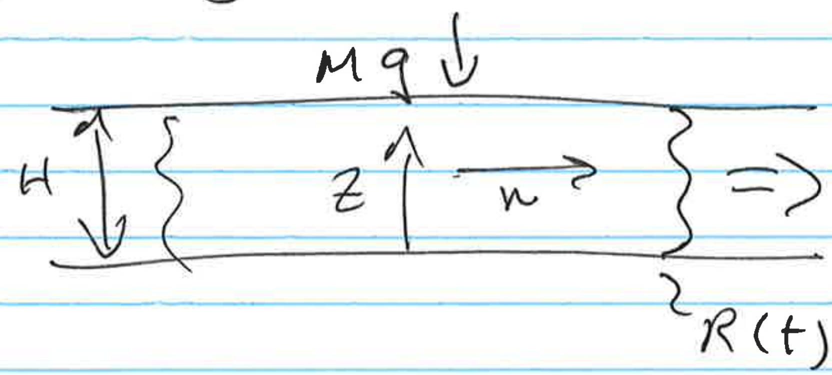
Problem of the Day 12:

squishing a fluid between two plates.



we have some $H(t)$, $R(t)$

we need u_r:

$$\frac{dR}{dt} = \langle u_r \rangle \Big|_{r=R} = \frac{1}{H}\int_0^H u_r \, dz$$

$$Mg = \int_0^R (p - p_0)\, 2\pi r \, dr$$

$$V_0 = \pi R^2 H \qquad \text{(volume of fluid)}$$

We start w/ CE: $\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = 0$

$$\therefore \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

Now we have axisym. problem!

$\therefore u_\theta = 0, \ \frac{\partial}{\partial \theta} = 0$ (all terms)

So $\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z} = 0$

Scale: $u_r^* = u_r / U_{rc}, \quad u_z^* = u_z / U_{zc}$

$r^* = r / R_o, \quad z^* = \frac{z}{H_o}$

where $\pi R_o^2 H_o = V_o$

So:

$$\frac{U_{rc}}{R_o}\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^* u_r^*) + \frac{U_{zc}}{H_o}\frac{\partial u_z^*}{\partial z^*} = 0$$

Divide out:

$$\left[\frac{H_o U_{rc}}{R_o U_{zc}}\right]\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^* u_r^*) + \frac{\partial u_z^*}{\partial z^*} = 0$$

$$\therefore U_{zc} = U_{rc}\frac{H_o}{R_o}$$

Let's scale radius eq'n:

$t^* = t/t_c$

$$\therefore \frac{R_o}{t_c}\frac{dR^*}{dt^*} = U_{r_c}\int_0^{H^*} u_r^*\Big|_{r^*=R^*} dz^*$$

where $R^* = R/R_o$, $H^* = H/H_o$

$$\therefore t_c = \frac{R_o}{U_{r_c}}$$

$$\text{and } \frac{dR^*}{dt^*} = \int_0^{H^*} u_r^*\Big|_{r^*=R^*} dz^*$$

Ok, now for radial momentum!

$$\left(\text{let } P^* = \frac{P-P_o}{\Delta P_c}\right)$$

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \cancelto{0}{\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}} - \cancelto{0}{\frac{u_\theta^2}{r}} + u_z\frac{\partial u_r}{\partial z}\right)$$

$$= -\frac{\partial P}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right) + \cancelto{0}{\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}}\right.$$

$$\left. - \cancelto{0}{\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}} + \frac{\partial^2 u_r}{\partial z^2}\right] + \cancelto{0}{\rho g_r}$$

Scale the rest! Note $\frac{U_{zc}}{H_0} = \frac{U_{rc}}{R_0}$

<u>and</u> $t_c = \frac{R_0}{U_{rc}}$ so <u>all</u> inertial

terms scale the same.

$$\frac{\rho U_{r_0}^2}{R_0}\left(\frac{\partial u_r^*}{\partial t^*} + u_r^*\frac{\partial u_r^*}{\partial r^*} + u_z^*\frac{\partial u_r^*}{\partial z^*}\right)$$

$$= -\frac{\Delta P_c}{R_0}\frac{\partial P^*}{\partial r^*} + \frac{\mu U_{rc}}{H_0^2}\left[\frac{H_0^2}{R_0^2}\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}\frac{\partial(r^* u_r^*)}{\partial r^*}\right) + \frac{\partial^2 u_r^*}{\partial z^{*2}}\right]$$

Divide by $\frac{\mu U_{rc}}{H_0^2}$:

$$\left[\frac{\rho U_{rc} H_0}{\mu}\frac{H_0}{R_0}\right]\left(\frac{\partial u_r^*}{\partial t^*} + u_r^*\frac{\partial u_r^*}{\partial r^*} + u_z^*\frac{\partial u_r^*}{\partial z^*}\right)$$

$$= -\left[\frac{\Delta P_c H_0^2}{R_0 \mu U_{rc}}\right]\frac{\partial P^*}{\partial r^*} + \frac{\partial^2 u_r^*}{\partial z^{*2}} + \frac{H_0^2}{R_0^2}\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}\frac{\partial(r^* u_r^*)}{\partial r^*}\right)$$

$$\underbrace{\left[\frac{\Delta P_c H_0^2}{R_0 \mu U_{rc}}\right]}_{=1}$$

$$\therefore U_{rc} = \frac{\Delta P_c H_0^2}{R_0 \mu} \; !$$

We can scale pressure too!

$$Mg = \Delta P_c \pi R_o^2 \int_0^{R^*} p^* \, 2r^* dr^*$$

$$\therefore \Delta P_c = \frac{Mg}{\pi R_o^2}$$

So

$$U_{rc} = \frac{Mg}{\pi R_o^2} \frac{H_o^2}{R_o \mu}$$

but

$$H_o = \frac{V_o}{\pi R_o^2}$$

$$\therefore U_{rc} = \frac{Mg V_o^2}{\pi^3 R_o^7 \mu}$$

and

$$t_c = \frac{R_o}{U_{rc}} = \frac{\pi^3 R_o^8 \mu}{V_o^2 M g}$$

So we've scaled everything!

$$0 = -\frac{\partial p^*}{\partial r^*} + \frac{\partial^2 u_r^*}{\partial z^{*2}}$$

(ignore $O\left(\frac{H_o^2}{R_o^2}\right)$ and $Re_H \frac{H_o}{R_o}$)

$$u_r^* \Big|_{z^*=0,\,H^*} = 0 \qquad u_z^* \Big|_{z^*=0} = 0$$

$$\frac{1}{r^*}\frac{\partial(r^* u_r^*)}{\partial r^*} + \frac{\partial u_z^*}{\partial z^*} = 0$$

$$\int_0^{R^*} p^* \, 2r^* \, dr^* = 1 \; ; \frac{dR^*}{dt^*} = \int_0^{H^*} u_r^* \, dz^*$$

While we've scaled relative to H_0, R_0 the scaling is valid at all R. This means that

$$\frac{dR}{dt} = \frac{M g V_0^2}{\pi^3 R^7 \mu}\lambda$$ where λ is some constant.

alternatively, $\frac{dR^*}{dt^*} = \frac{1}{R^{*7}}\lambda$, $R^*\Big|_{t^*=0} = 1$

So $R^{*7}\frac{dR^*}{dt^*} = \lambda$

$$\frac{1}{8}\frac{dR^{*8}}{dt^*} = \lambda$$

$$R^{*8} = 1 + 8\lambda t^*$$

solving the equations yields our usual parabola, but proportional to r^*:

some constant

$$u_r^* = C\,6\,r^*(z^* - z^{*2}) \qquad \underset{H^*=1,\,R^*=1}{(\text{at } t^*=0)}$$

$$\therefore \frac{\partial p^*}{\partial r^*} = \frac{\partial^2 u_r^*}{\partial z^{*2}} = -12\,C\,r^*$$

$$\therefore p^* = 6C(1-r^{*2}) \quad \text{since } p^*\Big|_{r^*=1} = 0$$

$$\int_0^1 p^*\,2r^*\,dr^* = 1 = \int_0^1 6C(1-r^{*2})\,2r^*\,dr^*$$

$$= 6C\left(1-\frac{1}{2}\right) = 3C \quad \therefore C = \frac{1}{3}$$

$$\text{and} \quad \langle u_r^* \rangle\Big|_{r^*=1} = C\int_0^1 6(z^* - z^{*2})\,dz^* = C$$

$$\text{so} \quad \langle u_r^* \rangle\Big|_{r^*=1} = \frac{1}{3} \equiv \lambda\,!$$

$$\therefore R^{*8} = 1 + \frac{8}{3} t^*$$

where $t^* = \dfrac{t}{\left(\dfrac{\pi^3 R_0^8 \mu}{V_0^2 M g}\right)}$

Now for the experiment!

We had $V_0 = 3\,ml$, $M = 582\,g$

glycerin, so $\mu = 14$ poise

We observed times at ring crossings:

n	t	$t - t_0$
6	9 s	0
7	35 s	26 s
8	94 s	85 s
9	238 s	229 s

ring location is $0.85\,cm \times n$

choose $n = 6$ as $R_0 = 5.1\,cm$

so we can plot this up!

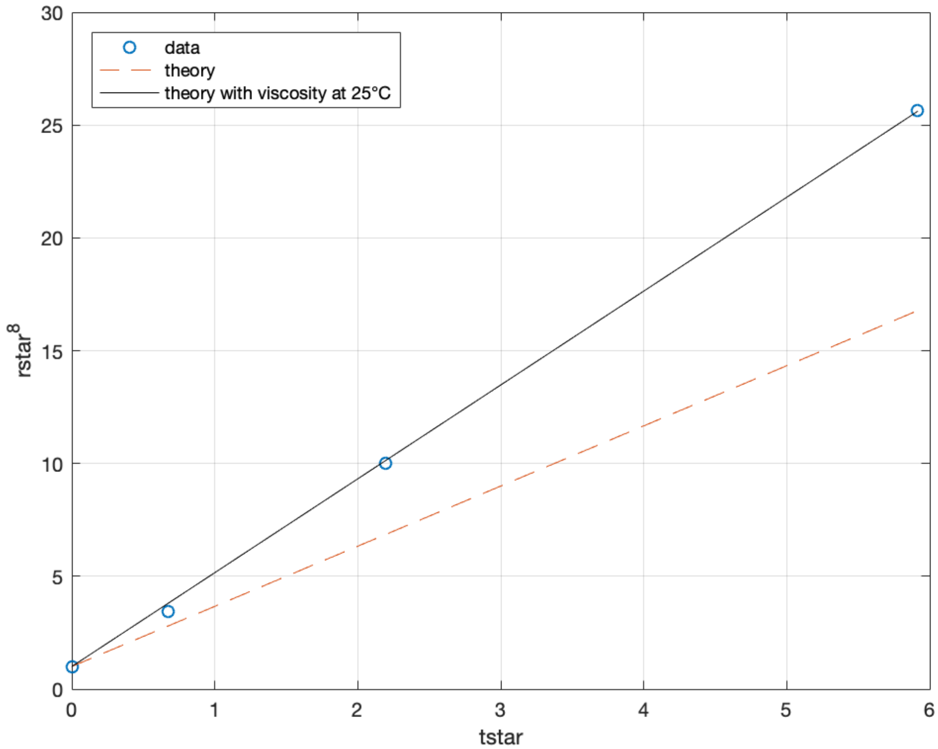
Contents

Problem of the Day 11: Radial Fluid Expansion

In this script we plot up the data from the class demonstration and compare it to our model. We expect that the radius of the fluid ring raised to the 8th power will grow linearly in time, with a proportionality constant that depends on the fluid viscosity, initial volume, weight of the glass plate, etc.

```
t = [9 35 94 238]; %Ring crossing times in seconds
r = [6 7 8 9]*0.85; %Ring locations in cm.

% Some parameters:

mu = 14; %viscosity of glycerin in poise at 20°C
m = 582; %weight of top plate in grams
v0 = 3; %fluid volume in ml

tc = pi^3*r(1)^8*mu/(v0^2*m*980) %the characteristic time

tstar = (t-t(1))/tc; %dimensionless times
rstar = r/r(1); %dimensionless radial positions

figure(1)
plot(tstar,rstar.^8,'o',tstar,1+8/3*tstar,'--',tstar,1+1.56*8/3*tstar,'k')
xlabel('tstar')
ylabel('rstar^8')
legend('data','theory','theory with viscosity at 25°C','Location','NorthWest')
grid on
```

```
tc =

   38.7033
```



Conclusion

The data certainly matches a linear increase in R^8 with time, however the growth is a little higher than would be predicted. It is highly likely that this discrepancy is due to the viscosity being just a bit less than expected. The viscosity of glycerin is a very strong function of temperature. The data matches that at 25°C exactly - a viscosity of 9 poise rather than 14 poise at 20°C. This is quite reasonable as the plate was sitting on a light box which was putting out a significant amount of heat...

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