CBE 30356 Transport Phenomena II
Final Exam

May 10, 2023

Closed Books and Notes (equations which *may* be useful are at the end)

Problem 1. (20 points) Convective Heat & Mass Transfer: In the midterm you solved for the time necessary for a sphere of ice to melt in a stream of air. In order to avoid mass transfer effects we looked at the specific case where the air was at a dew point equal to the melting temperature. Here we explore how mass transfer (e.g., the humidity of the air stream) affects the melting process.

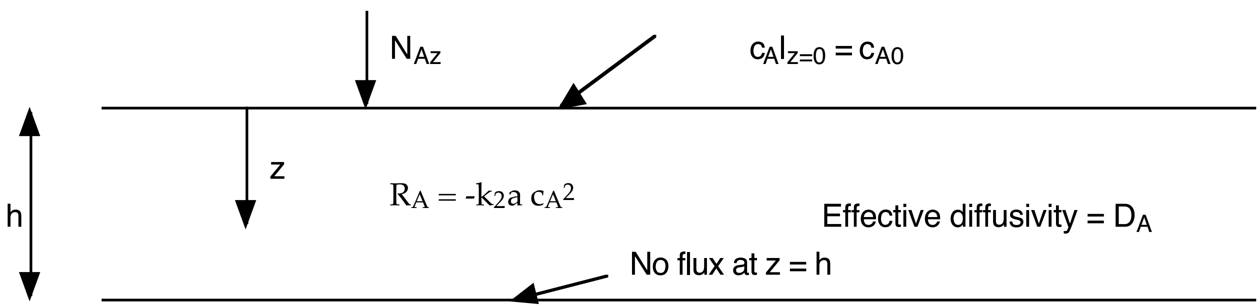
a. For this geometry and conditions (water vapor in air) we have the *extremely* convenient result that the Sherwood number and Nusselt number are essentially identical (e.g., the ratio is 1 to a good approximation at all Re because for water vapor in air Sc = 0.66 and Pr = 0.71, close enough to being equal). If the latent heat of vaporization of water is H, and the mole fraction of water in the air stream is $x_{A\infty}$ and at the surface is x_{A0}, determine the ratio of the heat flux to the ice due to mass transfer (e.g., condensation or evaporation) to that arising from heat transfer. Other parameters this ratio involves include c (the molar density of air), D_{AB} (the water vapor diffusion coefficient), k (the thermal conductivity of air) and ΔT (the temperature difference between the air and the ice). You don't need any other parameters because we can take Sh/Nu = 1 for this problem.

b. Take the air stream to be at 30°C. It is observed that the ice melts in 100s when the dew point of the air is at 0°C (e.g., the pure heat transfer result). Note that this time *would* depend on all sorts of parameters such as the latent heat of fusion, the radius, the air velocity, etc. as you showed in the midterm! How long would it take the ice to melt if the air stream were instead completely dry (e.g., 0% relative humidity) so that some of the melted ice evaporates into the air? Use the parameter values: $p_{sat}|_{0°C} = 0.61$ kPa, $p_{atm} = 101$ kPa, H = 45 kJ/mol, c = 44.6 mol/m^3, $D_{AB} = 2.19\times10^{-5}$ m^2/s, k = 0.0244 W/m°K.

c. If the air stream is at 30°C ($p_{sat}|_{30°C} = 4.25$ kPa) but is at 80% relative humidity, your ice melts significantly faster! How long would it take to melt now?

Problem 2. (20 points) The Thiele Problem. In class we showed how to obtain the *Thiele Effectiveness Factor* (the ratio of the mass flux to that which would occur with no mass transfer limitations) for a catalyst. The effectiveness factor was derived for first order reaction kinetics (as the math is simpler!), but as you will learn next fall many reactions are *not* first order! In this problem we shall determine the effectiveness factor for a second order reaction with a pseudo-homogeneous reaction rate $R_A = -k_2 a\, c_A^2$. The quantity "a" is the active catalyst surface area/volume, so the reaction rate is moles/(volume time) as usual.

a. Consider the catalyst layer depicted below. The layer is of thickness h, the concentration at the surface (z = 0) is c_{A0}, and the diffusivity is D_A. We have the pseudo-homogeneous reaction rate $R_A = -k_2 a\, c_A^2$ as discussed above. For simplicity, we shall consider the problem to be equimolar counter-diffusion (e.g., 2A -> C + D or similar) so that diffusion and reaction do not lead to convection. Write down the differential equation and boundary conditions for the reactant concentration c_A with these assumptions. Write down the expression for the molar flux of A at the surface z = 0.

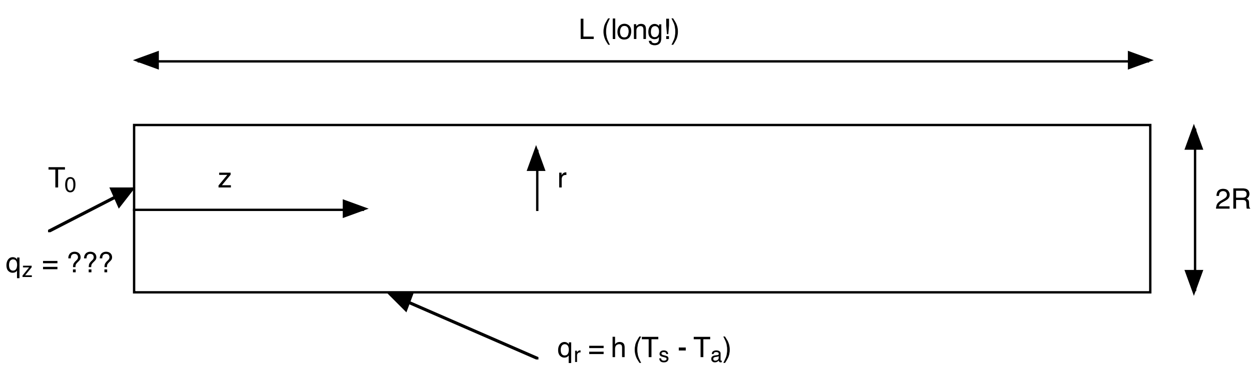


b. We are primarily interested in the fast reaction (e.g., diffusion limited) case, where the appropriate length scale for z is much less than h. Render the equations dimensionless and via scaling determine this length scale and the scale for the molar flux at the surface.

c. The effectiveness factor is the ratio of the molar flux to that which would occur if there were no diffusional limitations (e.g., where the entire catalyst layer of thickness h is exposed to the concentration c_{A0}). Develop an expression for the effectiveness factor in terms of the dimensionless derivative at the surface.

d. 2 points extra credit: While the differential equation is non-linear, in the limit where $h/z_c >> 1$ it is easily solved (Hint: use the trick of multiplying by the derivative to convert both terms into perfect differentials!). For a two points extra credit, determine the "number" for the effectiveness factor in this limit.

Problem 3. (20 points) A Cylindrical Cooling Fin: In class we demonstrated the combination of axial conduction and external heat transfer for a cylinder where there was a constant temperature at the base. For certain choices of parameters we can ignore radial temperature variations and just work with the temperature averaged over the cross-section. The geometry is depicted below:



a. Steady-state, long cooling rods. Develop an equation for the axial variation of the temperature averaged over the cross-section $\overline{T}$. Render this equation dimensionless and use it to determine the steady-state heat flux at the base of the rod (z = 0) in the limit $L/z_c >> 1$.

b. Your averaged expression required that the local surface temperature T_s be equal to the local average temperature $\overline{T}$, which of course isn't –quite- true! From scaling, what would be the characteristic magnitude of this temperature difference? (Hint: think about internal and external heat transfer resistances in the radial direction…)

c. It takes time for the temperature distribution and heat flux to reach steady-state. How does this time scale with the parameters of the problem?

Problem 4. (20 points) Dimensionless Groups: A recurring theme in this class has been the use of dimensionless groups of parameters to analyze transport problems. Using the parameters given below, determine the dimensionless groups corresponding to the following physical mechanisms **and state a problem or equation where it would play a role**.

a. $Re = \dfrac{Inertial\,Forces}{Viscous\,Forces}$

b. $Nu = \dfrac{Heat\,Transfer}{Conductive\,Heat\,Transfer}$

c. $Sh = \dfrac{Mass\,Transfer}{Diffusive\,Mass\,Transfer}$

d. $Bi = \dfrac{Internal\,Heat\,Transfer\,Resistance}{External\,Heat\,Transfer\,Resistance}$

e. $St = \dfrac{Heat\,Transfer}{Convective\,Heat\,Transfer}$

f. $Pr = \dfrac{Momentum\,Diffusivity}{Thermal\,Diffusivity}$

g. $Sc = \dfrac{Momentum\,Diffusivity}{Mass\,Diffusivity}$

h. $Pe = \dfrac{Convective\,Heat\,Transfer}{Conductive\,Heat\,Transfer}$

i. $Fr = \dfrac{Inertial\,Forces}{Gravitational\,Forces}$

j. $Eu = \dfrac{Pressure\,Differential}{Dynamic\,Pressure}$

Parameters:

$U\left[\frac{m}{s}\right]$; $d\,[m]$; $\Delta P\left[\frac{kg}{ms^2}\right]$; $\rho\left[\frac{kg}{m^3}\right]$; $g\left[\frac{m}{s^2}\right]$; $\alpha\left[\frac{m^2}{s}\right]$; $\upsilon\left[\frac{m^2}{s}\right]$; $D_{AB}\left[\frac{m^2}{s}\right]$; $h\left[\frac{W}{m^2\,°K}\right]$; $k\left[\frac{W}{m\,°K}\right]$; $k_{int}\left[\frac{W}{m\,°K}\right]$; $k_m\left[\frac{m}{s}\right]$; $\hat{C}_p\left[\frac{J}{kg\,°K}\right]$

Equation Sheet

Constitutive Relations

$$\underset{\sim}{q} = -k \underset{\sim}{\nabla} T$$

$$\underset{\approx}{\tau} = \mu\left(\underset{\sim}{\nabla}\underset{\sim}{u} + \left(\underset{\sim}{\nabla}\underset{\sim}{u}\right)^T\right)$$

$$\underset{\sim}{J_A} = -c\, D_{AB} \underset{\sim}{\nabla} x_A$$

Fundamental Equations

$$\underset{\sim}{N}_A = x_A\left(\underset{\sim}{N}_A + \underset{\sim}{N}_B\right) - c\, D_{AB} \underset{\sim}{\nabla} x_A$$

$$\frac{\partial c_A}{\partial t} + \underset{\sim}{\nabla} \bullet \underset{\sim}{N}_A = R_A$$

$$\frac{\partial c}{\partial t} + \underset{\sim}{u} \bullet \underset{\sim}{\nabla} c = D\nabla^2 c + R$$

$$\rho\left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \bullet \underset{\sim}{\nabla}\underset{\sim}{u}\right) = -\underset{\sim}{\nabla} p + \mu\nabla^2 \underset{\sim}{u} + \rho \underset{\sim}{g}$$

$$\rho \hat{C}_p\left(\frac{\partial T}{\partial t} + \underset{\sim}{u} \bullet \underset{\sim}{\nabla} T\right) = k\nabla^2 T + \dot{S}$$

Spherical Representation

$$\nabla^2 T \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2(\theta)}\frac{\partial^2 T}{\partial \phi^2}$$

$$\underset{\sim}{u} \bullet \underset{\sim}{\nabla} T \equiv u_r \frac{\partial T}{\partial r} + \frac{u_\theta}{r}\frac{\partial T}{\partial \theta} + \frac{u_\phi}{r\sin(\theta)}\frac{\partial T}{\partial \phi}$$

$$q_r = -k\frac{\partial T}{\partial r} \;;\; q_\theta = -k\frac{1}{r}\frac{\partial T}{\partial \theta} \;;$$
$$q_\phi = -k\frac{1}{r\sin(\theta)}\frac{\partial T}{\partial \phi}$$

$$\underset{\sim}{\nabla} \bullet \underset{\sim}{u} = \frac{1}{r^2}\frac{\partial\left(r^2 u_r\right)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{\partial\left(u_\theta \sin(\theta)\right)}{\partial \theta} + \frac{1}{r\sin(\theta)}\frac{\partial u_\phi}{\partial \phi}$$

Cylindrical Representation

$$\nabla^2 T \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}$$

$$\underset{\sim}{u} \bullet \underset{\sim}{\nabla} T \equiv u_r \frac{\partial T}{\partial r} + \frac{u_\theta}{r}\frac{\partial T}{\partial \theta} + u_z\frac{\partial T}{\partial z}$$

$$q_r = -k\frac{\partial T}{\partial r} \;;\; q_\theta = -k\frac{1}{r}\frac{\partial T}{\partial \theta} \;;\; q_z = -k\frac{\partial T}{\partial z}$$

$$\underset{\sim}{\nabla} \bullet \underset{\sim}{u} = \frac{1}{r}\frac{\partial\left(r u_r\right)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

Cartesian Representation

$$\nabla^2 T \equiv \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$$\underset{\sim}{u} \bullet \underset{\sim}{\nabla} T \equiv u_x \frac{\partial T}{\partial x} + u_y\frac{\partial T}{\partial y} + u_z\frac{\partial T}{\partial z}$$

$$q_x = -k\frac{\partial T}{\partial x} \;;\; q_y = -k\frac{\partial T}{\partial y} \;;\; q_z = -k\frac{\partial T}{\partial z}$$

$$\underset{\sim}{\nabla} \bullet \underset{\sim}{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

Mathematical Operations

If $\eta = \dfrac{y}{t^n}$ then $\dfrac{\partial \eta}{\partial t} = -n\dfrac{\eta}{t}$

$$\frac{1}{f}\frac{df}{d\eta} \equiv \frac{d\ln(f)}{d\eta}$$

$$f^n \frac{df}{d\eta} \equiv \frac{1}{n+1}\frac{d f^{n+1}}{d\eta}$$