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CBE 20258 Numerical & Statistical Analysis
Mid-Term Exam
March 5, 2015

You may have one hand-written sheet of notes for this exam

Problem 1. (10 points) Singular Value Decomposition. A non-square (nxm where n≠m) matrix $\underset{\approx}{A}$ undergoes SVD such that:

$$\underset{\approx}{A} = \underset{\approx}{U}\,\underset{\approx}{\Sigma}\,\underset{\approx}{V}^T$$

a. What are the dimensions of:

1) $\underset{\approx}{U}$

2) $\underset{\approx}{\Sigma}$

3) $\underset{\approx}{V}$

b. What is the inner product of the first and second columns of $\underset{\approx}{U}$ (e.g., U(: , 1)'*U(: , 2) in matlab)

c. What does matlab yield for the inner product of the first column of $\underset{\approx}{U}$ with the first column of $\underset{\approx}{V}$?

Problem 2. (15 points) Singular Value Decompostion. A **square** matrix $\underset{\approx}{A} = \underset{\approx}{U}\,\underset{\approx}{\Sigma}\,\underset{\approx}{V}^T$ is decomposed as depicted below. All norms in this problem are taken to be **2-norms**.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} = \begin{bmatrix} -0.2093 & 0.9644 & 0.1617 \\ -0.5038 & 0.0353 & -0.8631 \\ -0.8380 & -0.2621 & 0.4785 \end{bmatrix} \begin{bmatrix} 17.4125 & 0 & 0 \\ 0 & 0.8752 & 0 \\ 0 & 0 & 0.1969 \end{bmatrix} \begin{bmatrix} -0.4647 & -0.5538 & -0.6910 \\ -0.8333 & 0.0095 & 0.5528 \\ 0.2995 & -0.8326 & 0.4659 \end{bmatrix}$$

We are using this to solve the problem $\underset{\approx}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$

a. Suppose $\underset{\sim}{b} = \begin{bmatrix} 1.9288 \\ 0.0706 \\ -0.5243 \end{bmatrix}$ (e.g., twice the second column of $\underset{\approx}{U}$). Using SVD (rather than the matrix $\underset{\approx}{A}$ - show how you do this via orthogonality!) what is the solution vector $\underset{\sim}{x}$?

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b. If we have some error in $\underset{\sim}{b}$ given by $\Delta\underset{\sim}{b}$, this will yield some error in $\underset{\sim}{x}$ given by $\Delta\underset{\sim}{x}$. What is the maximum of $\|\Delta\underset{\sim}{x}\| / \|\Delta\underset{\sim}{b}\|$?

c. In control, we often go the other way – there is some error in the output $\Delta\underset{\sim}{b}$ due to error in the input $\Delta\underset{\sim}{x}$. What is the maximum value of $\|\Delta\underset{\sim}{b}\| / \|\Delta\underset{\sim}{x}\|$?

d. OK, now we put this together. Suppose we have some target vector $\underset{\sim}{b}$ (the desired output, not necessarily that from part a), and calculate some vector $\underset{\sim}{x}$ which solves the problem. If we then add some noise to $\underset{\sim}{x}$ such that $\Delta\underset{\sim}{x} = O(\varepsilon\,\underset{\sim}{x})$ (e.g., $\Delta\underset{\sim}{x}$ scales in magnitude with $\underset{\sim}{x}$), what is the maximum value of the 2-norm of the resulting error $\|\Delta\underset{\sim}{b}\| / \|\underset{\sim}{b}\|$?

e. Suppose that we find that this is too large. **Briefly** describe how you could use SVD to reduce the maximum of $\|\Delta\underset{\sim}{b}\| / \|\underset{\sim}{b}\|$.

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Problem 3. (10 points) Error Propagation: Suppose we have two random variables x_1 and x_2 which have the means and matrix of covariance:

$$\underset{\sim}{\mu_x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \underset{\approx}{\Sigma_x^2} = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 1 \end{bmatrix} * 10^{-4}$$

We wish to calculate two quantities $z_1 = f_1(\underset{\sim}{x})$ and $z_2 = f_2(\underset{\sim}{x})$, e.g.,

$$\underset{\sim}{z} = \begin{bmatrix} x_2 \ln x_1 \\ x_1^2 x_2 \end{bmatrix}$$

a. Write down the matrix used in error propagation $\underset{\sim}{\nabla} \underset{\sim}{f}$ in terms of the derivatives of f_1 and f_2 with respect to x_1 and x_2.

b. What is the matrix of covariance of $\underset{\sim}{z}$ in terms of these derivatives and the matrix of covariance of $\underset{\sim}{x}$ given above (numerical result, please)? Circle the covariance of z_1 and z_2 in this matrix.

Problem 4. (10 points) Averages and Distributions: A student is measuring the radioactivity of a sample and observes 100 counts in 1 minute of observation. Remember that radioactivity obeys the Poisson distribution you looked at for the tracer study in Project 5.

a. How long should the student observe the sample until the random uncertainty in the count rate (counts per minute, CPM) has a 2σ value of 1% of the count rate?

b. If other sources of error contribute an uncertainty of 5%, what is a reasonable choice for the observation time (recognizing that time = money)?

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Problem 5. (15 points) Linear Regression: You are interested in examining yearly temperature variations and determining what date the highest temperature is likely to occur. We have a record of the daily high temperature over a 6 year period and a corresponding series of times (ignore leap year issues for this exam!). We fit the data to the model:

$$T = C_1 + C_2 \cos\left((t - C_3)\frac{2\pi}{365}\right)$$

We want to get C_3, the day corresponding to the warmest day of the year.

a. Transform this model so that it can be used in linear regression. Hint: recall that cos(a+b) = cos(a)cos(b)-sin(a)sin(b).

b. What is the new matrix of modeling functions based on this linearization?

c. How can we calculate C_3 from our new modeling parameters?

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d. Making the usual assumptions of independent random error in linear regression, show how you would calculate the uncertainty in C_3.

e. A friend says "but what about heat waves, el nino and la nina?". What point is your friend trying to make, and what does it say about your error calculation in d?

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Problem 6. (10 points) Getting it right: How would you modify your calculations for problem 5 to get the correct estimate of the 95% confidence interval for C_3?