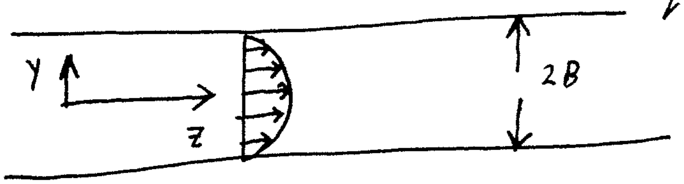
1) Start with the Navier-Stokes equation.



$$\rho\left(\frac{\partial V_z}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_z}{\partial y} + V_z \frac{\partial V_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}\right) + \rho g_z,$$

We can make a lot of assumptions, since this is a simple geometry.

- Steady state, so $\frac{\partial V_z}{\partial t} = 0$
- no flow in x- or y- directions, so $V_x = V_y = 0$
- From continuity, $\frac{\partial V_z}{\partial z} = 0$
- the system is semi-infinite in x-direction, so $\frac{\partial V_z}{\partial x} = 0$.

Therefore, N-S equation reduces to the following:

$$0 = -\frac{\partial P}{\partial z} + \mu \frac{\partial^2 V_z}{\partial y^2} + \rho g_z$$

We know that both $-\frac{\partial P}{\partial z}$ and ρg_z are constants, so let

$$\Gamma = -\frac{\partial P}{\partial z} + \rho g_z$$

$\longrightarrow$

1.) continued

Then we have the following 2nd order ODE:

$$\frac{\partial^2 V_z}{\partial y^2} = \frac{-\Gamma}{\mu}$$

with B.C. (no-slip) (i) $V_z\Big|_{y=\pm B} = 0$

(symmetry) (ii) $\left.\frac{\partial V_z}{\partial y}\right|_{y=0} = 0$

$$\frac{\partial}{\partial y}\left(\frac{\partial V_z}{\partial y}\right) = \frac{-\Gamma}{\mu}$$

$$\int \partial\left(\frac{\partial V_z}{\partial y}\right) = \int -\frac{\Gamma}{\mu}\, dy$$

$$\frac{\partial V_z}{\partial y} = \frac{-\Gamma}{\mu} y + C_1$$

$\Rightarrow$ from B.C. (ii), $C_1 = 0$

$$\int \partial V_z = \int \frac{-\Gamma}{\mu} y\, dy$$

$$V_z = -\frac{\Gamma}{\mu}\frac{1}{2} y^2 + C_2$$

$\Rightarrow$ from B.C. (i), $C_2 = \frac{\Gamma B^2}{2\mu}$

So

$$V_z = -\frac{\Gamma}{2\mu} y^2 + \frac{\Gamma B^2}{2\mu} = \frac{\Gamma B^2}{2\mu}\left(1 - \frac{y^2}{B^2}\right)$$

$\longrightarrow$

1.) <u>continued:</u>

Now for the flow rate:

$$Q = \int_A \underline{v} \cdot \underline{n} \, dA$$

$$= \int_0^W \int_{-B}^{B} \frac{\Gamma B^2}{2\mu} \left(1 - \frac{y^2}{B^2}\right) dx\, dy$$

$$= \frac{\Gamma B^2}{2\mu} W \left[y - \frac{y^3}{3B^2} \right]_{-B}^{B}$$

$$= \frac{\Gamma B^2}{2\mu} W \left[-(-B) + B + \left(\frac{-B^3}{3B^2}\right) - \frac{B^3}{3B^2} \right]$$

$$= \frac{\Gamma B^2}{2\mu} W \left[2B - \frac{2B}{3} \right]$$

$$\boxed{Q = 2\frac{\Gamma B^3 W}{3\mu}}$$

where W = width of channel (x-dir)

To calculate the force exerted by the fluid on the wall, use Newton's law of viscosity.

$$\tau_{yz} = -\mu \frac{\partial V_z}{\partial y}$$

$\longrightarrow$

1.) <u>continued:</u>

$$\tau_{yz}\Big|_{y=B} = -\mu\left[-\frac{\Gamma}{\mu}y\right]_{y=B}$$

$$\boxed{\tau_{yz} = \Gamma B}$$

This is the force on <u>one</u> wall, double it for both walls.

To include or neglect gravity, just remember that $\Gamma = -\frac{\partial P}{\partial z} + \rho g_z$. So, for each case:

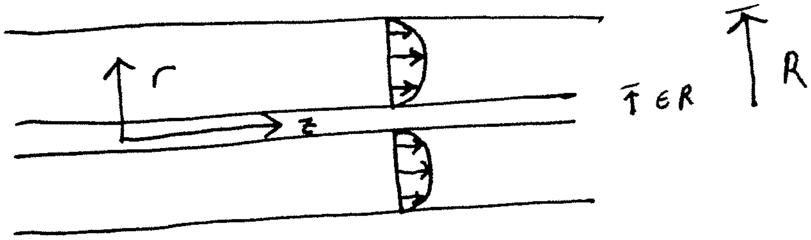
(a) no gravity:

- Γ decreases, so V_z also decreases.
- τ_{yz} also decreases by $\rho g_z B$.

(b) with gravity

- V_z increases
- τ_{yz} increases.

2. Pressure-driven flow through a pipe with a concentric wire:



Start with the Navier-Stokes equation in radial coordinates.

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

Assumptions:

- Steady state
- no velocity in r- or θ-directions
- Continuity
- no change in V_z in the z-direction, or θ-direction.

Then:

$$0 = -\frac{\partial P}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \rho g_z.$$

Again, let $\Gamma = -\frac{\partial P}{\partial z} + \rho g_z$, but gravity is unimportant, so: $\longrightarrow$

$$\Gamma = -\frac{\partial P}{\partial z}.$$

2.) <u>continued</u>

We are left with the following ODE:

$$\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = -\frac{\Gamma}{\mu} r$$

with B.C.: (no slip) (i) $V_z\Big|_{r=R} = 0$

(no slip) (ii) $V_z\Big|_{r=\epsilon R} = 0$

$$\int d\left(r\frac{\partial V_z}{\partial r}\right) = \int -\frac{\Gamma}{\mu} r\,dr$$

$$\frac{dV_z}{dr} = -\frac{\Gamma}{2\mu} r + \frac{C_1}{r}$$

$$\int dV_z = \int -\frac{\Gamma}{2\mu} r + \frac{C_1}{r}\,dr$$

$$V_z = -\frac{\Gamma}{4\mu} r^2 + C_1 \ln r + C_2$$

Apply both boundary conditions, we get 2 eq. & 2 unknowns, C_1 & C_2.

$$0 = \frac{-\Gamma}{4\mu} R^2 + C_1 \ln R + C_2 \qquad \text{eq. (i)}$$

$$0 = -\frac{\Gamma}{4\mu}\epsilon^2 R^2 + C_1 \ln(\epsilon R) + C_2 \qquad \text{eq. (ii)}$$

[subtract]:

→

2.) continued

$$-\frac{\Gamma}{4\mu} R^2 + C_1 \ln R = -\frac{\Gamma}{4\mu} \epsilon^2 R^2 + C_1 \ln(\epsilon R)$$

$$C_1 \left(\ln R - \ln(\epsilon R)\right) = \frac{\Gamma R^2}{4\mu}(1-\epsilon^2)$$

$$C_1 \left(\ln\left(\frac{R}{\epsilon R}\right)\right) = \frac{\Gamma R^2}{4\mu}(1-\epsilon^2)$$

$$\underline{C_1 = \frac{\Gamma R^2}{4\mu} \frac{(1-\epsilon^2)}{\ln(1/\epsilon)}}$$

Substitute back into eq. (i):

$$0 = -\frac{\Gamma}{4\mu} R^2 + \frac{\Gamma R^2}{4\mu} \frac{(1-\epsilon^2)}{\ln(1/\epsilon)} \ln R + C_2$$

So $\underline{C_2 = \frac{\Gamma R^2}{4\mu} - \frac{\Gamma R^2}{4\mu} \frac{(1-\epsilon^2)}{\ln(1/\epsilon)} \ln R}$

So
$$\boxed{V_z = -\frac{\Gamma}{4\mu} r^2 + \frac{\Gamma R^2}{4\mu} \frac{(1-\epsilon^2)}{\ln(1/\epsilon)} \ln r + \frac{\Gamma R^2}{4\mu} - \frac{\Gamma R^2}{4\mu} \frac{(1-\epsilon^2)}{\ln(1/\epsilon)} \ln R}$$

This can be simplified (a little), to:

$$V_z = \frac{\Gamma R^2}{4\mu}\left[1 - \frac{r^2}{R^2} + \frac{(1-\epsilon^2)}{\ln(1/\epsilon)} \ln r - \frac{(1-\epsilon^2)}{\ln(1/\epsilon)} \ln R\right]$$

and a little more: $\longrightarrow$

2. <u>Continued</u> ⑧

$$V_z = \frac{\Gamma R^2}{4\mu}\left[1 - \frac{r^2}{R^2} + \frac{(1-\epsilon^2)}{\ln(1/\epsilon)}\ln\left(\frac{r}{R}\right)\right]$$

Now for the flow rate: $\displaystyle Q = \int_A \vec{U}\cdot\vec{n}\,dA$

$$Q = \int_0^{2\pi}\int_{\epsilon R}^{R} V_z\, r\,dr\,d\theta = 2\pi\int_{\epsilon R}^{R} V_z\, r\,dr$$

$$= \frac{2\pi\,\Gamma R^2}{4\mu}\int_{\epsilon R}^{R}\left[r - \frac{r^3}{R^2} + \underbrace{\frac{(1-\epsilon^2)}{\ln(1/\epsilon)}\, r\ln\left(\frac{r}{R}\right)}\right]dr$$

→ Take a look at $\displaystyle\int_{\epsilon R}^{R} r\ln\left(\frac{r}{R}\right)dr.$

$$= \underbrace{\int_{\epsilon R}^{R} r\ln r\,dr}_{\text{Look it up!}} - \int_{\epsilon R}^{R} r\ln R\,dr$$

$$\int_{\epsilon R}^{R} r\ln\left(\frac{r}{R}\right)dr = \left[\frac{1}{2}r^2\ln r - \frac{1}{4}r^2 - \frac{1}{2}r^2\ln R\right]_{\epsilon R}^{R}$$

So: (substitute back into the whole integral)

$$Q = \frac{2\pi\,\Gamma R^2}{4\mu}\left[\frac{r^2}{2} - \frac{r^4}{4R^2} + \frac{(1-\epsilon^2)}{\ln(1/\epsilon)}\left(\frac{1}{2}r^2\ln r - \frac{1}{4}r^2 - \frac{1}{2}r^2\ln R\right)\right]_{\epsilon R}^{R}$$

→

2.) continued:

$$Q = \frac{2\pi\Gamma R^2}{4\mu}\left[\left(\frac{R^2}{2} - \frac{\epsilon^2 R^2}{2}\right) - \left(\frac{R^2}{4} - \frac{\epsilon^4 R^2}{4}\right) + \frac{(1-\epsilon^2)}{\ln(1/\epsilon)}\,(*)\right]$$

$$* = \left(\frac{R^2 \ln R}{2} - \frac{\epsilon^2 R^2 \ln(\epsilon R)}{2}\right) - \left(\frac{R^2}{4} - \frac{\epsilon^2 R^2}{4}\right)$$

$$- \left(\frac{R^2 \ln R}{2} - \frac{\epsilon^2 R^2 \ln R}{2}\right)$$

$$* = \left(\frac{R^2}{2} - \frac{\epsilon^2 R^2}{2}\right)\ln\left(\frac{\cancel{R}}{\epsilon \cancel{R}}\right) - \frac{R^2}{4}(1-\epsilon^2)$$

$$- \frac{R^2}{4}(2 - 2\epsilon^2)\ln R$$

$$* = \frac{R^2}{4}\left[(2-2\epsilon^2)\ln(1/\epsilon) - [1-\epsilon^2) - (2-2\epsilon^2)\ln R\right]$$

$$* = \frac{R^2}{4}\left[(2-2\epsilon^2)\ln\left(\frac{1}{\epsilon R}\right) - (1-\epsilon^2)\right]$$

Factor out $\frac{R^2}{4}$ from the entire expression for Q:

$$Q = \frac{2\pi\Gamma R^4}{16\mu}\left[(2-2\epsilon^2) - (1-\epsilon^4) + \frac{(1-\epsilon^2)}{\ln(1/\epsilon)}\left[(2-2\epsilon^2)\ln\left(\frac{1}{\epsilon R}\right) - (1-\epsilon^2)\right]\right]$$

2. Continued

$$Q = \frac{2\pi \Gamma R^2}{4\mu}\left[\left(\frac{R^2}{2} - \frac{\epsilon^2 R^2}{2}\right) - \left(\frac{R^2}{4} - \frac{\epsilon^4 R^2}{4}\right) + \frac{(1-\epsilon^2)}{\ln(1/\epsilon)}(*)\right]$$

$$* = \cancel{\frac{R^2 \ln R}{2}} - \frac{\epsilon^2 R^2 \ln(\epsilon R)}{2} - \left(\frac{R^2}{4} - \frac{\epsilon^2 R^2}{4}\right) - \cancel{\frac{R^2 \ln R}{2}} + \frac{\epsilon^2 R^2 \ln R}{2}$$

$$* = \frac{R^2}{4}\left[2\epsilon^2 \ln\left(1/\epsilon\right) - (1-\epsilon^2)\right]$$

Factor $\frac{R^2}{4}$ out of the whole expression for Q:

$$Q = \frac{2\pi \Gamma R^4}{16\mu}\left[(2-2\epsilon^2) - (1-\epsilon^4) + \frac{(1-\epsilon^2)}{\ln(1/\epsilon)}\left[2\epsilon^2 \ln(1/\epsilon) - (1-\epsilon^2)\right]\right]$$

Simplify

factor out $(1-\epsilon^2)$

$$Q = \frac{2\pi \Gamma R^4}{16\mu}\left[1 - 2\epsilon^2 + \epsilon^4 + \frac{(1-\epsilon^2)^2}{\ln(1/\epsilon)}\left(\frac{2\epsilon^2 \ln(1/\epsilon)}{(1-\epsilon^2)} - 1\right)\right]$$

$$= \frac{\pi \Gamma R^4}{8\mu}\left[1 - 2\epsilon^2 + \epsilon^4 + (1-\epsilon^2)2\epsilon^2 - \frac{(1-\epsilon^2)^2}{\ln(1/\epsilon)}\right]$$

simplify:

$$\boxed{Q = \frac{\pi \Gamma R^4}{8\mu}\left[1 - \epsilon^4 - \frac{(1-\epsilon^2)^2}{\ln(1/\epsilon)}\right]}$$

how much does Q change?

2.) <u>continued</u>

If $\epsilon = 0$, then it's just an open pipe with no wire.

$$\boxed{Q_{\epsilon=0} = \frac{\pi \Gamma R^4}{8\mu}}$$

if $\epsilon = 0.1$,

$$Q = (0.574)\frac{\pi \Gamma R^4}{8\mu}.$$

This is a 43 % drop in flowrate, which is large considering it is only a 1% change in area.

<u>Force per unit length:</u>

$$\tau_{rz} = -\mu \frac{dV_z}{dr}$$

recall from earlier that:

$$V_z = \frac{\Gamma R^2}{4\mu}\left[1 - \frac{r^2}{R^2} + \frac{1-\epsilon^2}{\ln(1/\epsilon)}\ln\left(\frac{r}{R}\right)\right]$$

$\longrightarrow$

2. <u>continued</u>

$$\tau_{rz} = \frac{\Gamma}{2} r + \frac{\cancel{\mu}}{r} \frac{\Gamma R^3}{4 \cancel{\mu}} \frac{1-\epsilon^2}{\ln(1/\epsilon)}$$

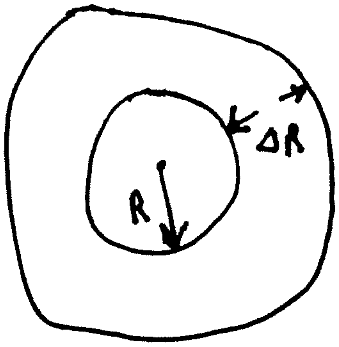
$$\tau_{rz}\Big|_{r=\epsilon R} = \frac{\Gamma \epsilon R}{2} + \frac{1}{\epsilon \cancel{R}} \frac{\Gamma \cancel{R^3}^{\,2}}{4} \frac{(1-\epsilon^2)}{\ln(1/\epsilon)}$$

$$F = \tau_{rz} A = \tau_{rz}\, 2\pi \epsilon R L$$

so: $$\frac{F}{L} = 2\pi \epsilon R\, \tau_{rz}$$

$$\boxed{F/L = 2\pi \epsilon \Gamma R^2 \left(\frac{\epsilon}{2} + \frac{1}{4\epsilon} \frac{(1-\epsilon^2)}{\ln(1/\epsilon)} \right)}$$

3. Couette Viscometer



Start with the Navier-Stokes equation in cylindrical coordinates.

$$\rho\left(\frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + V_z\frac{\partial V_\theta}{\partial z}\right) =$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r V_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta} + \frac{\partial^2 V_\theta}{\partial z^2}\right]$$

$$+ \rho g_\theta$$

Assumptions:

- Unidirectional flow, so $V_r = V_z = 0$
- Continuity, $\frac{\partial V_\theta}{\partial \theta} = 0$
- Steady state $\frac{\partial V_\theta}{\partial t} = 0$
- Semi-infinite in z-dir $\frac{\partial V_\theta}{\partial z} = 0$
- No pressure gradient in θ-dir, no gravity in θ-dir

→

3. Continued

We are left with:

$$0 = \frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r V_\theta)\right)$$

Boundary conditions:

no slip
$$\begin{cases} V_\theta\Big|_{r=R} = 0 & (i) \\ V_\theta\Big|_{r=R+\Delta R} = \Omega(R+\Delta R) & (ii) \end{cases}$$

Integrate once:

$$\frac{1}{r}\frac{d}{dr}(r V_\theta) = C_1 \Rightarrow$$

Integrate again:

$$r V_\theta = \frac{C_1 r^2}{2} + C_2$$

$$V_\theta = C_1 \frac{r}{2} + \frac{C_2}{r}$$

Apply B.C.

(i) $\Rightarrow$ $$0 = C_1 \frac{R}{2} + \frac{C_2}{R}$$

(ii) $\Rightarrow$ $$\Omega(R+\Delta R) = C_1 \frac{(R+\Delta R)}{2} + \frac{C_2}{R+\Delta R}$$

$\longrightarrow$

3. <u>Continued</u>

Solve 2 equations for C_1 and C_2.

$$C_2 = -C_1 \frac{R^2}{2}$$

$$\Omega (R+\Delta R)^2 = C_1 \frac{(R+\Delta R)^2}{2} - C_1 \frac{R^2}{2}$$

$$= C_1 \left(\frac{(R+\Delta R)^2 - R^2}{2}\right)$$

$$C_1 = \frac{2\Omega (R+\Delta R)^2}{2R\Delta R + \Delta R^2}$$

$$= \frac{2\Omega R^2}{2R\Delta R + \Delta R^2} + \frac{2\Omega \cancel{(2R\Delta R + \Delta R^2)}}{\cancel{2R\Delta R + \Delta R^2}}$$

$$\boxed{C_1 = 2\Omega\left[\frac{R^2}{2R\Delta R + \Delta R^2} + 1\right]}$$

$$\boxed{C_2 = -R^2\Omega\left[\frac{R^2}{2R\Delta R + \Delta R^2} + 1\right]}$$

$$V_\theta = C_1 \frac{r}{2} + \frac{C_2}{r}$$

$$V_\theta = r\Omega - \frac{R^2}{r}\Omega\left[\frac{R^2}{2R\Delta R + \Delta R^2} + 1\right]$$

$$= \Omega\left(r - \frac{R^2}{r}\right)\left[\frac{R^2}{2R\Delta R + \Delta R^2} + 1\right] = [\text{Constant}]\left(r - \frac{R^2}{r}\right)$$

3.) Continued

From Newton's Law of Viscosity:

$$\tau_{r\theta} = -\mu\left[r \frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)\right]$$

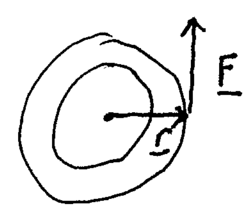
$$= -\mu r\,[\text{constant}]\,\frac{\partial}{\partial r}\left(1 - \frac{R^2}{r^2}\right)$$

$$= -\mu r\,[\text{constant}]\,\frac{-2R^2}{r^3}$$

$$= \frac{2\mu R^2}{r^2}\left(\frac{\Omega R^2}{2R\Delta R + \Delta R^2} + \Omega\right)$$

Recall that torque on the outer cylider is $\underline{M} = \underline{r} \times \underline{F}$

From def. of a cross product:



$$\underline{M} = |\underline{r}|\,|\underline{F}|\,\sin\theta\;\underline{n}$$

and since $\theta = \frac{\pi}{2}$,

$$|M| = |r|\,|F|, \qquad |F| = \tau_{r\theta}\Big|_{R+\Delta R} A$$

$$r = (R+\Delta R), \qquad A = \text{area} = 2\pi(R+\Delta R)L$$

So:

$$M = 2\pi(R+\Delta R)^2 L\left[\frac{2\mu R^2}{(R+\Delta R)^2}\left(\frac{\Omega R^2}{2R\Delta R + \Delta R^2} + \Omega\right)\right]$$

$$\boxed{M = 4\pi\mu L\Omega\left(\frac{R^4}{2R\Delta R + \Delta R^2} + R^2\right)}$$

→

3. continued

Recall from class the approximate solution:

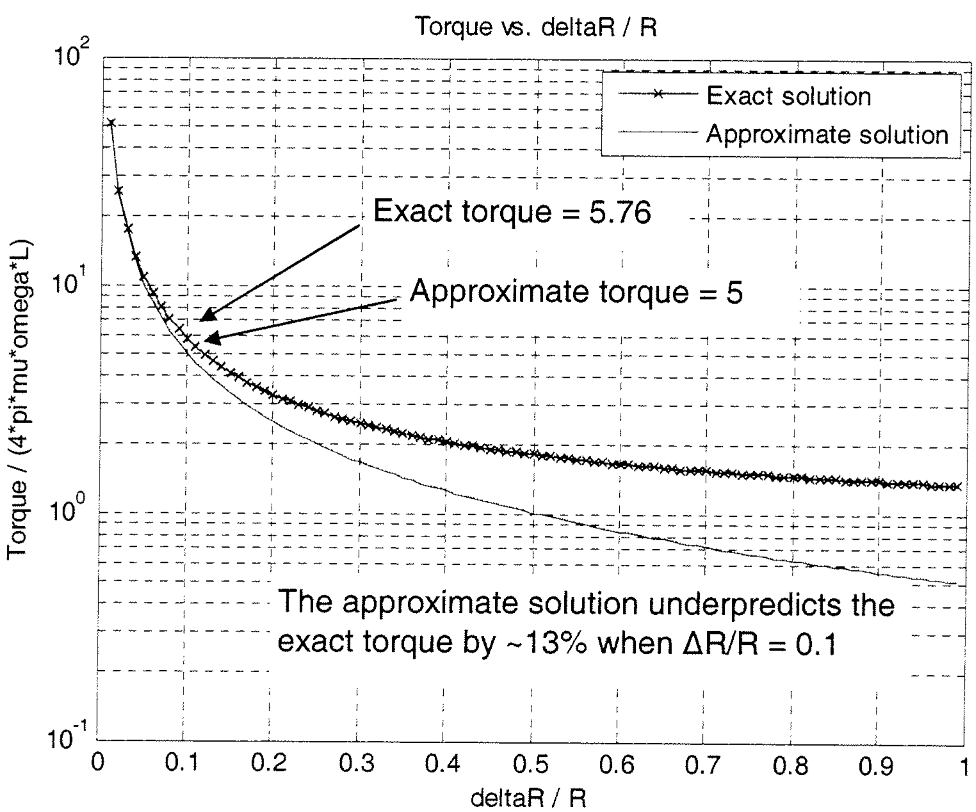
$$U_0 = \Omega R$$

$$\tau_{yx} = \mu \frac{U_0}{\Delta R}$$

$$|M|_{approx} \approx \tau_{yx} R A = \mu \frac{\Omega R}{\Delta R} R (2\pi R) L \Rightarrow \boxed{|M|_{approx} = \frac{2\pi\mu L \Omega R^3}{\Delta R}}$$

Plot both $\left(\frac{|M|_{exact}}{4\pi\mu\Omega L}\right)$ and $\left(\frac{|M|_{approx}}{4\pi\mu\Omega L}\right)$ vs. $\frac{\Delta R}{R}$.

Let $R = 1$.



For $\frac{\Delta R}{R} = 0.1$

Answer:

$$\frac{|M_{approx}|}{|M_{exact}|} = 0.87$$

4. From HW 4,

$$\frac{|U|}{|F|}\Bigg|_{\parallel} = A = 0.3 \qquad \frac{|U|}{|F|}\Bigg|_{\perp} = B = 0.1 \qquad , \qquad \begin{aligned} \lambda_1 &= B \\ \lambda_2 &= A - B \end{aligned}$$

$$P_i = \delta_{i3}\cos\theta + \delta_{i1}\sin\theta$$

$$A_{iJ} = \lambda_1 \delta_{iJ} + \lambda_2 P_i P_J$$

assume that the force is in the 3-direction

Now, if the velocity is given by:

$$U_i = A_{iJ} F_J = A_{iJ} \underbrace{F\delta_{J3}}$$

$$= \left(\lambda_1 \delta_{iJ} + \lambda_2 (\delta_{i3}\cos\theta + \delta_{i1}\sin\theta)(\delta_{J3}\cos\theta + \delta_{J1}\sin\theta)\right)$$

$$\times F\delta_{J3}$$

Distribute the δ_{J3}:

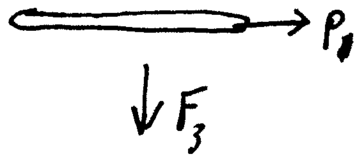
$$U_i = \left(\lambda_1 \delta_{i3} + \lambda_2(\delta_{i3}\cos\theta + \delta_{i1}\sin\theta)(\cancelto{1}{\delta_{33}}\cos\theta \;\; \cancelto{0}{\delta_{31}}\sin\theta)\right) F$$

$$= \left(\lambda_1 \delta_{i3} + \lambda_2 (\delta_{i3}\cos\theta + \delta_{i1}\sin\theta)\cos\theta\right) F.$$

$\longrightarrow$

4. continued

Now consider the velocity in the perpendicular case:



Then:

$$\frac{U_1}{F} = B\underbrace{\cancel{\delta_{13}}}_{0} + (A-B)\left(\underbrace{\cancel{\delta_{13}}}_{0}\cos\theta + \underbrace{\cancel{\delta_{11}}}_{1}\sin\theta\right)\cos\theta$$

$$\frac{|U_1|}{|F|} = (A-B)\sin\theta\cos\theta$$

Now for the parallel case (3-direction):

$$\frac{U_3}{F} = B\underbrace{\cancel{\delta_{33}}}_{1} + (A-B)\left(\underbrace{\cancel{\delta_{33}}}_{1}\cos\theta + \underbrace{\cancel{\delta_{31}}}_{0}\sin\theta\right)\cos\theta$$

$$= B + (A-B)\cos^2\theta$$

$$\frac{U_1}{U_3} = \frac{(A-B)\sin\theta\cos\theta}{B + (A-B)\cos^2\theta}.$$

$\longrightarrow$

4. Continued

At a maximum,

$$\frac{d\left(\frac{U_1}{U_3}\right)}{d\theta} = 0 .$$

if you grind through the derivative, you get this:

$$0 = \frac{2(A-B)^2 \sin^2\theta \cos^2\theta}{\left(B + (A-B)\cos^2\theta\right)^2} + \frac{(A-B)(\cos^2\theta - \sin^2\theta)}{B + (A-B)\cos^2\theta} .$$

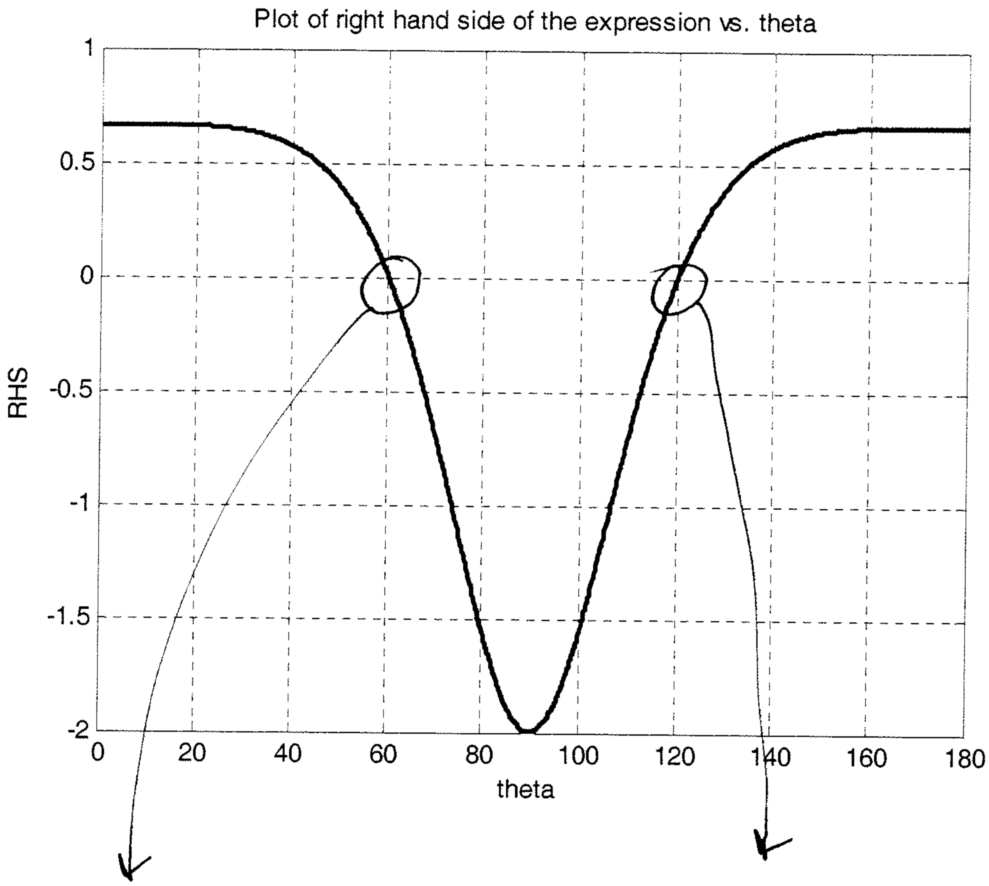
An acceptable answer would be to plot this expression vs. θ and find that $\theta_{max} = 60°$

Or, with a lot of work, you can get: $= \frac{\pi}{3}$.

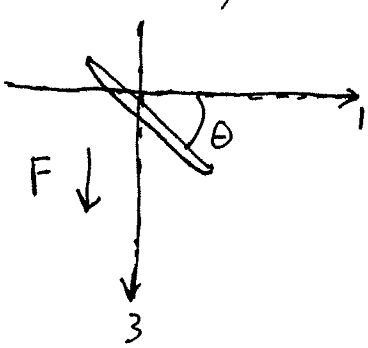
$$\cos\theta_{max} = \sqrt{\frac{B}{A+B}} , \quad \text{or} \quad \theta_{max} = \cos^{-1}\sqrt{\frac{0.1}{0.4}}$$

$$= \cos^{-1}\left(\frac{1}{2}\right)$$

$$\boxed{\theta_{max} = \frac{\pi}{3} = 60°}$$

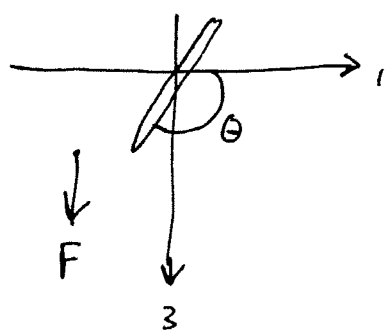


at $\theta = 60°$, we have the following situation:



- This is the angle for maximum velocity in the <u>positive</u> 1 direction.

at $\theta = 120°$:



- This will result in maximum $\frac{v_1}{v_3}$ in the <u>negative</u> 1 direction.