CBE 30356 Transport Phenomena II
Final Exam

May 2, 2022

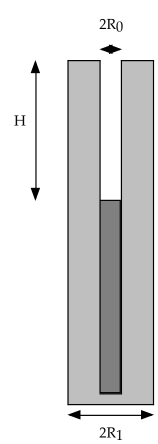
Closed Books and Notes (equations which *may* be useful are at the end)

Problem 1. (20 points) Heat and Mass Transfer in a Stefan Tube. As you all know, the Stefan tube is a useful way to measure the diffusivity of volatile compounds in air by determining the rate of liquid evaporation. In the implementation in Junior Lab, for example, a glass capillary is partially filled with diethyl ether. Air blows across the surface of the capillary to yield a mole fraction of zero at the top, while the mole fraction at the bottom is equal to the vapor pressure in equilibrium with the liquid divided by the atmospheric pressure. Because the partial pressure is a very strong function of temperature, the capillary is placed in a water bath so that the outer surface temperature of the capillary is controlled. So:

a. If the column of air is a height H above the liquid diethyl ether, derive the expression for the molar flux of the diethyl ether – the rate of evaporation of the liquid. Assume pseudo steady state.

b. As you also know, evaporation takes energy! That means that the interface where evaporation takes place will be colder than the surroundings, and that will depress the vapor pressure at equilibrium (e.g., the "wet bulb" effect we saw for a falling drop of water). If the latent heat of evaporation is L, the thermal conductivity of the glass is k, the inner radius is R_0 and the outer radius is R_1 (enough greater than R_0 that you have to use cylindrical coordinates), develop an equation for the depression in the temperature at the interface. For purposes of estimation it is reasonable to assume that the energy required to vaporize the diethyl ether diffuses in radially via the glass capillary through a cylindrical surface of radius R_0 and length $2R_0$.

c. Evaluate the magnitude of this temperature depression for the following properties: $R_0 = 0.05$cm, $R_1 = 0.33$cm, H = 1cm, $D_{AB} = 0.09\text{cm}^2/\text{s}$, k = 0.8 W/m °K = $0.8\text{x}10^5$ g cm/s^3 °K, c = 1/$24.4\text{x}10^3$ mol/cm^3, L = 27 kJ/mol = $27\text{x}10^{10}$ erg/mol, $x_{A0} = 0.7$. Note that the effect of temperature on this last (x_{A0}) is the primary source of error arising from the temperature depression effect!



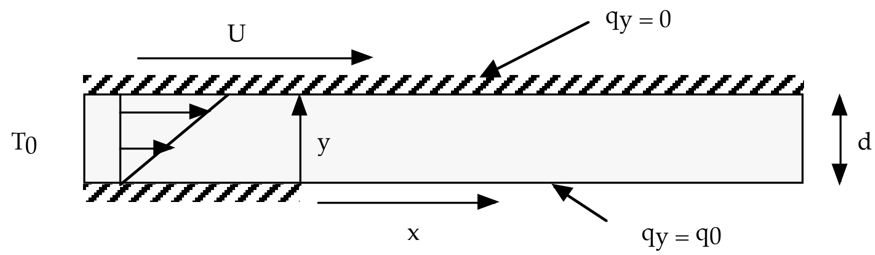
Problem 2. (20 points) Convective Heat Transfer: Consider the geometry depicted below. An upper plate at y = d is moving with a velocity U in the x direction, generating a simple shear flow between the two plates. The fluid is heated for all x>0 with a constant heat flux q_0, and the upper surface is insulated (so the fluid is getting hotter with x!). We are interested in the local Nusselt number for small x (e.g. the boundary layer where energy hasn't had a chance to diffuse very far from the lower wall).

a. Scale the equation and boundary conditions in this boundary layer limit, rendering the problem dimensionless.

b. From scaling (or from affine stretching) determine how the wall temperature depends on x.

c. Recognizing that the bulk temperature in this limit is just T_0 (e.g., the boundary layer is thin!) determine the local Nu as a function of x to within an O(1) constant.

d. What is the domain of validity (in x) over which this answer is valid (lower and upper limits)?



Problem 3. (20 points) Conduction and Radiation: A furnace in deep space is cooled by a very long fin of cross-section A, surface area per unit length P (e.g., the perimeter of the fin cross-section) and thermal conductivity k via radiation. The base of the fin is maintained at temperature T_0 and you may neglect all radiation to the fin from the surroundings (deep space is cold!).

a. Neglecting all gradients across the cross-section, write down the equation governing the axial temperature distribution in the fin. If you like, you can derive this by averaging the heat transfer equation over the cross-section, or obtain it via a shell balance.

b. (most of the points!) Via scaling analysis, determine both a characteristic fin length (e.g., the axial length scale over which the temperature fades away), and the characteristic heat flux at the base of the fin providing our cooling.

c. Solve the equation in the limit of an infinitely long fin to get the heat flux at the base. Note that the equation is non-linear, but can be easily solved by a trick: multiply both sides by dT^*/dz^* and show that *both* sides can be converted into perfect differentials. Integrate, apply the BC at infinity, and you are done! It is interesting to note that this trick works even if you include back radiation and (in an atmosphere) forced convection – but the solution would be a lot messier to deal with.

Problem 4. (20 points) Mass Transfer From a Falling Drop: A method for removing VOCs (volatile organic compounds) from ground water is a simple spray tower. The idea is that you spray the water into the air from some height over a pond and, if the drops are small enough, the VOCs will evaporate from the drop before it reaches the pond. In this problem I want you to determine the fraction of the VOCs remaining in the drop when it hits the pond.

a. It is proposed to use a spray nozzle which emits 400μm diameter drops. If the height of the tower is 10m, how long does it take for the drops to reach the pond (ignoring drop-drop interactions)? Use the following properties for air: $\mu = 1.8\text{x}10^{-2}$ cp, $\rho = 1.23\text{x}10^{-3}$ g/cm^3. Note that the drops are much too large for a significant fraction to evaporate over this distance and time.

b. If the Henry's Law coefficient is high enough that we are completely liquid side limited (the usual case for a VOC), the concentration of the pollutant at the surface of the drop is essentially zero. Assuming that there is no circulation in the drop (diffusion only!) write down the equation governing the transient concentration distribution and the total amount of pollutant remaining. Render the equations dimensionless and scale the problem!

c. At the temperature of the drop (reduced from that of the air due to "wet bulb" evaporation effects!) the diffusivity of our VOC is $1\text{x}10^{-5}$ cm^2/s. For this diffusivity and the fall time calculated in part a, what fraction of the pollutant would be remaining in the drop when it hits the pond? Note that while you –can- solve all this from scratch (and should know how to do so!), I would strongly recommend that you think about analogies with heat transfer and look at the equations page…

d. It is decided that this really isn't enough removal of the pollutant, however your colleague argues that while the viscosity of air is small, it isn't *that* small, particularly in relation to the viscosity of water. That means that there is likely some circulation inside the drop. Using your knowledge of fluid mechanics (and recalling what quantity involving viscosities must be continuous across an interface!) estimate the circulation time inside the drop, compare it to the answer in part a, and come up with a revised estimate of the fraction of pollutant remaining.

Equation Sheet

Constitutive Relations

$$\underset{\sim}{q} = -k \underset{\sim}{\nabla} T$$

$$\underset{\approx}{\tau} = \mu\left(\underset{\sim}{\nabla}\,\underset{\sim}{u} + \left(\underset{\sim}{\nabla}\,\underset{\sim}{u}\right)^T\right)$$

$$\underset{\sim}{J}_A = -c\, D_{AB} \underset{\sim}{\nabla} x_A$$

Useful Correlations and Equations

$$\frac{hD}{k} = 2 + \left(0.4\,\mathrm{Re}^{1/2} + 0.06\,\mathrm{Re}^{2/3}\right)\mathrm{Pr}^{0.4}\left(\frac{\mu_\infty}{\mu_0}\right)^{1/4}$$

$$St \equiv \frac{h}{\rho U_\infty \hat{C}_p} \equiv \frac{Nu}{\mathrm{Re}\,\mathrm{Pr}} \approx \frac{f_f}{2}\mathrm{Pr}^{-2/3}$$

$$\frac{1}{\sqrt{f_f}} = 4.0\log_{10}\left(\mathrm{Re}\sqrt{f_f}\right) - 0.40$$

$$f_f \equiv \frac{\tau_w}{\frac{1}{2}\rho U^2} \approx \frac{0.0791}{\mathrm{Re}^{1/4}}$$

$$\frac{F}{6\pi\mu U a} \approx 1 + \frac{1}{6}\mathrm{Re}^{1/2} + \frac{1}{60}\mathrm{Re}$$

$$U \equiv \frac{Q}{A} = \frac{a^2}{8\mu}\frac{\Delta P}{L}$$

$$q = e\sigma T^4$$

If Bi = ∞, $T^* = 2\sum_{n=1}^{\infty}(-1)^{n-1} e^{-n^2\pi^2 t^*} \frac{\sin(n\pi r^*)}{n\pi r^*}$

And $\overline{T}^* \equiv \dfrac{\int_0^1 T^* 4\pi r^{*2} dr^*}{\frac{4}{3}\pi} = 6\sum_{n=1}^{\infty} \dfrac{e^{-n^2\pi^2 t^*}}{n^2\pi^2}$

Fundamental Equations

$$\underset{\sim}{N}_A = x_A\left(\underset{\sim}{N}_A + \underset{\sim}{N}_B\right) - c\, D_{AB} \underset{\sim}{\nabla} x_A$$

$$\frac{\partial c_A}{\partial t} + \underset{\sim}{\nabla} \bullet \underset{\sim}{N}_A = R_A$$

$$\frac{\partial c}{\partial t} + \underset{\sim}{u} \bullet \underset{\sim}{\nabla} c = D\nabla^2 c + R$$

$$\rho\left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \bullet \underset{\sim}{\nabla}\,\underset{\sim}{u}\right) = -\underset{\sim}{\nabla} p + \mu\nabla^2 \underset{\sim}{u} + \rho \underset{\sim}{g}$$

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + \underset{\sim}{u} \bullet \underset{\sim}{\nabla} T\right) = k\nabla^2 T + \dot{S}$$

Spherical Representation

$$\nabla^2 T \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 T}{\partial\phi^2}$$

$$\underset{\sim}{u} \bullet \underset{\sim}{\nabla} T \equiv u_r\frac{\partial T}{\partial r} + \frac{u_\theta}{r}\frac{\partial T}{\partial\theta} + \frac{u_\phi}{r\sin(\theta)}\frac{\partial T}{\partial\phi}$$

$$q_r = -k\frac{\partial T}{\partial r}\;;\; q_\theta = -k\frac{1}{r}\frac{\partial T}{\partial\theta}\;;\; q_\phi = -k\frac{1}{r\sin(\theta)}\frac{\partial T}{\partial\phi}$$

$$\underset{\sim}{\nabla} \bullet \underset{\sim}{u} = \frac{1}{r^2}\frac{\partial\left(r^2 u_r\right)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{\partial\left(u_\theta\sin(\theta)\right)}{\partial\theta} + \frac{1}{r\sin(\theta)}\frac{\partial u_\phi}{\partial\phi}$$

Cylindrical Representation

$$\nabla^2 T \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}$$

$$\underset{\sim}{u} \bullet \underset{\sim}{\nabla} T \equiv u_r \frac{\partial T}{\partial r}+\frac{u_\theta}{r}\frac{\partial T}{\partial \theta}+u_z\frac{\partial T}{\partial z}$$

$$q_r = -k\frac{\partial T}{\partial r} \;;\; q_\theta = -k\frac{1}{r}\frac{\partial T}{\partial \theta} \;;\; q_z = -k\frac{\partial T}{\partial z}$$

$$\underset{\sim}{\nabla} \bullet \underset{\sim}{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}$$

Cartesian Representation

$$\nabla^2 T \equiv \frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}$$

$$\underset{\sim}{u} \bullet \underset{\sim}{\nabla} T \equiv u_x \frac{\partial T}{\partial x}+u_y\frac{\partial T}{\partial y}+u_z\frac{\partial T}{\partial z}$$

$$q_x = -k\frac{\partial T}{\partial x} \;;\; q_y = -k\frac{\partial T}{\partial y} \;;\; q_z = -k\frac{\partial T}{\partial z}$$

$$\underset{\sim}{\nabla} \bullet \underset{\sim}{u} = \frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}$$

Mathematical Operations

If $\eta = \frac{y}{t^n}$ then $\frac{\partial \eta}{\partial t} = -n\frac{\eta}{t}$

$$\frac{1}{f}\frac{df}{d\eta} \equiv \frac{d\ln(f)}{d\eta}$$

$$f^n\frac{df}{d\eta} \equiv \frac{1}{n+1}\frac{df^{n+1}}{d\eta}$$

$$e^{it} \equiv \cos(t)+i\sin(t)$$

If $\frac{dy}{dx}+p(x)\,y = f(x)$

Then $y = e^{-\int p(x)dx}\left[\int e^{+\int p(x')dx'} f(x)\,dx + K\right]$