1) Lawn chair Larry:

$$F_b = \Delta\rho\, g\, V_{b\,total}$$

$$F_w = g\, m_{total}$$

- We know from hydrostatics that Larry will remain stationary when his weight (plus the chair, plus the balloons) balances the buoyancy force of the air displaced by the balloons,

Fw: Let Larry = 80 kg
chair = 5 kg
balloons = 5 kg.

then: $F_w = (90\text{ kg})\, g$

Fb: The density of air changes with elevation, because it's a compressible fluid. To make things easier, we'll assume that the density of helium is constant, →

1) continued

... and we'll assume helium is an ideal gas.

Then, from I G law: $PV = nRT$

$$\rho_{He} = \frac{n M_{He}}{V} = \frac{P}{RT} M_{He}$$

$$= \frac{(1\,mol)\left(\frac{4\,g}{mol}\right)}{1\,L} = \frac{1\,\cancel{atm}}{\left(0.08206\,\frac{L\,\cancel{atm}}{mol\,\cancel{K}}\right)(288\,\cancel{K})} \frac{4\,g}{mol}$$

$$\rho_{He} = 0.169\,\frac{g}{L} = 0.169\,\frac{kg}{m^3}$$

Now for ρ_{air}. If you google "air density as a function of altitude" you'll find some great references, including the following correlation for height above sealevel (H) as a function of air density (D):

$H \to$ km

$D \to$ kg/m^3

$$H = 44.3308 - 42.2665 \times D^{0.234969}$$

(see attached plot)

~~at H = 0.009 3 km~~ ~~D = 1.221767 $\frac{kg}{m^3}$~~

~~= 30 m,~~

at $H = 0.0093$ km
$= 9.3$ m
$= 30$ ft

$$\left.\right\} \quad \underline{D = 1.224202\ \frac{kg}{m^3}}$$

→

1.) Continued

$$F_b = F_W$$

$$\Delta\rho \, \cancel{g} \, V_{b\ total} = \cancel{g} \, m_{total}$$

$$(\rho_{air} - \rho_{He}) \, V_{b\ total} = m_{total}$$

$$V_{b\ total} = \frac{m_{total}}{\rho_{air} - \rho_{He}}$$

$$V_{b\ total} = \frac{90 \text{ kg}}{1.224202 \frac{kg}{m^3} - 0.169 \frac{kg}{m^3}}$$

$$= 85.292 \text{ m}^3$$

$$= 3012 \text{ ft}^3$$

$$\boxed{N_b = \frac{V_{b\ total}}{V_b} = 26.65 \text{ balloons}}$$

answer:
20 - 30 balloons depending on assumptions

$$\text{radius} = \left(\frac{3}{4\pi} V_b\right)^{1/3} = 3 \text{ ft}$$

$$\text{so } \boxed{\text{diameter} = 6 \text{ ft}}$$

1) Continued

Question: Can Larry control his elevation?

Interpretation: What happens if he adds one extra balloon?

$$V'_{b_{tot}} = 3012\,ft^3 + 113\,ft^3$$
$$= 3125\,ft^3$$
$$= 88.49\ meters^3$$

Then: Larry will equilibrate at a new elevation H' corresponding to a new density ρ'_{air}:

$$D' = \rho'_{air} = \frac{m_{Total}}{V'_{b_{Total}}} + \rho_{He}$$

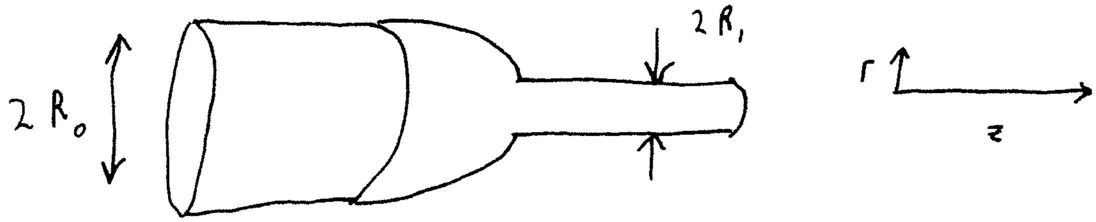
$$= \frac{90\ kg}{88.49\ m^3} + 0.169\,\frac{kg}{m^3} = 1.186\ \frac{kg}{m^3}$$

$$H' = 44.3308 - 42.2665\,D'^{\,0.23469}$$

$$H' = 338\,m$$
$$= 1091\ ft\ \ !!!$$

$$\boxed{\text{Answer: Larry has almost no control. A change by 1 balloon results in a thousand-foot change in elevation.}}$$

2) Fluid flow through a pipe constriction:



a. Velocity profile: $U_z = D\left(1 - \frac{r^2}{R_0^2}\right)$

Integrate over the cross-section to get the flow. Remember that flow = (velocity) · (area).

$$Q_z = \int_0^{R_0} U_z \, 2\pi r \, dr$$

$$= \int_0^{R_0} D\left(1 - \frac{r^2}{R_0^2}\right) 2\pi r \, dr$$ – factor out constants

$$= 2\pi D \int_0^{R_0} \left(1 - \frac{r^2}{R_0^2}\right) r \, dr$$ – multiply and divide by R_0^2

$$= 2\pi D R_0^2 \int_0^{R_0} \left(1 - \frac{r^2}{R_0^2}\right) \frac{r}{R_0} \frac{dr}{R_0}$$

Now let $\eta = \frac{r}{R_0}$. $d\eta = \frac{dr}{R_0}$.

Substitute:

$$Q_z = 2\pi D R_0^2 \int_0^{1} (1 - \eta^2) \, \eta \, d\eta$$ →

2.) Continued

$$Q_z = 2\pi D R_o^2 \left[\frac{\eta^2}{2} - \frac{\eta^4}{4} \right]_0^1$$

$$= 2\pi D R_o^2 \left(\frac{1}{2} - \frac{1}{4} \right)$$

a.) $\boxed{Q_z = \frac{\pi D R_o^2}{2}}$

b.) By conservation of mass, and since the fluid is incompressible, the flow rate in the second section is the same as in the first section.

$$\boxed{Q_1 = \frac{\pi D R_o^2}{2}}$$

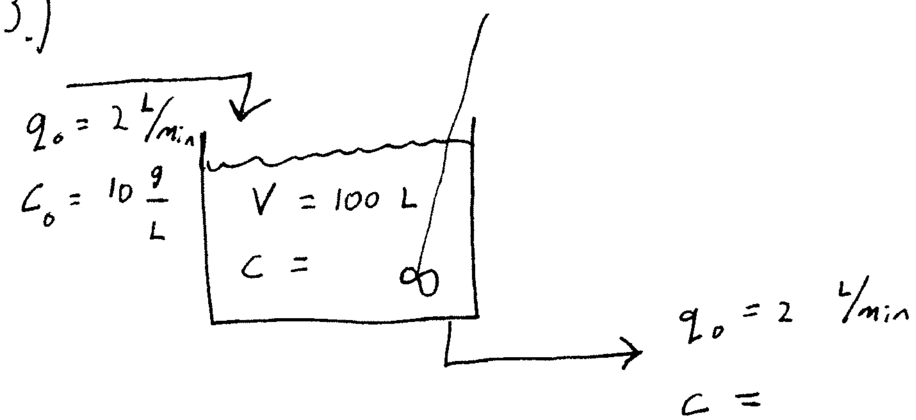
c.) $$U_{z\,avg} = \frac{\text{flow rate}}{\text{area}} = \frac{\frac{\cancel{\pi} D \cancel{R_o^2}}{2}}{\cancel{\pi} \cancel{R_o^2}} = \frac{D}{2}$$

$$\boxed{U_{z\,avg} = \frac{D}{2}}$$

d.) $$U_{1\,avg} = \frac{\text{flow rate}}{\text{area}} = \frac{\frac{\cancel{\pi} D R_o^2}{2}}{\cancel{\pi} R_1^2} = \frac{D}{2}\left[\frac{R_o^2}{R_1^2}\right]$$

$$\boxed{U_{1\,avg} = \frac{D}{2}\,\frac{R_o^2}{R_1^2}}$$

3.)



- Since this is a CSTR, the concentration out = conc. in tank, and flow in = flow out.

- Assume density is constant.

Concentration of salt is changing. if we apply conservation of mass for salt:

$$(\text{in}) - (\text{out}) = (\text{accumulation})$$

$$q_0 C_0 - q_0 C = \frac{d}{dt}(V C)$$

$$q_0 (C_0 - C) = V \frac{dC}{dt}$$

- This is a separable first-order ODE.

$$\frac{q_0}{V} dt = \frac{dC}{(C_0 - C)}$$

- integrate this from $t=0$ to $t=t$, or $C=0$ to $C=C$.

$$\int_0^t \frac{q_0}{V} dt = \int_0^C (C_0 - C)^{-1} dC$$

$\longrightarrow$

3.) Continued

$$\frac{q_0}{V} t = \left[-\ln(C_0 - C)\right]_0^C$$

$$\frac{q_0}{V} t = -\ln(C_0 - C) + \ln(C_0)$$

– Now take exponential

$$\exp\left(\frac{q_0}{V} t\right) = \exp\left(-\ln(C_0 - C) + \ln C_0\right)$$

$$= +(C_0 - C)^{-1} C_0$$

Solve for (C) as a function of (t):

$$C_0 - C = C_0 \exp\left(-\frac{q_0}{V} t\right)$$

$$\boxed{C = C_0\left[1 - \exp\left(-\frac{q_0}{V} t\right)\right]}$$

Note: $\frac{q_0}{V}$ is just the residence time τ^{-1}

So:

$$\boxed{C = C_0\left[1 - \exp\left(-\frac{t}{\tau}\right)\right]}$$

What time will $C = 0.9 C_0$?

$$0.9 \cancel{C_0} = \cancel{C_0}\left[1 - \exp\left(-\frac{t}{\tau}\right)\right]$$

$$\exp\left(-\frac{t}{\tau}\right) = 0.1$$

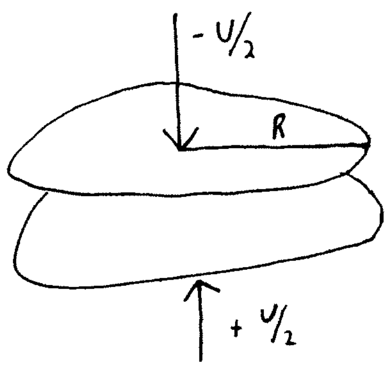
$$-\frac{t}{\tau} = \ln(0.1)$$

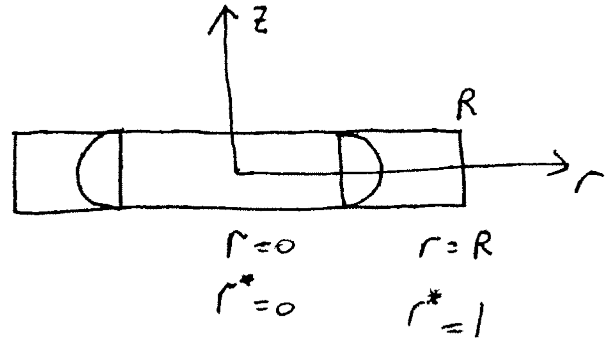
$$t = -\tau \ln(0.1)$$

$$t = \tau \ln(10)$$

$$\boxed{t = 115 \text{ min}}$$

4.) Two disks of radius R:



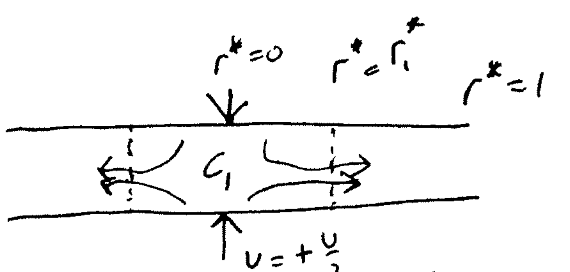


a.) Calculate the average radial velocity as a function of dimensionless radius.

$$r^* = \frac{r}{R}.$$

Remember that the continuity equation is just the conservation of an incompressible fluid.

Imagine we cut the disk at some radius r_1^* making a cylinder C_1:



Apply conservation of mass to this volume C_1. Since the volume decreases as the plates squeeze together, and the continuity equation tells us that the fluid is incompressible, then:

(Vol. rate of change in C_1) = (avg flow rate at $r=r_1^*$).

4.) Continued:

rate of squeezing fluid:

$$2\left(\frac{U}{2}\right) r_1^{*2} \pi = \langle U_{r_1^*} \rangle_{avg} \cdot 2\pi r_1^* 2b\, R$$

$2b$ = distance between plates

$$\langle U_{r_1^*} \rangle = \frac{U}{2} \frac{1}{2 b r_1^*} r_1^{*2} R$$

$$= \frac{UR}{4} \frac{r_1^*}{b}$$

, but since r_1^* is arbitrary,

$$\boxed{\langle U_{r^*} \rangle = \frac{UR}{4} \frac{r^*}{b}}$$

b.) Quasi-parallel flow approximation

$$U_r = U_{r,max}\left(1 - \frac{z^2}{b^2}\right)$$

$$U_{r,max} = fxn\left(\frac{r}{R}\right)$$

$$b = fxn(t)$$

If U_r is parabolic, it must satisfy a quadratic form:

$$U_r = a' + b'z + c'z^2$$

boundary conditions →

4. Continued

Boundary conditions.

No-slip at each plate:

at $z = \pm b(t)$, $\quad u_r = 0 \qquad$ b.c. (i)

From symmetry:

at $z = 0$, $\quad \dfrac{\partial u_r}{\partial z} = 0 \qquad$ b.c. (ii)

Apply b.c. (ii):

$$\frac{\partial u_r}{\partial z} = b' + 2c' z \Big|_{z=0} = 0 ,$$

so $b' = 0$. $\qquad U_r = a' + c' z^2$

Apply b.c. (i):

$$U_r \Big|_{z=\pm b(t)} = a' + c' b^2(t) = 0$$

or $\quad c' = -\dfrac{a'}{b^2(t)}$

Substitute:

$$U_r = a' \left(1 - \frac{z^2}{b^2(t)} \right)$$

Now compute the average velocity:

$$\langle u \rangle = \frac{1}{b} \int_0^b U_r \, dz \longrightarrow$$

4. <u>Continued</u>

$$\langle v \rangle = \frac{a'}{b} \int_0^b \left(1 - \frac{z^2}{b^2}\right) dz$$

Let $\eta = \frac{z}{b}$. $\quad d\eta = \frac{dz}{b}$

then:

$$\langle v \rangle = a' \int_0^b (1 - \eta^2)\, d\eta$$

$$= a' \left[\eta - \frac{\eta^3}{3}\right]_{\eta=0}^{\eta=1}$$

$$= \frac{2a'}{3}, \quad \text{or} \quad a' = \frac{3\langle v \rangle}{2}.$$

Recall that $\langle v \rangle = \frac{UR}{4b} r^*$, so $a' = \frac{3}{8}\frac{UR}{b}(r^*)$

and

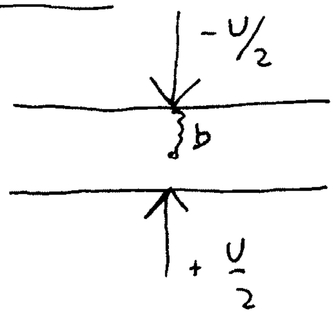
$$U_r = a'\left(1 - \frac{z^2}{b^2}\right)$$

$$\boxed{U_r = \frac{3}{8}\frac{UR}{b} r^* \left(1 - \frac{z^2}{b^2}\right)}$$

Velocity profile as a function of $r^* = \frac{r}{R}$ and separation distance b.

Now for b: $\longrightarrow$

4. continued



$$\frac{db}{dt} = -\frac{U}{2}$$

$$b = b_o - \int \frac{U}{2}\,dt$$

$$b = b_o - \frac{Ut}{2}$$

So

Answer:

$$\boxed{U_r = \frac{3}{8}\frac{UR}{b}\, r^* \left(1 - \frac{z^2}{\left(b_o - \frac{U}{2}t\right)^2}\right)}$$

Velocity profile as a function of $r^* = \frac{r}{R}$ and t

5) Using the concept of symmetry, isotropy, and index notation, evaluate the following integrals over a spherical surface.

a. $\displaystyle\int_{r=a} x_i\, x_j\, x_k\, x_\ell\, dA$

Let:

$$I_{ijk\ell} = \int_{r=a} x_i\, x_j\, x_k\, x_\ell\, dA = \int_S x_i\, x_j\, x_k\, x_\ell\, dA,$$

Where S is a spherical surface of radius a.

- A sphere is isotropic, so $I_{ijk\ell}$ must be isotropic as well. $(x_i x_j x_k x_\ell)$ is a 4th order <u>physical</u> tensor, so $I_{ijk\ell}$ must be physical as well.
- Summary:

$$I_{ijk\ell} = \int_S x_i\, x_j\, x_k\, x_\ell\, dA$$

$I_{ijk\ell}$ is a 4th order isotropic physical tensor.

- The most general form of a 4th order isotropic physical tensor is:

$$I_{ijk\ell} = \lambda_1\, \delta_{ij}\, \delta_{k\ell} + \lambda_2\, \delta_{ik}\, \delta_{j\ell} + \lambda_3\, \delta_{i\ell}\, \delta_{jk}$$

5. <u>continued</u>

Note that it doesn't matter for I_{ijkl} what order you write the indicies. (i.e):

$$I_{ijkl} = I_{ikjl} = I_{iljk} = I_{jikl} \dots \text{etc.}$$

<u>So:</u>

$$\lambda_1 \delta_{ij} \delta_{kl} + \lambda_2 \delta_{ik} \delta_{jl} + \lambda_3 \delta_{il} \delta_{jk} = \int_S x_i x_j x_k x_l \, dA.$$

<u>is the same as:</u>

(i) $$\lambda \left[\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right] = \int_S x_i x_j x_k x_l \, dA$$

because λ_1, λ_2, and λ_3 must be the same.

$$\lambda = \lambda_1 = \lambda_2 = \lambda_3 .$$

<u>Multiply eqn. (i) by $\delta_{ij} \delta_{kl}$:</u>

$$\lambda \left[\delta_{ij} \delta_{kl} \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} \delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk} \delta_{ij} \delta_{kl} \right]$$

$$= \delta_{ij} \delta_{kl} \int_S x_i x_j x_k x_l \, dA$$

→

5.) continued:

$$\lambda\left[\delta_{ii}\,\delta_{kk} + \delta_{il}\,\delta_{il} + \delta_{jl}\,\delta_{jl}\right] = \int_S x_i x_i \; x_k x_k \, dA$$

- Remember that $\delta_{ii} = 3$, so:

$$\lambda\left[3 \cdot 3 + 3 + 3\right] = \int_S x_i x_i \; x_k x_k \, dA$$

$$15\lambda = \int_S x_i x_i \; x_k x_k \, dA.$$

- $x_i x_i = \sum_{i=1}^{i=3} x_i x_i = x_1^2 + x_2^2 + x_3^2$

$= (\text{radius})^2$ for a sphere,

so: $x_i x_i = a^2$.

$$15\lambda = a^4 \int_S dA, \qquad \int_S dA = \text{surface area of a sphere} = 4\pi r^2$$

$$15\lambda = a^4\, 4\pi a^2,$$

$$\lambda = \frac{4\pi}{15} a^6$$

a.) $\boxed{I_{ijkl} = \frac{4\pi}{15} a^6 \left[\delta_{ij}\,\delta_{kl} + \delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\right]}$

5. Continued

b.) $$I_{ijk} = \int_S x_i\, x_j\, x_k\, dA$$

Let:

$$I_{ijk} = \lambda\, \epsilon_{ijk} = \int_S x_i\, x_j\, x_k\, dA \, ,$$

Since ϵ_{ijk} is the most general 3rd order isotropic tensor.

Now some symmetry:

- ϵ_{ijk} is antisymmetric in i & j,

  which is to say: $\epsilon_{ijk} = -\epsilon_{jik}$

- $\int_S x_i\, x_j\, x_k\, dA$ is symmetric in i & j,

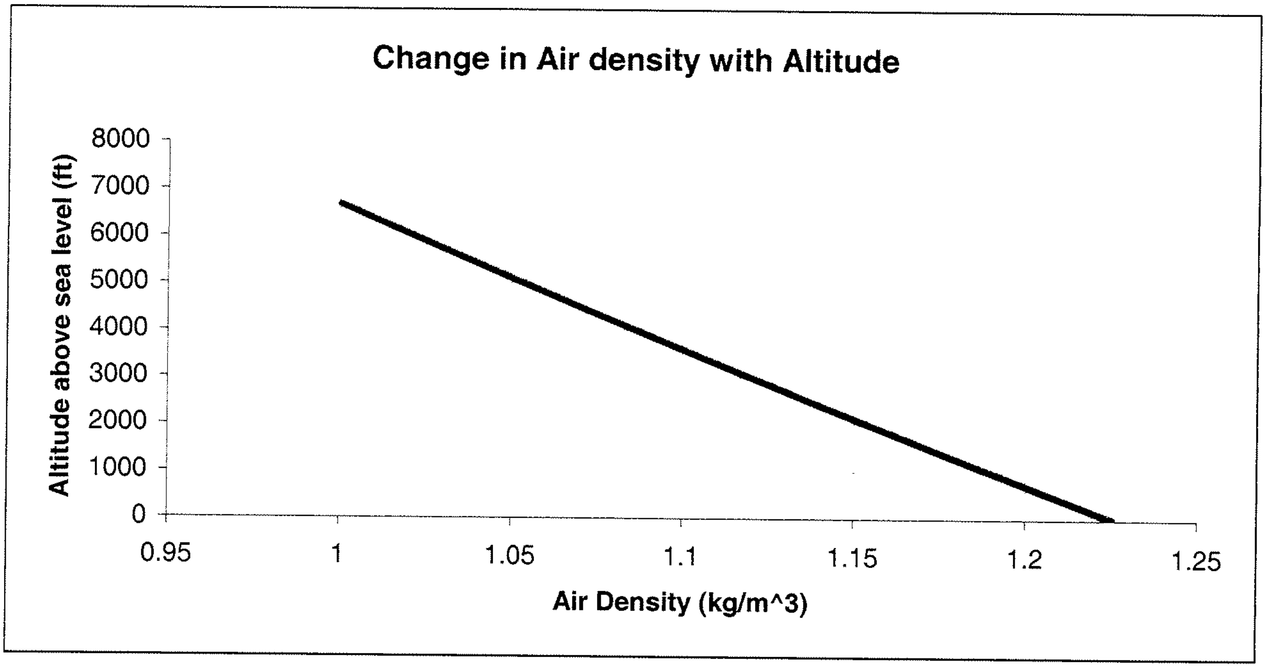
  which is to say: $\int_S x_i\, x_j\, x_k\, dA = \int_S x_j\, x_i\, x_k\, dA$

Therefore:

$$\lambda\, \epsilon_{ijk} = \lambda\, \epsilon_{jik} \, ,$$

which is only true if $\lambda = 0$.

$$\boxed{I_{ijk} = \int_S x_i\, x_j\, x_k\, dA = 0}$$


Change in Air density with Altitude
Altitude above sea level (ft)
8000
7000
6000
5000
4000
3000
2000
1000
0
0.95
1
1.05
1.1
1.15
1.2
1.25
Air Density (kg/m^3)