So far we have focused on ht transfer in solids, but we need to consider fluids too! The necessary eq'ns are derived in ch 11 of BS&L.

Key is that total energy is conserved!

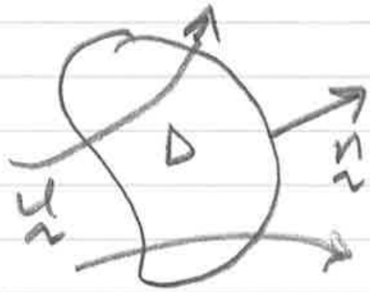
Let $\hat{U}$ be the internal energy per unit mass.

Then $\rho\hat{U}$ is internal energy/vol

Likewise $\frac{1}{2}\rho u^2$ is the KE/vol of the fluid

So $\left(\frac{1}{2}\rho u^2 + \rho\hat{U}\right)$ is the total energy/volume!

Now let's draw a control volume



$$\left\{\begin{matrix}\text{accum of } E \\ \text{in } D\end{matrix}\right\} + \left\{\begin{matrix}E \text{ out by} \\ \text{convection}\end{matrix}\right\}$$

$$+ \left\{\begin{matrix}E \text{ out by} \\ \text{conduction}\end{matrix}\right\} = \left\{ \text{sources} \right\}$$

$$\left\{\begin{matrix}\text{accum of } E \\ \text{in } D\end{matrix}\right\} \equiv \frac{\partial}{\partial t} \int_D \left(\tfrac{1}{2}\rho u^2 + \rho \hat{U}\right) dV$$

$$\left\{\begin{matrix}E \text{ out by} \\ \text{convection}\end{matrix}\right\} \equiv \int_{\partial D} \left(\tfrac{1}{2}\rho u^2 + \rho \hat{U}\right) \underset{\sim}{u} \cdot \underset{\sim}{n} \, dA$$

vol flux normal to ∂D

$$\left\{\begin{matrix}E \text{ out by} \\ \text{conduction}\end{matrix}\right\} = \int_{\partial D} \underset{\sim}{q} \cdot \underset{\sim}{n} \, dA$$

What are the sources?

This is total energy (including KE) so it includes forces as well as thermal sources:

$$\int_{\partial D} -\underset{\sim}{u}\, p \cdot \underset{\sim}{n}\, dA \quad \text{(pressure forces)}$$

shear forces (use dif. sign convention

$$+\int_{\partial D} \underset{\sim}{u} \cdot \underset{\approx}{\tau} \cdot \underset{\sim}{n}\, dA \quad \text{(viscous forces)}$$

$$+\int_{D} \rho \underset{\sim}{g} \cdot \underset{\sim}{u}\, dV \quad \text{(gravity forces)}$$

$$+\int_{D} \dot{s}\, dV \quad \text{(thermal sources like rxn, dissip, etc.)}$$

To get an eq'n at a point in the fluid, apply divergence theorem to surface integrals to convert to vol. integrals. Since D is arb. eq'n is valid at every point in flow!

So we get eq'n 11.1-7:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho\hat{U}\right) + \nabla\cdot\left[\left(\frac{1}{2}\rho u^2 + \rho\hat{U}\right)\underset{\sim}{u}\right]$$
$$= -\nabla\cdot\underset{\sim}{q} - \nabla\cdot(p\underset{\sim}{u}) + \nabla\cdot(\underset{\approx}{\tau}\cdot\underset{\sim}{u})$$
$$+ \rho(\underset{\sim}{u}\cdot\underset{\sim}{g}) + \dot{S}$$

we want to get rid of KE term & just have it involve <u>internal</u> <u>energy</u>

we <u>subtract</u> <u>off</u> the <u>mechanical</u> <u>energy</u> balance (from F=ma) used last term!

You thus get eq. 11.2-1:

$$\frac{\partial}{\partial t}(\rho\hat{U}) + \nabla\cdot(\rho\hat{U}\underset{\sim}{u}) = -\nabla\cdot\underset{\sim}{q}$$
$$-p(\nabla\cdot\underset{\sim}{u}) + \underset{\approx}{\tau}:\nabla\underset{\sim}{u} + \dot{S}$$

↑ <u>reversible</u> conversion of mech E to thermal Energy (pressure work)

↑ Irreversible conversion (viscous dissipation)

so $-p(\nabla \cdot u)$ => if you compress a gas it gets hot!
(pos or neg!)

$\tau : \nabla u$ => viscous heating due to shear - always positive (but usually small - not always!)

From Thermo we can relate $\hat{U}$ to T & P

This yields eq'n 11.2-5:

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + u \cdot \nabla T \right) = -\nabla \cdot q + \tau : \nabla u - \left(\frac{\partial \ln \rho}{\partial \ln T} \right) \frac{DP}{Dt} + \dot{S}$$

Note: $\frac{DP}{Dt} \equiv \frac{\partial P}{\partial t} + u \cdot \nabla P$

(material Derivative, Lagrangian perspective)

For a fluid at constant pressure (usual case), neglecting viscous dissip. as it is usually small, we get (w/ Fourier's Law, const. k):

eq'n 11.2-8:

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla} T \right) = k \nabla^2 T + \dot{S}$$

This is the eq'n we will use - but if you work w/ compressible gases (e.g., gas turbines!) you would use a different form! Table 11.4-1 has lots of versions for different applications!

So: let's review $\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla} T$

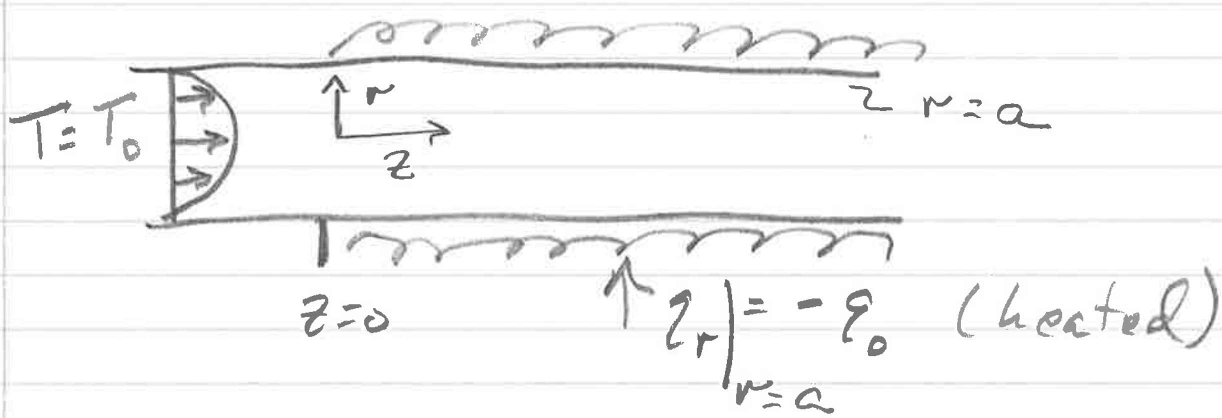
This is material derivative: there are 2 ways a fluid element can change temp. (1) $\frac{\partial T}{\partial t}$ = time deriv

(2) $\underset{\sim}{u} \cdot \underset{\sim}{\nabla} T$ ≡ convected in direction of temp. gradient!

Material derivative has both pieces!

OK, let's apply this!

Probably **the** classic problem in convective transport is the Graetz (or Nusselt-Graetz) problem: laminar flow through a heated pipe!



we want to calculate the heat transfer coefficient:

$$q_0 = h\left(T\big|_{r=a} - T_b\right)$$

T_b is the <u>bulk</u> or cup-mixing temperature ⇒ what you would get for the temp. in a cup holding the fluid coming out of the pipe!

$$\bar{T}_b \equiv \frac{\int_0^a T\, u_z\, 2\pi r\, dr}{\int_0^a u_z\, 2\pi r\, dr}$$

This is much more useful than the area avg T or centerline T!

What are the equations governing this problem? Assume incompressible flow w/ constant properties!

CE: $\quad \underset{\sim}{\nabla} \cdot \underset{\sim}{u} = 0$

NS: $\quad \rho \left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \underset{\sim}{u} \right) = -\underset{\sim}{\nabla} p + \mu \nabla^2 \underset{\sim}{u} + \rho \underset{\sim}{g}$

Energy: $\quad \rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla} T \right) = k \nabla^2 T + \mu \phi_v$

$\left(\underset{\approx}{\tau} : \underset{\sim}{\nabla} \underset{\sim}{u} \right)$

Viscous Dissip.

How do we solve?

1) choose coord system in which bdy has convenient rep! $\Rightarrow$ cyl. coord!

2) Get rid of terms that are zero

3) scale the rest to further simplify!

Start with C.E. & velocity profile!

$\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = 0$

$$\therefore \quad \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

Assume unidirectional flow!

$$\therefore \quad u_r = u_\theta = 0 \quad \text{and} \quad \frac{\partial u_z}{\partial z} = 0!$$

This is valid for fully developed flow!
otherwise (if $\frac{\partial u_z}{\partial z} \neq 0$) then $u_r \neq 0$!

Now for z-component of NS eq'ns:

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right)$$

$$= -\frac{\partial P}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right] + \rho g_z$$

Ok, assume unidirectional flow in z dir., also axisymmetric.

We can ignore ρg_z term as it is the difference from hydrostatics that drives flow $\left(-\frac{\partial P}{\partial z} + \rho g_z\right) \equiv -\frac{\partial \mathcal{P}}{\partial z}$ (augmented pressure!)

since from CE $\frac{\partial u_z}{\partial z} = 0 \therefore \frac{\partial^2 u_z}{\partial z^2} = 0$!

We're left with:

$$\rho\cancel{\frac{\partial u_z}{\partial t}} = -\frac{\partial P}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right]$$

Assume SS too!

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In general we impose a flow rate or avg. vel U which yields the pressure gradient!

Anyway, we get Poiseuille flow:

$$u_z = 2U\left(1 - r^2/a^2\right)$$

e.g., CL velocity is twice average! (geometry specific: this is for circular tube only!)

This yields $\dfrac{\partial P}{\partial z} = -\dfrac{8\mu U}{a^2}$

with flow rate $Q = U\pi a^2$

Ok, now for energy!

$$\rho \hat{C}_p\left(\frac{\partial T}{\partial t} + u_r\frac{\partial T}{\partial r} + \frac{u_\theta}{r}\frac{\partial T}{\partial \theta} + u_z\frac{\partial T}{\partial z}\right)$$

$$= k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right]$$

$$+ \mu\phi_v$$

Again, assume SS, so $\frac{\partial T}{\partial t} = 0$

unidirectional flow $\therefore u_r = u_\theta = 0$

axi-symmetric so $\frac{\partial^2 T}{\partial \theta^2} = 0$

$$\therefore \rho \hat{C}_p \, 2U\left(1 - \frac{r^2}{a^2}\right)\frac{\partial T}{\partial z} = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right] + \mu\left(\frac{\partial u_z}{\partial r}\right)^2$$

viscous dissip.

Ok, now for BC's:

$$T\Big|_{z \to -\infty} = T_0 \qquad \frac{\partial T}{\partial r}\Big|_{r=0} = 0 \quad \text{(finite)}$$

$$-k\frac{\partial T}{\partial r}\Big|_{\substack{r=a \\ z>0}} = -q_0 \quad \text{(heating)}$$

So let's scale this!

Let $r^* = r/a$, $z^* = \frac{z}{z_c}$, $T^* = \frac{T - T_0}{\Delta T_c}$

Look at inhomogeneous BC:

$$-k \frac{\Delta T_c}{a} \left.\frac{\partial T^*}{\partial r^*}\right|_{\substack{r^*=1 \\ z^*>0}} = -q_o$$

$$\therefore \quad \Delta T_c = \frac{q_o a}{k}$$

and $\left.\frac{\partial T^*}{\partial r^*}\right|_{\substack{r^*=1 \\ z^*>0}} = 1$

And for DE:

$$\rho \hat{C}_p U \frac{\Delta T_c}{z_c} 2(1-r^{*2}) \frac{\partial T^*}{\partial z^*} = k \frac{\Delta T_c}{a^2} \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right)$$

$$+ k \frac{\Delta T_c}{z_c^2} \frac{\partial^2 T^*}{\partial z^{*2}} + \mu\left(\frac{4Ur}{a^2}\right)^2$$

$$\hookrightarrow \left(\frac{16\mu U^2}{a^2}\right) r^{*2}$$

OK - next divide by scaling of important term!

Heating from wall, so radial cond. has to matter!

Divide by $\frac{k \Delta T_c}{a^2}$

So:

$$\left[\frac{\rho \hat{C}_p U a^2}{k Z_c}\right] 2\left(1-r^{*2}\right)\frac{\partial T^*}{\partial Z^*}$$

$$= \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right) + \left[\frac{a^2}{Z_c^2}\right]\frac{\partial^2 T^*}{\partial Z^{*2}}$$

$$+ \left[\frac{16\mu U^2}{k\,\Delta T_c}\right] r^{*2}$$

Choose Z_c so that we balance axial convection w/ radial conduction!

$$\therefore\ Z_c = \frac{\rho \hat{C}_p a^2}{k} U \equiv \underbrace{\left(\frac{a^2}{\alpha}\right)}_{\text{diff}^n\ \text{time}} U$$

$\therefore$ Z_c is the distance fluid is convected down the tube during radial diff^n t!

So we are left w/ two groups.

Look at $$\frac{a^2}{Z_c^2} = \frac{a^2}{\left(\frac{a^2}{\alpha}U\right)^2} = \left(\frac{\alpha}{Ua}\right)^2 = \left[\frac{\nu}{Ua}\,\frac{\alpha}{\nu}\right]^2$$

$$= \left(\frac{1}{Re}\frac{1}{Pr}\right)^2$$ normally Re is D, not a!

In general, $Re\,Pr >> 1$!!

Say, water in a 2 cm dia pipe at slow speed of 10 cm/s

$$\frac{Ua}{\nu} = \frac{(10)(1)}{0.01} = 1000!$$

$Pr \cong 7$ (20°C)

$$\therefore \frac{\alpha}{Ua} = \frac{1}{7000} \qquad \left(\frac{\alpha}{Ua}\right)^2 = \underline{2 \times 10^{-8}}!$$

That is why you neglect thermal dif^n^ in direction of motion! It's small!

Ok, what about viscous dissipation?

$$\frac{16\mu U^2}{k\Delta T_c} = \frac{16\mu U^2}{q_0 a} \approx \frac{(16)(0.01)(10)^2}{(10^3)(1)} = 0.016$$

e.g., $1\ W/m^2 = 1000\ erg/s\,cm^2$

This is actually high - usually q_0 is bigger!

Except for specific cases, if you are heating things, viscous dissipation is negligible!

So:
$$2(1-r^{*2})\frac{\partial T^*}{\partial z^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right)$$

$$T^*\Big|_{z^*=0} = 0 \qquad \frac{\partial T^*}{\partial r^*}\Big|_{r^*=1} = 1 \qquad \frac{\partial T^*}{\partial r^*}\Big|_{r^*=0} = 0 \text{ (finite)}$$

We want the asymptotic solution!

It gets hotter w/ z^*, so expect

$$T_\infty^* = z^* f_1(r^*) + f_2(r^*)$$

$$\therefore \frac{1}{r^*}(r^* f_1')' = 0 \qquad f_1'(0) = 0 \quad f_1'(1) = 0$$

This just yields $f_1 = c$ (cst!)

Now for f_2:

$$\frac{1}{r^*}(r^* f_2')' = 2(1-r^{*2})f_1; \quad f_2'(0)=0, \ f_2'(1)=1$$