epsilon$ in place of $\lambda$.

Set
\[
\mathcal{J}_\epsilon = \{\text{primes } \ell \text{ such that } \epsilon \equiv 0 \pmod{\ell} \text{ and } z_1 \not\equiv 0 \pmod{\ell}\}.
\]

Note that all primes in $\mathcal{J}_\epsilon$, if any, are odd. If $\mathcal{J}_\epsilon \neq \emptyset$, write $\mathcal{J}_\epsilon = \mathcal{X}_\epsilon \cup \mathcal{Y}_\epsilon$, where
\[
\mathcal{X}_\epsilon = \{\text{primes } \ell \text{ in } \mathcal{J}_\epsilon \text{ such that } z_2 \equiv 0 \pmod{\ell}\},
\]
and
\[
\mathcal{Y}_\epsilon = \{\text{primes } \ell \text{ in } \mathcal{J}_\epsilon \text{ such that } z_2 \not\equiv 0 \pmod{\ell}\}.
\]

Since $\ell$ is odd for every prime $\ell$ in $\mathcal{Y}_\epsilon$, one can choose, for each prime $\ell$ in $\mathcal{Y}_\epsilon$, an integer $b_\ell$ such that $b_\ell \not\equiv -z_2^{-1}w_1 \pmod{\ell}$ and $b_\ell \not\equiv -z_2^{-1}w_2 \pmod{\ell}$. For each prime $\ell$ in $\mathcal{X}_\epsilon$, choose an integer $a_\ell$ such that $a_\ell \not\equiv -z_2^{-1}w_1 \pmod{\ell}$. By the Chinese Remainder Theorem, there exists an integer $u_\epsilon$ such that
\[
\begin{cases}
u_\lambda \equiv a_\ell \pmod{\ell} \text{ for every prime } \ell \in \mathcal{X}_\epsilon, \\
u_\lambda \equiv b_\ell \pmod{\ell} \text{ for every prime } \ell \in \mathcal{Y}_\epsilon.
\end{cases}
\]

Note that if exactly one of $\mathcal{X}_\epsilon$ and $\mathcal{Y}_\epsilon$ is empty, $u_\epsilon$ is chosen so as to satisfy exactly one of the above congruence conditions which corresponds to the nonempty set.

Set
\[
P_\epsilon = \prod_{\ell \in \mathcal{J}_\epsilon} \ell.
\]

If $\mathcal{J}_\epsilon = \emptyset$, we set $P_\epsilon = u_\epsilon = 1$. We claim that $\gcd(z_2P_\epsilon, z_2u_\epsilon + w_2) = 1$. Since $\gcd(z_2, w_2) = 1$, it is clear that $\gcd(z_2, z_2u_\epsilon + w_2) = 1$. By the choice of $u_\epsilon$, it is also clear that $\gcd(P_\epsilon, z_2u_\epsilon + w_2) = 1$, and thus
\[
\gcd(z_2P_\epsilon, z_2u_\epsilon + w_2) = 1.
\]

By Dirichlet’s theorem on primes in arithmetic progressions, there exist infinitely many integers $f$ such that $p = (z_2P_\epsilon)f + z_2u_\epsilon + w_2$ is a prime for which $p \not\in \mathcal{I}$ and $\gcd(p, \lambda) = 1$. Take such an integer $f$, and set $p = (z_2P_\epsilon)f + z_2u_\epsilon + w_2 = z_2e + w_2$, where $e = P_\epsilon f + u_\epsilon$.

We see that
\[
[x_1 + x_2\sqrt{\alpha}, y_1 + y_2\sqrt{\alpha}] \to_{c_{1,2}} [x_1 + x_2\sqrt{\alpha}, x_1e + y_1 + (x_2e + y_2)\sqrt{\alpha}] = [x_1 + x_2\sqrt{\alpha}, \lambda(z_1e + w_1) + e(z_2e + w_2)\sqrt{\alpha}] = [x_1 + x_2\sqrt{\alpha}, \lambda(z_1e + w_1) + pe\sqrt{\alpha}]
\]
\[
= [x_1 + x_2\sqrt{\alpha}, y'_1 + y'_2\sqrt{\alpha}],
\]
where
\[
y'_1 = \lambda(z_1e + w_1),
\]
\[
y'_2 = pe.
\]
We contend that $\gcd(y_1',y_2') = 1$. Indeed, we first prove that $p$ does not divide $z_1e + w_1$. Assume the contrary, i.e., $p$ divides $z_1e + w_1$, and thus $p = z_1e + w_2 \equiv 0 \pmod{p}$ and $z_1e + w_1 \equiv 0 \pmod{p}$. Thus $z_1w_2 - z_2w_1 \equiv 0 \pmod{p}$, which implies that $p \in \mathcal{I}$, a contradiction to the choice of $p$. Thus $\gcd(p,z_1e + w_1) = 1$. By the choice of $p$, $\gcd(p,\lambda) = 1$, and thus

\begin{equation}
\gcd(p,\lambda(z_1e + w_1)) = 1.
\end{equation}

We now prove that $\gcd(\epsilon,z_1e + w_1) = 1$. Assume the contrary, i.e., there exists a prime factor $\ell$ of $\epsilon$ such that $z_1e + w_1 \equiv 0 \pmod{\ell}$. If $z_1 \equiv 0 \pmod{\ell}$, then it follows that $w_1 \equiv 0 \pmod{\ell}$, which is a contradiction since $\gcd(z_1,w_1) = 1$. Hence $z_1 \not\equiv 0 \pmod{\ell}$, and thus $\ell \not\in \mathcal{J}$.

Recall that $e = P_f + u_\epsilon$. Since $\ell$ divides $P_\epsilon$, we deduce that

$$z_1e + w_1 = z_1u_\epsilon + w_1 \equiv 0 \pmod{\ell}.$$ 

Thus $u_\epsilon \equiv -z_1^{-1}w_1 \pmod{\ell}$, which is a contradiction to the choice of $u_\epsilon$. Thus $\gcd(\epsilon,z_1e + w_1) = 1$. Since $\gcd(\lambda,\epsilon) = 1$, we deduce that

\begin{equation}
\gcd(\epsilon,\lambda(z_1e + w_1)) = 1.
\end{equation}

By (14), (15), we deduce that $\gcd(y_1',y_2') = \gcd(\lambda(z_1e + w_1),p\epsilon) = 1$. From (13), we can use the result in Case 2 for $(y_1',y_2')$ in place of $(y_1,y_2)$ to deduce that there exists an integer $h \in \mathbb{Z}$ and an element $\beta \in \mathcal{O}_k$ such that if $[h,\beta] \in \text{EBIG}^n_m(\mathcal{O}_k)$, then $[x_1 + x_2\sqrt{\alpha},y_1' + y_2'\sqrt{\alpha}]$ is an element in $\text{EBIG}^{\min(19,n+2)}_{m+2}(\mathcal{O}_k)$. It follows from (9) that there exists an integer $h \in \mathbb{Z}$ and an element $\beta \in \mathcal{O}_k$ such that if $[h,\beta] \in \text{EBIG}^n_m(\mathcal{O}_k)$, then $[x_1 + x_2\sqrt{\alpha},y_1 + y_2\sqrt{\alpha}]$ is an element in $\text{EBIG}^{\min(20,n+3)}_{m+2}(\mathcal{O}_k)$

- Subcase 3B. $\delta = \gcd(s,r) > 1$.

Write

\begin{align*}
s &= \delta s', \\
r &= \delta r',
\end{align*}

where $\gcd(s',r') = 1$. Let

\begin{align*}
x_1 &= sx_1', \\
x_2 &= sx_2', \\
y_1 &= ry_1', \\
y_2 &= ry_2',
\end{align*}

where $\gcd(x_1',x_2') = \gcd(y_1',y_2') = 1$.

We see that

\begin{equation}
[x\ y] = [\delta s'(x_1' + x_2'\sqrt{\alpha})\ \delta r'(y_1' + y_2'\sqrt{\alpha})] = \left(\begin{array}{cc}
1 & -1 \\
0 & 0
\end{array}\right) \left(\begin{array}{cc}
1 & 0 \\
1 & -\delta
\end{array}\right) \left(\begin{array}{cc}
s'x_1' + s'x_2'\sqrt{\alpha} & ry_1' + ry_2'\sqrt{\alpha}
\end{array}\right) = \left(\begin{array}{cc}
1 & -1 \\
0 & 0
\end{array}\right) \left(\begin{array}{cc}
1 & 0 \\
1 & -\delta
\end{array}\right) \left(\begin{array}{cc}
x''_1 + x''_2\sqrt{\alpha} & y''_1 + y''_2\sqrt{\alpha}
\end{array}\right),
\end{equation}

where

\begin{align*}
x''_1 &= s'x_1', \\
x''_2 &= s'x_2', \\
y''_1 &= r'y_1', \\
y''_2 &= r'y_2'.
\end{align*}

Note that $\gcd(x''_1,x''_2) = s'$, $\gcd(y''_1,y''_2) = r'$, and $\gcd(s',r') = 1$. So applying the result from Subcase 3A, with $x_1,x_2,y_1,y_2$ replaced by $x''_1,x''_2,y''_1,y''_2$, respectively, we deduce that there exists an integer $h \in \mathbb{Z}$ and an element $\beta \in \mathcal{O}_k$ such that if $[h,\beta] \in \text{EBIG}^n_m(\mathcal{O}_k)$, then $[x''_1 + x''_2\sqrt{\alpha}, y''_1 + y''_2\sqrt{\alpha}]$ is
an element in $\text{EBIG}_{m+22}^{\min(20,n+3)}(O_k)$. By Lemmas 2.12 and 2.13, and since $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 1 - \delta & 0 \end{pmatrix}$ are idempotent matrices, we deduce from (16) that there exists an integer $h \in \mathbb{Z}$ and an element $\beta \in O_k$ such that if $[h, \beta] \in \text{EBIG}_m^n(O_k)$, then $[x, y]$ is an element in $\text{EBIG}_{m+24}^{\min(20,n+3)}(O_k)$.

By all of what we have showed above and Lemma 2.12, there exists an integer $h \in \mathbb{Z}$ and an element $\beta \in O_k$ such that if $[h, \beta] \in \text{EBIG}_m^n(O_k)$, then $[x, y]$ is an element in $\text{EBIG}_{m+24}^{n+3}(O_k)$.

\begin{lemma}
Let $k = \mathbb{Q}(\sqrt{\alpha})$, where $\alpha$ is a positive square-free integer such that $\alpha \equiv 1 \pmod{4}$, and let $O_k$ be its ring of integers. Let $x, y$ be elements in $O_k$. Then there exist an integer $h \in \mathbb{Z}$ and an element $\beta \in O_k$ such that if $[h, \beta] \in \text{EBIG}_m^n(O_k)$, then $[x, y]$ is an element in $\text{EBIG}_{m+24}^{n+3}(O_k)$.

\begin{proof}
It is well-known that $O_k = \mathbb{Z}[\frac{1+\sqrt{\alpha}}{2}]$ (see Borevich and Shafarevich [1]). Hence each element in $O_k$ can be written in the form $a + b \left(\frac{1+\sqrt{\alpha}}{2}\right)$, where $a, b \in \mathbb{Z}$. Equivalently each element in $O_k$ is of the form

$$\frac{2a + b + b\sqrt{\alpha}}{2}$$

for some integers $a, b$.

Write $x = \frac{2x_1 + x_2 + x_2\sqrt{\alpha}}{2}$ and $y = \frac{2y_1 + y_2 + y_2\sqrt{\alpha}}{2}$, where the $x_i, y_i$ are integers.

Suppose that $x_2 = 0$. Letting $h = x = x_1 \in \mathbb{Z}$ and $\beta = y \in O_k$, we deduce that if $[h, \beta] \in \text{EBIG}_m^n(O_k)$, then $[x, y]$ is an element in $\text{EBIG}_{m+24}^{n+3}(O_k)$, and thus by Lemma 2.12, $[x, y]$ is an element in $\text{EBIG}_{m+24}^{n+3}(O_k)$.

Suppose that $y_2 = 0$. We see that

$$[x, y] = [x, y_1] \rightarrow_{02, 2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} [-y_1, x].$$

Letting $h = -y_1 \in \mathbb{Z}$ and $\beta = x \in O_k$, we deduce from the above equation that if $[h, \beta] \in \text{EBIG}_m^n(O_k)$, then $[x, y]$ is an element in $\text{EBIG}_{m+1}^{n+1}(O_k)$, and thus by Lemma 2.12, $[x, y]$ is an element in $\text{EBIG}_{m+24}^{n+3}(O_k)$.

Suppose that both $x_1, y_1$ are zero. Then

$$[x, y] = [x_2(1+\sqrt{\alpha})/2, y_2(1+\sqrt{\alpha})/2]$$

$$= [(1+\sqrt{\alpha})/2, 0] [x_2, y_2]$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 - \sqrt{\alpha}/2 & 0 \end{pmatrix} [x_2, y_2]$$

Letting $h = x_2 \in \mathbb{Z}$ and $\beta = y_2 \in O_k$, and since $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 1 - \sqrt{\alpha}/2 & 0 \end{pmatrix}$ are idempotent matrices, we deduce that if $[h, \beta] \in \text{EBIG}_m^n(O_k)$, then $[x, y]$ is an element in $\text{EBIG}_{m+2}^{n+3}(O_k)$, and thus by Lemma 2.12, $[x, y]$ is an element in $\text{EBIG}_{m+24}^{n+3}(O_k)$.

For the rest of the proof, without loss of generality, we can assume that the following assumptions are true:

(i) both $x_2, y_2$ are nonzero;
(ii) at least one of $x_1, y_1$ is nonzero.

It suffices to consider the following cases.

\textbf{Case 1. gcd}(x_1, x_2) = 1.

Since $\text{gcd}(x_1, x_2) = 1$, it follows that $\text{gcd}(x_1, x_1 + x_2) = 1$, and thus there exist integers $a_0, b_0$ such that

$$a_0(x_1 + x_2) + b_0x_2 = 1.$$
Thus
\begin{equation}
(17) \quad a(x_1 + x_2) + bx_2 = -y_2,
\end{equation}
where \( a = -a_0y_2 \in \mathbb{Z} \) and \( b = -b_0y_2 \in \mathbb{Z} \).

We see that
\[
[x, y] = \left[ \frac{2x_1 + x_2 + x_2\sqrt{\alpha}}{2}, \frac{2y_1 + y_2 + y_2\sqrt{\alpha}}{2} \right] \rightarrow \left[ \frac{2x_1 + x_2 + x_2\sqrt{\alpha}}{2}, \frac{2b + a + a\sqrt{\alpha}}{2} \right] \frac{2x_1 + x_2 + x_2\sqrt{\alpha}}{2} + \frac{2y_1 + y_2 + y_2\sqrt{\alpha}}{2} \]
\[
= \left[ \frac{2x_1 + x_2 + x_2\sqrt{\alpha}}{2}, bx_1 + \frac{bx_2}{2} + \frac{ax_1}{2} + \frac{ax_2}{4} + \frac{ax_2\alpha}{4} + y_1 + \frac{y_2}{2} + \frac{(2b + a)x_2 + (2x_1 + x_2)a + 2y_2\sqrt{\alpha}}{4} \right] \]
\[
\rightarrow \lambda \left[ \begin{array}{cc}
0 & 0 \\
1 & 0 \\
\end{array} \right] \left[ \begin{array}{c}
-(bx_1 + ax_2(a - 1)/4 + y_1) \\
\frac{2x_1 + x_2 + x_2\sqrt{\alpha}}{2} \\
\end{array} \right]
\]

By Corollary 3.4, \( \left[ \begin{array}{cc}
0 & 0 \\
1 & 0 \\
\end{array} \right] \in \text{EBIG}^{19}_{11}(O_k) \). Since \( h = -(bx_1 + ax_2(a - 1)/4 + y_1) \in \mathbb{Z} \) and \( \beta = \frac{2x_1 + x_2 + x_2\sqrt{\alpha}}{2} \) is an element in \( O_k \), we deduce from Lemma 2.13 that if \( h, \beta \in \text{EBIG}^n_m(O_k) \), then
\[
[x, y] \in \text{EBIG}^{\min(19,n)+2}_{m+11}(O_k).
\]
* Case 2. \( \gcd(y_1, y_2) = 1 \).
We see that
\[
[x, y] \rightarrow \lambda \left[ \begin{array}{cc}
0 & 0 \\
1 & 0 \\
\end{array} \right] \left[ \begin{array}{c}
-y \\
x \\
\end{array} \right]
\]
\[
= \left[ \begin{array}{cc}
0 & 0 \\
1 & 0 \\
\end{array} \right] \left[ \begin{array}{c}
\frac{-2y_1 - y_2 - y_2\sqrt{\alpha}}{2} \\
\frac{2x_1 + x_2 + x_2\sqrt{\alpha}}{2} \\
\end{array} \right].
\]

Since \( \gcd(-y_1, -y_2) = 1 \), repeating the same arguments as in Case 2 of Lemma 3.5, and Case 1 above, we deduce that there exists an integer \( h \in \mathbb{Z} \) and an element \( \beta \in O_k \) such that if \( h, \beta \in \text{EBIG}^n_m(O_k) \), then \( [x, y] \) is an element in \( \text{EBIG}^{\min(19,n)+2}_{m+22}(O_k) \) which is equivalent to \( [x, y] \in \text{EBIG}^{\min(19,n)+2}_{m+22}(O_k) \).

* Case 3. \( s = \gcd(x_1, x_2) > 1 \) and \( r = \gcd(y_1, y_2) > 1 \).
  • Subcase 3A. \( \gcd(s, r) = 1 \).
Set \( \lambda = \gcd(x_1, y_1) \), and \( \epsilon = \gcd(x_2, y_2) \). Since \( x_2, y_2 \) are nonzero, and at least one of \( x_1, y_1 \) is nonzero, \( \lambda, \epsilon \) are positive integers.
Write \( x_1 = \lambda z_1, y_1 = \lambda w_1, x_2 = \epsilon z_2, \) and \( y_2 = \epsilon w_2 \), where \( z_1, z_2, w_1, w_2 \) are all integers such that \( \gcd(z_1, w_1) = \gcd(z_2, w_2) = 1 \).
Assume first that \( z_1 w_2 - z_2 w_1 = 0 \).
We see that \( z_1 w_2 = z_2 w_1 \). Since \( x_2, y_2 \) are nonzero and at least one of \( x_1, y_1 \) is nonzero, the last identity implies that all of \( z_1, z_2, w_1, w_2 \) are nonzero.
Since \( \gcd(z_1, w_1) = 1 \) and \( z_1 \) divides \( z_2 w_1 \), we deduce that \( z_1 \) divides \( z_2 \). On the other hand, since \( \gcd(z_2, w_2) = 1 \) and \( z_2 \) divides \( z_1 w_2 \), it follows that \( z_2 \) divides \( z_1 \). Thus \( z_1 = \delta z_2 \), where \( \delta \in \{ \pm 1 \} \). Thus
Thus set \( p = (z_1 P_\lambda) f + z_1 u_\lambda + w_1 \) is a prime for which \( p \notin \mathcal{I} \) and \( \gcd(p, e) = 1 \). Take such an integer \( f \), and set \( P = (z_1 P_\lambda) f + z_1 u_\lambda + w_1 = z_1 e + w_1 \), where \( e = P_\lambda f + u_\lambda \).
We see that
\[
\begin{align*}
\left[ \frac{2x_1 + x_2 + x_2 \sqrt{\alpha}}{2} \right] & \to_{e_1, 2} \left[ \frac{2x_1 + x_2 + x_2 \sqrt{\alpha}}{2} \right] \\
&= \left[ \frac{2x_1 + x_2 + x_2 \sqrt{\alpha}}{2} \right] \\
&= \left[ \frac{2x_1 + x_2 + x_2 \sqrt{\alpha}}{2} \right] \\
&= \left[ \frac{2x_1 + x_2 + x_2 \sqrt{\alpha}}{2} \right] \\
&= \left[ \frac{2x_1 + x_2 + x_2 \sqrt{\alpha}}{2} \right] \\
&= \left[ \frac{2x_1 + x_2 + x_2 \sqrt{\alpha}}{2} \right]
\end{align*}
\]
where
\[
\begin{align*}
y'_1 &= p \lambda, \\
y'_2 &= \epsilon(z_2 e + w_2).
\end{align*}
\]

Using the same arguments as in Subcase 3A in the proof of Lemma 3.5, we deduce that \( \gcd(y'_1, y'_2) = 1 \). Using the result in Case 2 for \((y'_1, y'_2)\) in place of \((y_1, y_2)\), we deduce that there exists an integer \( h \in \mathbb{Z} \) and an element \( \beta \in \mathcal{O}_k \) such that \([h, \beta] \in \text{EBIG}^n_m(\mathcal{O}_k)\), then \[ \left[ \frac{2x_1 + x_2 + x_2 \sqrt{\alpha}}{2} \right] \] is an element in \( \text{EBIG}^{\min(19, n+2)}_{m+22}(\mathcal{O}_k) \). It follows from (19) that there exists an integer \( h \in \mathbb{Z} \) and an element \( \beta \in \mathcal{O}_k \) such that \([h, \beta] \in \text{EBIG}^n_m(\mathcal{O}_k)\), then \([x, y]\) is an element in \( \text{EBIG}^{\min(20, n+3)}_{m+22}(\mathcal{O}_k) \).

Suppose now that \( \epsilon \) is odd, i.e., every prime factor of \( \epsilon \) is odd. We use a similar argument as above with \( \epsilon \) in place of \( \lambda \) to deduce the lemma. Indeed define \( J, X, Y, s, r, e, P, \) and \( \epsilon \) as in Subcase 3A of the proof of Lemma 3.5. Recall that \( e = P, f + u, \) and \( \epsilon = (z_2 P) f + z_2 u, w_2 = z_2 e + w_2 \) for some integer \( f \) such that \( p \) is a prime.

We see that
\[
\begin{align*}
\left[ \frac{2x_1 + x_2 + x_2 \sqrt{\alpha}}{2} \right] & \to_{e_1, 2} \left[ \frac{2x_1 + x_2 + x_2 \sqrt{\alpha}}{2} \right] \\
&= \left[ \frac{2x_1 + x_2 + x_2 \sqrt{\alpha}}{2} \right] \\
&= \left[ \frac{2x_1 + x_2 + x_2 \sqrt{\alpha}}{2} \right] \\
&= \left[ \frac{2x_1 + x_2 + x_2 \sqrt{\alpha}}{2} \right] \\
&= \left[ \frac{2x_1 + x_2 + x_2 \sqrt{\alpha}}{2} \right] \\
&= \left[ \frac{2x_1 + x_2 + x_2 \sqrt{\alpha}}{2} \right]
\end{align*}
\]
where
\[
\begin{align*}
y'_1 &= \lambda(z_1 e + w_1), \\
y'_2 &= p \epsilon.
\end{align*}
\]

Following the same arguments as in the last part of Subcase 3A in the proof of Lemma 3.5, one can prove that \( \gcd(y'_1, y'_2) = 1 \). Thus using (20) and Case 2 above, there exists an integer \( h \in \mathbb{Z} \) and an element \( \beta \in \mathcal{O}_k \) such that \([h, \beta] \in \text{EBIG}^n_m(\mathcal{O}_k)\), then \([x, y]\) is an element in \( \text{EBIG}^{\min(20, n+3)}_{m+22}(\mathcal{O}_k) \).

- **Subcase 3B.** \( \delta = \gcd(s, r) > 1 \).
  Write
  \[
  s = \delta s', \\
r = \delta r',
  \]
where $s', r'$ are integers such that $\gcd(s', r') = 1$. Let
\[
\begin{align*}
x_1 &= sx'_1, \\
y_1 &= ry'_1, \\
x_2 &= sx'_2, \\
y_2 &= ry'_2,
\end{align*}
\]
where $x'_1, x'_2, y'_1, y'_2$ are integers such that $\gcd(x'_1, x'_2) = \gcd(y'_1, y'_2) = 1$.

We see that
\[
\begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} \frac{2x_1 + x_2 + x_2\sqrt{\alpha}}{2} & \frac{2y_1 + y_2 + y_2\sqrt{\alpha}}{2} \\
\frac{\delta s'(2x'_1 + x'_2 + x'_2\sqrt{\alpha})}{2} & \frac{\delta r'(2y'_1 + y'_2 + y'_2\sqrt{\alpha})}{2} \end{bmatrix}
\]
\[(21)\]
where
\[
\begin{align*}
x''_1 &= s'x'_1, \\
x''_2 &= s'x'_2, \\
y''_1 &= r'y'_1, \\
y''_2 &= r'y'_2.
\end{align*}
\]

Note that $\gcd(x''_1, x''_2) = s'$, $\gcd(y''_1, y''_2) = r'$, and $\gcd(r', s') = 1$. So applying the result from Subcase 3A, with $x_1, x_2, y_1, y_2$ replaced by $x''_1, x''_2, y''_1, y''_2$, respectively, we deduce that there exists an integer $h \in \mathbb{Z}$ and an element $\beta \in \mathcal{O}_k$ such that $[h, \beta] \in \text{EBIG}^n_{m+22}(\mathcal{O}_k)$, then $\begin{bmatrix} 2x''_1 + x''_2 + x''_2\sqrt{\alpha} & 2y''_1 + y''_2 + y''_2\sqrt{\alpha} \end{bmatrix}$ is an element in $\text{EBIG}^\text{min(20,n+3)}_{m+22}(\mathcal{O}_k)$. By Lemmas 2.12 and 2.13, and since $\begin{bmatrix} 1 & -1 \\
0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\
1-\delta & 0 \end{bmatrix}$ are idempotent matrices, we deduce from (21) that there exists an integer $h \in \mathbb{Z}$ and an element $\beta \in \mathcal{O}_k$ such that $[h, \beta] \in \text{EBIG}^n_{m}(\mathcal{O}_k)$, then $[x \ y]$ is an element in $\text{EBIG}^\text{min(20,n+3)}_{m+24}(\mathcal{O}_k)$.

By all of what we have showed above and Lemma 2.12, there exists an integer $h \in \mathbb{Z}$ and an element $\beta \in \mathcal{O}_k$ such that $[h, \beta] \in \text{EBIG}^n_{m}(\mathcal{O}_k)$, then $[x \ y]$ is an element in $\text{EBIG}^\text{min(20,n+3)}_{m+24}(\mathcal{O}_k)$.

\[\square\]

The following lemma is a slightly modified version of Theorem 3.1 in Cossu and Zanardo [3].

**Lemma 3.7.** Let $\mathcal{O}_k$ be the ring of integers of a real quadratic field $k = \mathbb{Q}(\sqrt{\alpha})$, where $\alpha$ is a positive square-free integer. Let $x, y$ be elements in $\mathcal{O}_k$ such that $x\mathcal{O}_k + y\mathcal{O}_k = z\mathcal{O}_k$ for some nonzero element $z \in \mathcal{O}_k$, i.e. $x, y$ generates the principal ideal of $z\mathcal{O}_k$. Then $[x \ y] \in \text{EBIG}^\text{13}_{13}(\mathcal{O}_k)$.

**Proof.** By assumption, there exist elements $x_1, y_1 \in \mathcal{O}_k$ such that $x = x_1 z$ and $y = y_1 z$. Since $x\mathcal{O}_k + y\mathcal{O}_k = z\mathcal{O}_k$, there exist elements $a, b \in \mathcal{O}_k$ such that
\[
(x_1 z)a + (y_1 z)b = z,
\]
and thus
\[
x_1 a + y_1 b = 1.
\]
Thus $\begin{bmatrix} x_1 & y_1 \\
a & b \end{bmatrix} \in \text{SL}_2(\mathcal{O}_k)$. By Corollary 3.2, $[x_1 \ y_1] \in \text{EBIG}^\text{13}_{13}(\mathcal{O}_k)$. Since
\[
[x \ y] = \begin{bmatrix} 1 & -1 \\
0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\
1-\delta & 0 \end{bmatrix} [x_1 \ y_1],
\]

and \(\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 - z & 0 \end{pmatrix}\) are idempotent matrices, we deduce from Lemmas 2.11 and 2.13 that \([x \ y] \in \text{EBIG}_{13}^{18}(\mathcal{O}_k)\), which proves the lemma.

\[\square\]

**Theorem 3.8.** Let \(k = \mathbb{Q}(\sqrt{\alpha})\) be a real quadratic number field, where \(\alpha\) is a positive square-free integer.

Let \(\mathcal{O}_k\) be the ring of integers of \(k\). Let \(\mathcal{M}\) be the set of \(2 \times 2\) matrices over \(\mathcal{O}_k\) of the form \(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\), where \(x, y\) are elements in \(\mathcal{O}_k\). Then \(\mathcal{M}\) is a subset of \(\text{EBIG}_{15}^{19}(\mathcal{O}_k)\), i.e., every matrix in \(\mathcal{M}\) belongs to \(\text{EBIG}_{15}^{19}(\mathcal{O}_k)\).

**Proof.** Throughout the proof, for each prime \(p\), we denote by \(v_p\) the \(p\)-adic valuation on \(\mathbb{Q}\).

By Lemmas 3.5 and 3.6, it suffices to prove that the subset \(\mathcal{M}_0\) of \(\mathcal{M}\) consisting of matrices of the form \([x \ y]\), where \(x \in \mathbb{Z}\) and \(y \in \mathcal{O}\) is a subset of \(\text{EBIG}(\mathcal{O}_k)\). In order to prove this, we will use the techniques in the proof of Theorem 3.2 in Cossu and Zanardo [3].

Suppose first that there exists a non-unit element \(z \in \mathcal{O}_k\) such that \(x = x_1z\) and \(y = y_1z\), where \(x_1, y_1\) are elements in \(\mathcal{O}_k\) such that \(x_1, y_1\) have no common non-unit factors in \(\mathcal{O}_k\). Then

\[\begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z & 0 \end{pmatrix} [x_1 \ y_1].\]

Since \(\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -z & 0 \end{pmatrix}\) are idempotent matrices, we deduce from Lemmas 2.13 and 2.12 that if \([x_1 \ y_1] \in \text{EBIG}_{13}^{18}(\mathcal{O}_k)\) then \([x \ y] \in \text{EBIG}_{15}^{19}(\mathcal{O}_k)\).

On the other hand, note that Lemma 3.7 implies that if \(x\mathcal{O}_k + y\mathcal{O}_k\) is a principal ideal in \(\mathcal{O}_k\), then \([x \ y] \in \text{EBIG}_{13}^{18}(\mathcal{O}_k)\). By Lemma 2.12, we deduce that \([x \ y] \in \text{EBIG}_{15}^{19}(\mathcal{O}_k)\) if \(x\mathcal{O}_k + y\mathcal{O}_k\) is a principal ideal in \(\mathcal{O}_k\).

So without loss of generality, for the rest of the proof, we can further assume that the following are true:

(i) \(x, y\) have no common non-unit factors in \(\mathcal{O}_k\);

(ii) \(x\mathcal{O}_k + y\mathcal{O}_k\) is not a principal ideal; especially \(x\mathcal{O}_k + y\mathcal{O}_k \neq \mathcal{O}_k\), which implies that

\[
m = \gcd(x, ||y||) \neq 1,
\]

where for the rest of this paper, \(||y||\) denotes the norm of \(y\) in \(\mathcal{O}_k\), i.e., \(||y|| = \bar{y}\bar{y}\), where \(\bar{y}\) is the conjugate element of \(y\) (see [1]).

Our aim is to show that if conditions (i) and (ii) are satisfied, then \([x \ y] \in \text{EBIG}_{13}^{18}(\mathcal{O}_k)\). We consider the following cases.

**Case 1.** \(s = \gcd(x, ||y||) = 1\).

Following the same arguments as in Step 1 of the proof of Theorem 3.2 in Cossu and Zanardo [3], one can write

\[\begin{pmatrix} x & y \end{pmatrix} = [x' \ y'] \begin{pmatrix} a & b \\ c & 1 - a \end{pmatrix},\]

where \(a, b, c \in \mathcal{O}_k\) such that \(\begin{pmatrix} a & b \\ c & 1 - a \end{pmatrix}\) is an idempotent matrix, and \(x', y' \in \mathcal{O}_k\) such that \(\begin{pmatrix} x' & y' \\ u & v \end{pmatrix} \in \text{SL}_2(\mathcal{O}_k)\) for some elements \(u, v \in \mathcal{O}_k\). By Corollary 3.2, \([x' \ y'] \in \text{EBIG}_{11}^{17}(\mathcal{O}_k)\), and it thus follows from Lemmas 2.13 and 2.12 that \([x \ y] \in \text{EBIG}_{12}^{18}(\mathcal{O}_k)\).

**Case 2.** \(s = \gcd(x, ||y||) \neq 1\).

In this case, we consider the following subcases.

**Subcase 2A.** \(\alpha \equiv 2, 3 \pmod{4}\).

In this subcase, \(\mathcal{O} = \mathbb{Z}[\sqrt{\alpha}]\), and each element in \(\mathcal{O}\) can be written in the form \(a + b\sqrt{\alpha}\) for some integers \(a, b \in \mathbb{Z}\).
Write \( y = y_1 + y_2 \sqrt{\alpha} \), where \( y_1, y_2 \) are integers. By assumption, we know that \( x, y \) have no common non-unit factors in \( \mathcal{O}_k \), and thus \( \gcd(x, y_1, y_2) = 1 \). One can write
\[
x = x_0 m,
\]
\[
\|y\| = \lambda m,
\]
where \( x_0 \) and \( \lambda \) are integers such that \( \gcd(x_0, \lambda) = 1 \).

By Fact 2 in the proof of Theorem 3.2 in [3], there exists an integer \( e \in \mathbb{Z} \) such that
\[
(23) \quad \gcd(x, \|y + ex\|/m) = 1.
\]

By computation, we see that
\[
\|y + ex\| = m(\lambda + 2x_0 ey_1 + mx_0^2 e^2) = \|y\| + 2x_0 ey_1 + x_0^2 e^2.
\]

Set \( \gamma = \gcd(x, \|y + ex\|) \). Since \( x = mx_0 \), we see from the above equation that \( m \) divides \( \gamma \). By (23), there exist integers \( a, b \) such that
\[
a x + b (\|y + ex\|/m) = 1,
\]
and thus
\[
(\lambda a + b) x + b \|y + ex\| = m.
\]
Thus \( \gamma \) divides \( m \), and therefore \( m = \gamma = \gcd(x, \|y + ex\|) \). Using the results from Case 1 with \( x, y + ex \) in the roles of \( x,y \), respectively, we deduce that \( [x \ y + ex] \in \text{EBIG}^{18}_{12}(\mathcal{O}_k) \). Since
\[
[x \ y]^{e_{1.2}} = [x \ y + ex],
\]
we deduce that \( [x \ y] \in \text{EBIG}^{19}_{12}(\mathcal{O}_k) \).

Subcase 2B. \( \alpha \equiv 1 \pmod{4} \).

In this subcase, \( \mathcal{O} = \mathbb{Z}[(1 + \sqrt{\alpha})/2] \), and each element in \( \mathcal{O} \) can be written in the form \( a + b \sqrt{\alpha} \) for some integers \( a, b \in \mathbb{Z} \) with \( a \equiv b \pmod{2} \).

By Facts 2(a) and 2(b) in Step 3 of the proof in Theorem 3.2 in [3], there exists an integer \( e \in \mathbb{Z} \) such that
\[
(24) \quad \gcd(x, \|y + ex\|/m) = 1.
\]

Using (24), and the same arguments as in Subcase 2A, we deduce that \( [x \ y] \in \text{EBIG}^{19}_{12}(\mathcal{O}_k) \).

By what we have verified in Cases 1 and 2, it follows from Lemma 2.12 that if \( x,y \) are elements in \( \mathcal{O}_k \) that satisfy conditions (i) and (ii) above, then \( [x \ y] \in \text{EBIG}^{19}_{13}(\mathcal{O}_k) \). By the discussion at the beginning of the proof, the theorem follows immediately.

\[\Box\]

REFERENCES
