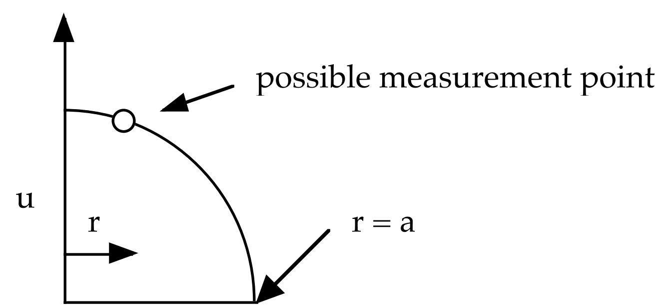
CBE 20258 Numerical & Statistical Analysis
Final Exam
May 10, 2017

You may have one hand-written sheet of notes for this exam

Problem 1. (10 points) Error Propagation: It's surprisingly tricky to measure the inside diameter of a small tube such as a straw with any degree of accuracy. Rather than, say, holding the end up to a ruler, it is proposed to instead fill the tube with water, let it drain out into a beaker, and then weigh the water. Using the weight of the water and the density, you can calculate the ID – if you know the fill length. If we measure the mass of water to a precision of 1% and we know the density to a precision of 0.1% (it does vary a bit with temperature), how accurately do we need to measure the filled length to get the diameter to a precision of 1% as well?

Problem 2. (20 points) Quadrature: You are trying to measure the flow rate for unsteady flow in a tube of radius a as depicted below. At steady-state the profile is just a parabola (as you will show next semester!), but for unsteady flows such as occur in blood flow through arteries, etc., it isn't. It is still going to be an even function of r (e.g., only even powers of r), and the velocity is still always zero at the wall. You have a single probe which you can place at an arbitrary radial position r_c.



The flow rate is given by $Q = \int_0^a u\, 2\pi r\, dr$ where u is the fluid velocity.

a. If the probe is located at the center (r = 0) determine the quadrature rule that achieves the highest polynomial accuracy (e.g., your best estimate of the flow rate given this probe location).

b. As you know, you can do much better: Determine the optimal location for the probe and the corresponding quadrature rule. Hint: Don't forget to scale everything before starting: it makes it much easier to solve!!!

Problem 3. (20 points) Systems of Differential Equations: A classic problem in heat transfer is the cooling of a fin or rod due to natural convection. The dimensionless temperature along a rod is governed by the second order differential equation:

$$\frac{d^2T}{dx^2} - C\,T^{5/4} = 0$$

where T is the local dimensionless temperature. The dimensionless constant C depends on the diameter, length, and thermal conductivity of the rod, as well as properties of the air. The boundary conditions are:

$$T\Big|_{x=0} = 1 \qquad ; \qquad \frac{dT}{dx}\Big|_{x=1} = 0$$

a). Recast this problem in terms of a system of first order equations, carefully defining all variables, initial conditions, any shooting parameters, etc.

b). Is the system of differential equations stable or unstable? Show your work.

c). You can most easily solve this problem in matlab using two of the canned routines (such as has been demonstrated in class). What are they, and how would they be used?

Problem 4. (20 points) Model Linearization: Power-law correlations are very useful for many engineering applications. In fluid mechanics, for example, complex fluids such as polymers or suspensions often exhibit a power law relationship for fluid viscosity as a function of shear rate (the rate with which you deform the fluid). In these sorts of experiments, you change the shear rate γ and measure the resulting shear stress τ. We fit this to the empirical relationship:

$$\tau = C\gamma^n$$

a. Show how you can use unweighted linear regression to determine the unknown parameters C and n from a set of measurements of τ as a function of γ. Set it up in the form Ax = b, defining A, x, and b.

b. If you are using unweighted regression, what are you assuming about the measurement error in τ?

c. How would you calculate the standard deviation of the power law exponent n?

Problem 5. (30 points) Short Answer: Please answer the following questions *briefly.*

1. What is a slack variable and when is it useful?

2. What is the primary benefit of implicit integration methods over explicit ones?

3. What is the primary disadvantage of implicit integration methods relative to explicit ones?

4. What is the difference between a 2-stage Runge-Kutta method and the Trapezoidal Rule method for integration? For what case would they yield identical results?

5. What is a penalty function and when would you use it?

6. t-statistics (or the t-distribution) are closely related to z-statistics. How do they differ, and under what conditions?

7. You make N measurements x_i. How do you calculate the error in the **mean** of these measurements? What assumption are you making?

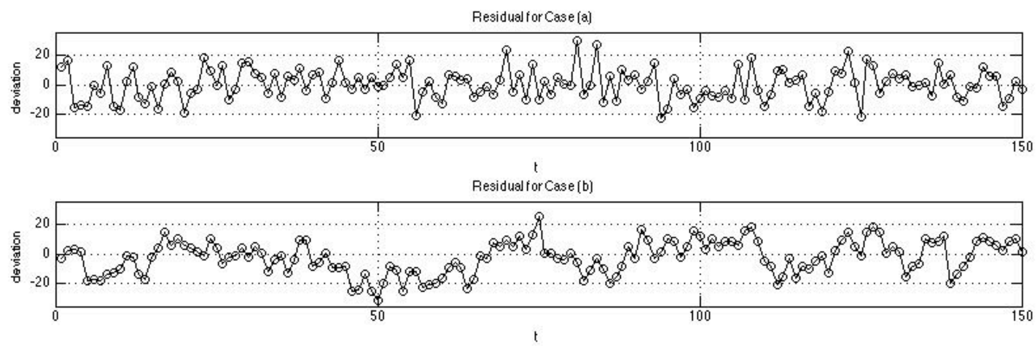
8. A controller has the model $\underset{\approx}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$ where $\underset{\sim}{b}$ are the measurements (deviations from the target values) and $\underset{\sim}{x}$ are the calculated control actions. It is found that the $\underset{\sim}{x}$ are way too large (unstable controller). Why might this occur and how could you fix it?

9. What are the polynomial degrees of three point Gaussian Quadrature and Simpson's Rules? Why is GQ more accurate?

10. A well-posed linear regression problem has 10 data points and 3 modeling functions. How many degrees of freedom are there in the problem, and what is the rank of the matrix of modeling functions $\underset{\approx}{A}$?

11. If we regard the system of equations $\underset{\approx}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$ as a transfer function and the SVD of $\underset{\approx}{A}$ is $\underset{\approx}{A} = \underset{\approx}{U}\,\underset{\approx}{\Sigma}\,\underset{\approx}{V}^T$, what are the interpretations of $\underset{\approx}{U}$, $\underset{\approx}{V}$, and $\underset{\approx}{\Sigma}$?

12. The residuals depicted below are observed after performing linear regression for two different time series. What can you conclude about the data in the two cases, and how might it influence your calculation of the regression coefficient error?



13. You are trying to calculate the concentration of a biomarker by tagging it with a radioactive isotope. The nominal count rate is 400cpm (counts per minute). How long do you need to observe to get the concentration right to +/- 5% at the 95% confidence level? Show your work!

14. You are estimating parameters from experimental measurements via linear regression and your boss wants you to reduce the error bounds. If you just repeat all your measurements again, by what factor would your error bounds change?

15. In the answer to the above question you are making a really key assumption that is often in error. What is it?