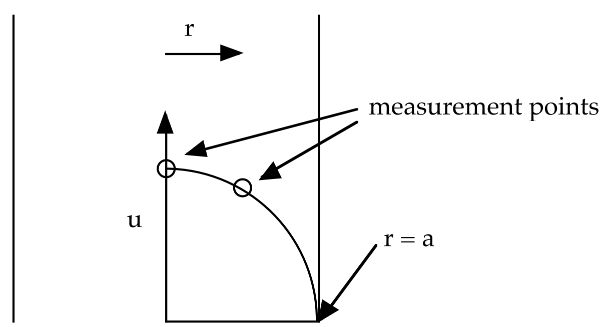
CBE 20258 Numerical & Statistical Analysis
Final Exam
May 3, 2016

You may have one hand-written sheet of notes for this exam

Problem 1. (10 points) Error Propagation: It's tricky to measure the diameter of a cylinder without calipers, but a student proposes to do it by using a ruler to measure the length, weighing it, and knowing what the density of the material is. If we know both the mass and density to a precision of 1%, how accurately do we need to measure the length to get the diameter to a precision of 1% as well?

Problem 2. (20 points) Quadrature: You are trying to measure the flow rate for start-up flow in a tube of radius a as depicted below. While the profile will eventually become parabolic (as in your homework problem), it doesn't start out that way! It is still going to be an even function of r (e.g., only even powers of r), and the velocity is still always zero at the wall. You have two probes, one which you place at the center of the tube and one which you can place at an arbitrary radial position r_c.



a. If the flow rate is given by:

$$Q = \int_0^a u \, 2\pi r \, dr$$

determine the optimum location of the second probe and the corresponding quadrature rule that achieves the highest polynomial accuracy. Hint: the velocity is just the linear combination of polynomials of the form $r^{2i}(a^2-r^2)$ where i = 0, 1, 2, 3, 4, etc. Also, don't forget to scale everything before starting: it makes it much easier to solve!!!

b. If there is some independent random error in the measurements σ_u, what will be the corresponding random error in Q? Which probe measurement dominates the error calculation (e.g., which one would it really pay to get right)?

Problem 3. (20 points) Systems of Differential Equations: A common chemical reaction is a liquid catalyzed second order reaction such as is depicted below. Basically, two gases A and B dissolve at the surface of a liquid film of thickness L and react with a rate $K\, c_A\, c_B$ where K is a second order rate constant. Their diffusion coefficients are D_A and D_B, respectively. The species concentrations at the surface are given by the equilibrium concentrations c_{A0} and c_{B0}, while the derivatives of the concentrations at the impermeable inner wall are zero. The concentrations of the species at steady-state are governed by the differential equations:

$$D_A \frac{d^2 c_A}{dx^2} - K c_A c_B = 0 \quad ; \quad D_B \frac{d^2 c_B}{dx^2} - K c_A c_B = 0$$

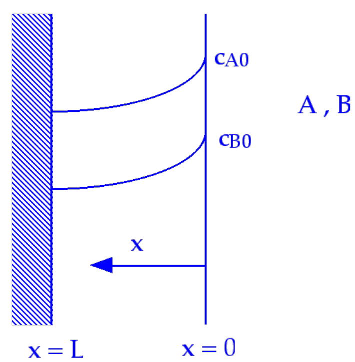
and by the boundary conditions:

$$c_A|_{x=0} = c_{A0} \; ; \; c_B|_{x=0} = c_{B0} \; ; \; \left.\frac{dc_A}{dx}\right|_{x=L} = \left.\frac{dc_B}{dx}\right|_{x=L} = 0$$

a). Recast this problem in terms of a system of first order equations, carefully defining all variables, initial conditions, any shooting parameters, etc.

b). Briefly describe how you would determine the stability characteristics of the system, as well as the maximum step size allowable for explicit integration schemes (e.g., RK techniques), defining any matrices and techniques you would use.

c). You can most easily solve this problem in matlab using two of the canned routines (such as has been demonstrated in class). What are they, and how would they be used?



Problem 4. (20 points) Investing Strategies: You have been given a sum to invest for one year, and can choose between two possible investments A & B. You will put a fraction x in A and 1-x in B. You would like to figure out some rational basis for picking x!

a. If the annual yields on each of these investments over the last 10 years are the series A_i and B_i, describe how you could estimate the probability distribution the annual yield of your investment strategy as a function of x. Be as complete as possible!

b. If the expected yields of A and B are the same, develop an expression for x that minimizes the uncertainty in your yield in terms of variances and covariances.

Problem 5. (30 points) Short Answer: Please answer the following questions *briefly.*

1. What is a slack variable and when is it useful?

2. What is the primary benefit of implicit integration methods over explicit ones?

3. What is the primary disadvantage of implicit integration methods relative to explicit ones?

4. What is the difference between a 2-stage Runge-Kutta method and the Trapezoidal Rule method for integration? For what case would they yield identical results?

5. What is a penalty function and when would you use it?

6. t-statistics (or the t-distribution) are closely related to z-statistics. How do they differ, and under what conditions?

7. What is the shooting method and when do you need to use it?

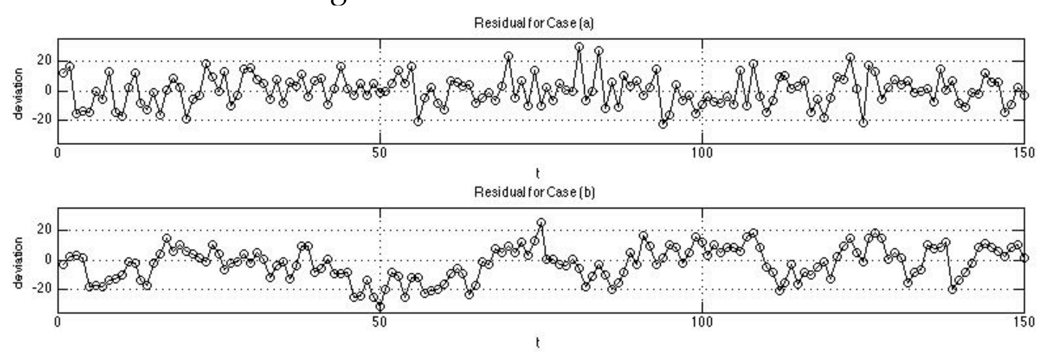
8. A controller has the model $\underset{\approx}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$ where $\underset{\sim}{b}$ are the measurements and $\underset{\sim}{x}$ are the calculated control actions. It is found that the $\underset{\sim}{x}$ are way too large (unstable controller). How could you fix it?

9. What are the polynomial degrees of three point Gaussian Quadrature and Simpson's Rules? Why is GQ more accurate?

10. A well-posed linear regression problem has 20 data points and 4 modeling functions. How many degrees of freedom are there in the problem, and what is the rank of the matrix of modeling functions $\underset{\approx}{A}$?

11. In a system of equations $\underset{\approx}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$, $\underset{\sim}{x}$ has units of time and $\underset{\sim}{b}$ has units of length. If the SVD of $\underset{\approx}{A}$ is $\underset{\approx}{A} = \underset{\approx}{U}\,\underset{\approx}{\Sigma}\,\underset{\approx}{V}^T$, what are the units of $\underset{\approx}{U}$ and $\underset{\approx}{\Sigma}$?

12. The residuals depicted below are observed after performing linear regression for two different time series. What can you conclude about the data in the two cases, and how might it influence your calculation of the regression coefficient error?



13. You are trying to calculate the concentration of a biomarker by tagging it with a radioactive isotope. How many counts do you need to observe to get the concentration right to +/- 10% at the 95% confidence level?

14. You are estimating parameters from experimental measurements via linear regression and your boss wants you to reduce the error bounds. If you just repeat all your measurements again, by what factor would your error bounds change?

15. In the answer to the above question you are making a really key assumption that is often in error. What is it?