CBE 30355 Problem Set 1

1) From definition of a dot product:

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta \longrightarrow \text{Solve for } \theta:$$

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z \qquad \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

$$|\vec{b}| = \sqrt{b_x^2 + b_y^2 + b_z^2}$$

(a.) $\underbrace{(1,0,1)}_{\vec{a}}, \underbrace{(1,0,0)}_{\vec{b}}$

$$\left.\begin{aligned} \vec{a}\cdot\vec{b} &= 1 + 0 + 0 = 1 \\ |\vec{a}| &= \sqrt{1+0+1} = \sqrt{2} \\ |b| &= \sqrt{1+0+0} = 1 \end{aligned}\right\} \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}\cdot 1}\right) \quad \boxed{\theta = \frac{\pi}{4}} = 45^\circ$$

(b.) $\underbrace{(1,0,1)}_{\vec{a}}, \underbrace{(0,1,0)}_{\vec{b}}$

$$\left.\begin{aligned} \vec{a}\cdot\vec{b} &= 0 + 0 + 0 = 0 \\ |\vec{a}| &= \sqrt{1+0+1} = \sqrt{2} \\ |\vec{b}| &= \sqrt{0+1+0} = 1 \end{aligned}\right\} \quad \theta = \cos^{-1}\left(\frac{0}{\sqrt{2}\cdot 1}\right) \quad \boxed{\theta = \frac{\pi}{2}} = 90^\circ$$

(c.) $\underbrace{(1,2,3)}_{\vec{a}}, \underbrace{(3,2,1)}_{\vec{b}}$

$$\left.\begin{aligned} \vec{a}\cdot\vec{b} &= 3 + 4 + 3 = 10 \\ |\vec{a}| = |\vec{b}| &= \sqrt{1+4+9} = \sqrt{14} \end{aligned}\right\} \quad \theta = \cos^{-1}\left(\frac{10}{\sqrt{14}\cdot\sqrt{14}}\right) \quad \boxed{\theta = 0.247} = 44.4^\circ$$

2.) (a) $\underbrace{(1,0,1)}_{\vec{a}} \cdot \underbrace{(1,0,0)}_{\vec{b}}$

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z = 1 + 0 + 0 = \boxed{1}$$

(b) $\underbrace{(1,0,1)}_{\vec{a}} \cdot \underbrace{(0,1,0)}_{\vec{b}}$

$$\vec{a}\cdot\vec{b} = 0 + 0 + 0 = \boxed{0}$$

(c) $\underbrace{(1,2,3)}_{\vec{a}} \times \underbrace{(3,2,1)}_{\vec{b}} = \det\begin{bmatrix} \hat{i} & \hat{j} & \vec{k} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

$$-6\vec{k} - 6\vec{i} - \vec{j}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = (2-6)\vec{i} + (9-1)\vec{j} + (2-6)\vec{k}$$

$$+2\vec{i} + 9\vec{j} + 2\vec{k}$$

Compute the determinant via diagonals

$$\boxed{\vec{a} \times \vec{b} = (-4, 8, -4)}$$

3. $\phi = (x^2 + y^2 + z^2)^2$

$\vec{u} = (y^2, z, x^2)$

(a) $\vec{\nabla}$ = "divergence", or matrix of first derivatives.

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\vec{\nabla}\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$$

↳ scalar
↳ vector

$$\frac{\partial \phi}{\partial x} = 2(x^2+y^2+z^2)\,2x, \qquad \frac{\partial \phi}{\partial z} = 2(x^2+y^2+z^2)(2z).$$

$$\frac{\partial \phi}{\partial y} = 2(x^2+y^2+z^2)\,2y$$

$$\boxed{\vec{\nabla}\phi = 2(x^2+y^2+z^2)(2x, 2y, 2z)}$$

3. (a) Continued

$$\vec{\nabla} \cdot \vec{U} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(y^2, z, x^2\right) = \underbrace{\cancel{\frac{\partial y^2}{\partial x}}}_{0} + \underbrace{\cancel{\frac{\partial z}{\partial y}}}_{0} + \underbrace{\cancel{\frac{\partial x^2}{\partial z}}}_{0}$$

$$= \boxed{0}$$

3. (b) $\underbrace{\nabla^2 \phi}_{} = \underbrace{(\vec{\nabla} \cdot \vec{\nabla})}_{} \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \phi$

Scalar, since it's the dot product of two vectors

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial \phi}{\partial x}\right] = \frac{\partial}{\partial x}\left[\underbrace{2(x^2 + y^2 + z^2)2x}\right]$$

see prob. 3(a)

$$= \frac{\partial}{\partial x}\left[4(x^3 + xy^2 + xz^2)\right]$$

$$= 12x^2 + 4y^2 + 4z^2$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial \phi}{\partial y}\right] = 4x^2 + 12y^2 + 4z^2$$

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{\partial}{\partial z}\left[\frac{\partial \phi}{\partial z}\right] = 4x^2 + 4y^2 + 12z^2$$

$$\boxed{\nabla^2 \phi = 20(x^2 + y^2 + z^2)}$$

3.(b) <u>Continued</u>

$$\nabla^2 \vec{u} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\vec{u}$$

$$= \left(\frac{\partial^2 y^2}{\partial x^2} + \frac{\partial^2 y^2}{\partial y^2} + \frac{\partial^2 y^2}{\partial z^2},\ \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial z^2},\ \frac{\partial^2 x^2}{\partial x^2} + \frac{\partial^2 x^2}{\partial y^2} + \frac{\partial^2 x^2}{\partial z^2}\right)$$

$$= (0 + 2 + 0,\ 0 + 0 + 0,\ 2 + 0 + 0)$$

$$\boxed{\nabla^2 \vec{u} = (2, 0, 2)}$$

3 (c.) $\vec{\nabla} \times \vec{u} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (y^2, z, x^2)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z & x^2 \end{vmatrix} = \left(\frac{\partial x^2}{\partial y} - \frac{\partial z}{\partial z}\right)\vec{i} + \left(\frac{\partial y^2}{\partial z} - \frac{\partial x^2}{\partial x}\right)\vec{j} + \left(\frac{\partial z}{\partial x} - \frac{\partial y^2}{\partial y}\right)\vec{k}$$

Cross product using diagonals

$$= (0 - 1)\vec{i} + (0 - 2x)\vec{j} + (0 - 2y)\vec{k}$$

$$\boxed{\vec{\nabla} \times \vec{u} = (-1, -2x, -2y)}$$

4. Prove that for arbitrary scalar function ϕ:

$$\vec{\nabla} \times (\vec{\nabla} \phi) = 0.$$

$$\vec{\nabla} \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$

$$\vec{\nabla} \times (\vec{\nabla} \phi) = \begin{vmatrix} \vec{i} & \vec{J} & \vec{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial z} \right) \vec{i} + \left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x \partial z} \right) \vec{J} + \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y} \right) \vec{K}$$

$$= (0)\vec{i} + (0)\vec{J} + (0)\vec{K} = \boxed{0}$$

Recall: $\frac{\partial^2 \phi}{\partial y \partial z} = \frac{\partial^2 \phi}{\partial z \partial y}$, and the same for $\vec{J}$ and $\vec{K}$, so all go to zero.

5. Prove the following vector identity for arbitrary vector $\vec{u}$:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{u}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) - \nabla^2 \vec{u}$$

Step by step in index notation:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{u}) = \epsilon_{ijk} \nabla_j (\vec{\nabla} \times \vec{u})_k$$

$$= \epsilon_{ijk} \frac{\partial}{\partial x_j} (\vec{\nabla} \times \vec{u})_k$$

$$= \epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{klm} \nabla_l u_m$$

$$= \epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{klm} \frac{\partial u_m}{\partial x_l}$$

$$= \epsilon_{ijk} \epsilon_{klm} \frac{\partial^2 u_m}{\partial x_j \partial x_l}$$

Recall: you may permutate indices for ϵ_{ijk} as follows:

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312}$$

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$$

So: $$\vec{\nabla} \times (\vec{\nabla} \times \vec{u}) = \epsilon_{kij} \epsilon_{klm} \frac{\partial^2 u_m}{\partial x_j \partial x_l}$$

Now apply the identity: $\epsilon_{123} \epsilon_{145} = \delta_{24} \delta_{35} - \delta_{25} \delta_{34}$

$$\epsilon_{kij} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

So:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{u}) = \delta_{il} \delta_{jm} \frac{\partial^2 u_m}{\partial x_j \partial x_l} - \delta_{im} \delta_{jl} \frac{\partial^2 u_m}{\partial x_j \partial x_l}$$

5. Continued:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{u}) = \delta_{il}\,\delta_{jm} \frac{\partial^2 u_m}{\partial x_j \partial x_l} - \delta_{im}\,\delta_{jl} \frac{\partial^2 u_m}{\partial x_j \partial x_l}$$

$$= \frac{\partial^2 u_j}{\partial x_j \partial x_i} - \frac{\partial^2 u_i}{\partial x_j^2}$$

$$= \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) - \frac{\partial^2 u_i}{\partial x_j^2}$$

$$= \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) - \nabla^2 \vec{u}$$

Remember:

$$\vec{\nabla} = \frac{\partial}{\partial x_i}$$

$$\nabla^2 = \frac{\partial^2}{\partial x_j^2}$$

$$\frac{\partial u_j}{\partial x_j} = \vec{\nabla} \cdot \vec{u}$$

So:

$$\boxed{\vec{\nabla} \times (\vec{\nabla} \times \vec{u}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) - \nabla^2 \vec{u}}$$

Problem Set 1
#5 solution
Long winded version

Prove:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{u}\right) = \vec{\nabla}\left(\vec{\nabla} \bullet \vec{u}\right) - \nabla^2 \vec{u}$$

Step by step in index notation.
Start by changing the first cross product into index notation.

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{u}\right) = \varepsilon_{ijk} \vec{\nabla}_j \left(\vec{\nabla} \times \vec{u}\right)_k$$

Remember that: $\vec{\nabla}_j = \dfrac{\partial}{\partial x_j}$

$$= \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left(\vec{\nabla} \times \vec{u}\right)_k$$

Change 2nd dot product to index notation:

$$= \varepsilon_{ijk} \frac{\partial}{\partial x_j} \varepsilon_{klm} \nabla_l u_m$$

And since: $\nabla_l u_m = \dfrac{\partial u_m}{\partial x_l}$

$$= \varepsilon_{ijk} \frac{\partial}{\partial x_j} \varepsilon_{klm} \frac{\partial u_m}{\partial x_l}$$

Once in index notation, products are commutative. Rearranging:

$$= \varepsilon_{ijk} \varepsilon_{klm} \frac{\partial^2 u_m}{\partial x_j \partial x_l}$$

Take a look at $\varepsilon_{ijk}\varepsilon_{klm}$ There are four free indices, so this must be a 4th order tensor. And since ε_{ijk} is isotropic, $\varepsilon_{ijk}\varepsilon_{klm}$ must also be isotropic. There are only three possible ways to express a 4th order isotropic tensor with the free indices *i,j,l*,and *m*. These are:

$$\delta_{ij}\delta_{lm} \qquad \delta_{il}\delta_{jm} \qquad \delta_{im}\delta_{jl}$$

Therefore, we can put the expression above in terms of these three isotropic 4th order tensors, with some scalar constants $\lambda_1, \lambda_2, \lambda_3$

$$\varepsilon_{ijk}\varepsilon_{klm} = \lambda_1\delta_{ij}\delta_{lm} + \lambda_2\delta_{il}\delta_{jm} + \lambda_3\delta_{im}\delta_{jl}$$

Now multiply by: $\delta_{ij}\delta_{lm}$

$$(\varepsilon_{ijk}\varepsilon_{klm})\delta_{ij}\delta_{lm} = (\lambda_1\delta_{ij}\delta_{lm} + \lambda_2\delta_{il}\delta_{jm} + \lambda_3\delta_{im}\delta_{jl})\delta_{ij}\delta_{lm}$$

Which simplifies to:

$$\varepsilon_{iik}\varepsilon_{kmm} = \lambda_1\delta_{ii}\delta_{ll} + \lambda_2\delta_{jl}\delta_{jl} + \lambda_3\delta_{jm}\delta_{jm}$$

Remember that: $\delta_{ii} = 3$

$\varepsilon_{iik} = 0$

So:

$$0 = 9\lambda_1 + 3\lambda_2 + 3\lambda_3$$

Similarly, if you multiply by the other two 4th order isotropic tensors:
Multiply by $\delta_{il}\delta_{jm}$, remembering that $\varepsilon_{ijk}\varepsilon_{ijk} = 6$

$$6 = 3\lambda_1 + 9\lambda_2 + 3\lambda_3$$

and by: $\delta_{im}\delta_{jl}$

$$-6 = 3\lambda_1 + 3\lambda_2 + 9\lambda_3$$

Thus we have a system of 3 equations and 3 unknowns. Solve for:

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = -1$$

Now go back to the original expression, and substitute for $\varepsilon_{ijk}\varepsilon_{klm}$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{u}\right) = \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\frac{\partial^2 u_m}{\partial x_j \partial x_l}$$

Distribute:

$$= \delta_{il}\delta_{jm}\frac{\partial^2 u_m}{\partial x_j \partial x_l} - \delta_{im}\delta_{jl}\frac{\partial^2 u_m}{\partial x_j \partial x_l}$$

Now use the kroneker deltas to change indices in each term:

$$= \frac{\partial^2 u_j}{\partial x_j \partial x_i} - \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

Factor out the $\frac{\partial}{\partial x_i}$ in the first term, and combine x_j's in the second:

$$= \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) - \frac{\partial^2 u_i}{\partial x_j^{\,2}}$$

Convert back to vector notation:

$$= \vec{\nabla}\left(\vec{\nabla} \bullet \vec{u}\right) - \nabla^2 \vec{u}$$

And you're done.

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{u}\right) = \vec{\nabla}\left(\vec{\nabla} \bullet \vec{u}\right) - \nabla^2 \vec{u}$$