Ok, let's review error calc

Sample mean $\bar{x} = \frac{1}{N} \sum x_i$

population mean $\mu = E(x)$

(or $E(\bar{x})$)

Sample variance

$$s_x^2 = \frac{1}{N-1} \sum (x_i - \bar{x})^2$$

$$= \frac{1}{N-1} \left(\sum x_i^2 - N \bar{x}^2 \right)$$

Pop. variance $\sigma^2 = E(s_x^2)$

st dev $= \sigma$ ($\sqrt{\ }$ of variance)

Matrix of covariance:

$$\underset{\sim}{V} = \begin{bmatrix} s_{x_1}^2 & s_{x_1 x_2}^2 & \cdots \\ \vdots & \ddots & \\ \vdots & & \ddots \\ & \cdots & s_{x_n}^2 \end{bmatrix}$$

where $s_{x_1 x_2}^2 = \frac{1}{N-1} (x_1 - \bar{x}_1)(x_2 - \bar{x}_2)$

where x_1, x_2 are dif types of meas (say, apples & oranges)

②

$$\underset{\sim}{\Sigma}_x^2 = E(\underset{\sim}{V})$$

Error propagation:

<u>If</u> $y = c_1 x_1 + c_2 x_2$

then $\sigma_y^2 = c_1^2 \sigma_{x_1}^2 + c_2^2 \sigma_{x_2}^2 + 2 c_1 c_2 \underbrace{\sigma_{x_1 x_2}}_{\text{covariance}}$

<u>If</u> $y = x_1 x_2$

then $\dfrac{\sigma_y^2}{y^2} \underset{\text{approx.}}{\cong} \dfrac{\sigma_{x_1}^2}{x_1^2} + \dfrac{\sigma_{x_2}^2}{x_2^2} + 2 \dfrac{\sigma_{x_1 x_2}^2}{x_1 x_2}$

↳ ignore higher order terms

In general, <u>if</u>

$y = f(\underset{\sim}{x})$

Then $\sigma_y^2 \cong \underset{\sim}{\nabla} f \, \underset{\sim}{\Sigma}_x^2 (\underset{\sim}{\nabla} f)^T$

For a vector function:

$$\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{x})$$

we get:

$$\underset{\approx}{\Sigma}_{\underset{\sim}{y}}^{2} \cong (\underset{\sim}{\nabla}\underset{\sim}{f}) \underset{\approx}{\Sigma}_{x}^{2} (\underset{\sim}{\nabla}\underset{\sim}{f})^{T}$$

$$\text{where } \underset{\sim}{\nabla}\underset{\sim}{f} \equiv \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

OK, given all this, what do we use it for?

— Usually hypothesis testing!

⇒ Do measurements match theory within error? To what conf. level,

⇒ Is a disease (say cancer) assoc. w/ diet? Lifestyle?

⇒ Is a new drug regimen superior to standard treatment? Placebo?

We usually test this by examining the <u>Null Hypothesis</u>

What is the prob. that the meas. mean/response/value, etc. is really from the same dist?
— If prob of null hyp. H_0 < threshold value (usually 0.05) <u>reject</u> null hyp. & accept alt hyp.

How do we do this? Suppose we <u>know</u> pop. mean & S.D., what is prob. meas <u>not</u> from a population?

Form <u>z-statistic</u>

$$z = \frac{x - \mu}{\sigma}$$

For multivariable system: col vector

$$z^2 = (\underset{\sim}{x} - \underset{\sim}{\mu})^T \underset{\sim}{\Sigma}_x^{-2} (\underset{\sim}{x} - \underset{\sim}{\mu})$$

inverse of $\underset{\sim}{\Sigma}_x^2$

What is the <u>level</u> <u>of</u> <u>significance</u>?

α	conf interval	z
0.10	90%	± 1.645
0.05	95%	± 1.96 ($\sim 2\sigma$ level)
0.01	99%	± 2.575
0.005	99.5%	± 2.81
0.001	99.9%	± 3.27

Ok, how can we use this?
Suppose we want to test a theory: Stokes sedimentation velocity
At low Re,

$$U_s = \frac{2}{9}\frac{\Delta\rho g a^2}{\mu}$$

Lets test this!

Let $$y = \frac{U_s}{\frac{\Delta\rho g a^2}{\mu}} = f(\underset{\sim}{x})$$

We <u>measure</u> each of these values!

Say $x_1 = U_s$, $x_2 = \Delta\rho$, $x_3 = g$, etc.

We figure out uncertainty in each!

$\sigma_{U_s}^2$, σ_a^2, etc...

$$\underset{\sim}{\Sigma}_x^2 = \begin{bmatrix} \sigma_{U_s}^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_\mu^2 \end{bmatrix}$$

(assume indep)

$$\underset{\sim}{\nabla} f = \left(\frac{\partial f}{\partial U_s}, \frac{\partial f}{\partial \Delta\rho}, \ldots \right)$$

$\Rightarrow$ take numerically or analytically

$$\sigma_y^2 \approx \underset{\sim}{\nabla} f \ \underset{\sim}{\Sigma}_x^2 \left(\underset{\sim}{\nabla} f\right)^T$$

form z-statistic $z \doteq \dfrac{y - \frac{2}{9}}{\sigma_y}$

Usually, det. if "agrees to within 2σ" — or not!

Note: random chance will cause it to be outside 2σ interval 5% of time!

Actually, it's more complex than this! Usually the ~~#~~ meas. is fairly small. Thus, we don't know σ, instead have est s_x

We can form the t-statistic

$$t = \frac{y-\mu}{s_x}$$

If the sample size is small, t is not normally distributed.

This is because s_x is a random variable too!

There is a sig chance that s_x is too small, which makes t too big → increase prob of tail of dist

Normally, prob of $|Z| > 3$ is very small — but if s_x is underestimated, t can easily be > 3!

Run Example

t-test has interesting history:
May know as "student t-test" or "Student's t-test"
⇒ Really should be Gosset t-test
— was a brewer working for Guinness, had corporate freeze on publication. Had to publish anonomously under name "Student" in '08

Ok, what does this mean?

If we know σ, 95% meas within $\pm 2\sigma$ of mean

If est s_x from 2 meas, (1 deg freedom)

95% prob within $\pm 12.7 s_x$!

If $n > 5$ and look at 1σ limit, nearly identical — only affects tails, at small n

Ok, what about other tests?

Suppose we have 2 samples (say, response from 2 treatments) — is the response any dif?

$\Rightarrow$ Calc. $\bar{x}_1$, $\bar{x}_2$

var: $s_{x_1}^2$, $s_{x_2}^2$

Null Hyp: $\Delta x = \bar{x}_1 - \bar{x}_2 = 0$

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What is the SD in Δx?

$$\Delta x = \bar{x}_1 - \bar{x}_2$$

$$\therefore\ S^2_{\Delta x} = \underbrace{S^2_{\bar{x}_1} + S^2_{\bar{x}_2}}_{\text{var}_2 \text{ in } \underline{\text{mean}}}$$

$$S^2_{\bar{x}_1} = \frac{S^2_{x_1}}{n_1}, \quad S^2_{\bar{x}_2} = \frac{S_{x_2}}{n_2}$$

We <u>define</u> $t = \dfrac{\Delta x}{S_{\Delta x}}$

If > threshold, <u>reject</u> null hyp.

If n_1, n_2 <u>large</u> 95% conf $|t| \gtrsim 2$

for smaller n_1, n_2 use t-tables

Actually, easier to use matlab $f^{\underline{n}}$ (stat. tool box)

<u>Common functions</u>

cdf, normcdf, norminv

$\Rightarrow$ based on Gaussian dist.

tcdf – t cum dist fⁿ

⇒ can do hyp. testing w/
ztest, ttest, ttest2

2 samples

Ok, what if we want to test many samples? suppose we look to see if there is a difference among many groups

We could do a t-test between each sample in a combinatoric fashion, but you are guaranteed a false positive! In text, talks about looking at relation between 8 groups ⇒ $7 \times 8/2 = 28$ combinations

Even random chance at 95% conf. level would lead to at least 1 false pos.

A better way is ANOVA, analysis of variance

Basically, you look at the variance of "supergroup" and variance of all the subgroups.

If the variance of supergroup is larger than expected from subgroups, then at least one of the subgroups doesn't belong.

This ratio yields the F-statistic which has a dist. depending on the number of degrees of freedom.

Matlab has a canned routine anova1(A) which does this to a matrix of values.

Most of these formulas require independent normally distributed data

If the data is not indep, $s_{\bar{x}}$ will be underestimated! (Effectively, n is reduced)

If the data isn't normally dist, sometimes you can fix it!

Example: Dist of particle size
- often have # density of small particles >> larger ones
- could look at radius vs mass fraction, not # of particles
- Also - log normal dist - work w/ log of variable rather than direct meas
- plot dist & see what works!