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This means that the columns of $\underset{\sim}{U}$ and $\underset{\sim}{V}$ are *also* orthogonal!

This is easy to show. Let's look at the product:

$$\underset{\sim}{C} = \underset{\sim}{U}^T \underset{\sim}{U} = \underset{\sim}{I}$$

Now the element c_{ij} is the inner product of the i^{th} column of $\underset{\sim}{U}$ (e.g., $\underset{\sim}{u}_i$) w/ the j^{th} column!

$$C_{ij} = \underset{\sim}{u}_i^T \underset{\sim}{u}_j = I_{ij}$$

But if $i \neq j$ this is an off-diag. element of the identity matrix!

$$\text{So} : \quad \underset{\sim}{u}_i^T \underset{\sim}{u}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

The same applies to $\underset{\sim}{V}$!

We can use the values (magnitude) of σ_{ii} to explore the sensitivity of $\underset{\sim}{z}$ to variations in $\underset{\sim}{X}$.

Suppose the model $\underset{\sim}{z} = \underset{\sim}{f}(\underset{\sim}{x})$ represents a multiple input / multiple output model of plant operations.

$\underset{\sim}{x}$ might be the mass flow rate of input streams

$\underset{\sim}{z}$ might be the resulting products needed elsewhere in the plant.

In general, you want to minimize $\underset{\sim}{z}'$ - the deviation from some optimal output $\underset{\sim}{z}_0$.

Question: what fluctuations $\underset{\sim}{x}'$ give rise to the <u>maximum</u> change in $\underset{\sim}{z}'$?

This is addressed by SVD, looking at the <u>largest</u> singular value σ_{11}.

The most dangerous mode is any vector of disturbances $\underset{\sim}{x}'$ proportional to the <u>first</u> <u>column</u> of $\underset{\approx}{\tilde{V}}$

The resulting output vector is proportional to the first column of $\underset{\sim}{U}$

The amplification factor is the corresponding singular value σ_{11}

If the i^{th} singular value is zero, then ~~that~~ a disturbance vector $\underset{\sim}{x}' \sim V(:,i)$ (i^{th} column of $\underset{\sim}{V}$) has no effect on $\underset{\sim}{z}$!

In the homework project 3 you are doing the same thing, only in reverse: what is the magnitude of $\underset{\sim}{x}$ to get some output $\underset{\sim}{z}$.

Thus, you are interested in the smallest singular values, rather than (as in the usual case) the largest.

This all is an example of Principal Component Analysis, or PCA.

Data Compression

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Let $\underset{\sim}{u}_i$ be the columns of $\underset{\approx}{U}$
and let $\underset{\sim}{v}_i$ be the columns of $\underset{\approx}{V}$

(or the rows of $\underset{\approx}{V}^T$)

Then:

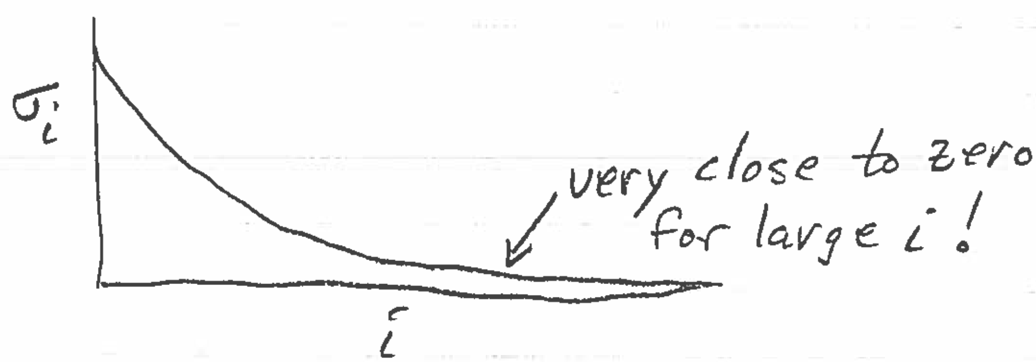
$$\underset{\approx}{A} = \sum_{i=1}^{n} \sigma_i \, \underset{\sim}{u}_i \, \underset{\sim}{v}_i^T$$

vector composition or outer product

Suppose $\underset{\approx}{A}$ is a huge matrix, say

$\underset{\approx}{A} \equiv 512 \times 512$ (usual picture size)

If we plot σ_i vs. i we get:



Thus if we truncate picture after $i \cong 20$ we can reproduce the original picture!

Take $$\underset{\approx}{A} \cong \sum_{i=1}^{20} \sigma_i \underset{\sim}{u}_i \underset{\sim}{v}_i^T$$

This has a total of

$$20 \times (1 + 512 + 512) = 20{,}500$$

vs. 262,144 elements!

For larger matrices you get even greater reduction in size

Now SVD requires a large # of computations, so it's a tradeoff between storage/transmission considerations and comp. speed!

Recently, transmission rates have not kept up w/ processor speed, so equation has shifted toward computationally expensive compression

An extreme example: the Gallileo Jupiter probe

- stuck high gain antenna, so must use wimpy low gain antenna
- transmission rate = $O(10 \text{ bps})$!

$\Rightarrow$ Use on-board computers to compress images, then transmit very slowly!

They are using JPEG compression - not quite SVD, but related. It's using fourier transform of data & only transmitting some of the modes.

You really appreciate data compression when downloading images over a modem!

~~Run matlab example~~

we'll look at SVD again later

Systems of linear equations

Summary of Solution approaches

$$A x = b$$

1) If A is square & well conditioned

   a) PLU factorization (Gaussian elimination

   b) A^{-1} so $x = A^{-1} b$

      - convenient, but
      - squares cond(A) & takes 3x longer...

   c) If A is sparse special techniques are much fas

2) If A is square but close to singularity

   a) QRP factorization
      (zero out rows of R which are very small)

   b) SVD
      (zero out singular values which are very small)

      ⇒ computationally intensive, but guaranteed to get sol'n w/ min 2-norm

3) If $\underset{\approx}{A}$ is overdetermined $(n > m)$
- often occurs in regression

a) QRP

b) $\underset{\approx}{A}^T \underset{\approx}{A} \underset{\sim}{x} = \underset{\approx}{A}^T \underset{\sim}{b}$

both minimize 2-norm of residual

- square. $m \times m$ problem
- solve via LU, inverse, etc.
- if m is large, cond maybe a problem

4) If $\underset{\approx}{A}$ is underdetermined $(m > n)$

- use SVD (min 2-norm of $\underset{\sim}{x}$)

SVD: Uses

1) Solve linear equations, particularly those close to singularity

2) Get <u>approx</u> solutions which yield more robust control by zeroing out smallest singular values

3) Analyze transfer functions:

$\underset{\approx}{U}$ ≡ orthonormal basis set of output vectors

$\underset{\approx}{V}$ = orthonormal basis set of input vectors

$\underset{\approx}{\Sigma}$ = amplification factors

4) Compress images:

outer product yields matrix

singular values

$$\underset{\approx}{A} = \operatorname*{sum}_{i} \left\{ \underset{\sim}{u}_i \, \underset{\sim}{v}_i^T \, \sigma_{ii} \right\}$$

col. of $\underset{\approx}{U}$ col of $\underset{\approx}{V}$

– computationally expensive, but good for very large $\underset{\approx}{A}$

5) Face Detection:

The columns of U are an ordered (by importance) set of eigenfaces

6). Principal Component Analysis:
By examining the <u>projection</u> of a data set on $\underset{\approx}{U}$, can distinguish pop.