CBE 30355 Problem Set 6 $\boxed{\text{Solutions}}$

1. Great Molasses Flood simulation through ~~the~~ a scale model.

Kinematic viscosity of molasses is: $\nu = \frac{\mu}{\rho} = 44$ centistokes

For the scale model, both Re and Fr must remain the same.

$$Re_1 = Re_2$$

$$\text{(i)} \quad \frac{V_1 L_1}{\nu_1} = \frac{V_2 L_2}{\nu_2}$$

$\nu_1 = 44$ cSt (molasses)
$\nu_2 = 1.004$ cSt (H_2O)

$$Fr_1 = Fr_2$$

$$\text{(ii)} \quad \frac{V_1^2}{g L_1} = \frac{V_2^2}{g L_2}$$

Rearrange (ii):

$$\frac{L_1}{L_2} = \frac{V_1^2}{V_2^2}$$

rearrange (i)

$$\frac{V_1 L_1}{V_2 L_2} = \frac{\nu_1}{\nu_2}$$

Combine, solve for $\frac{V_1}{V_2}$:

$$\boxed{\frac{V_1}{V_2} = \left(\frac{\nu_1}{\nu_2}\right)^{1/3}}$$

(Substitute)

$$\boxed{\frac{L_1}{L_2} = \left(\frac{\nu_1}{\nu_2}\right)^{2/3}}$$

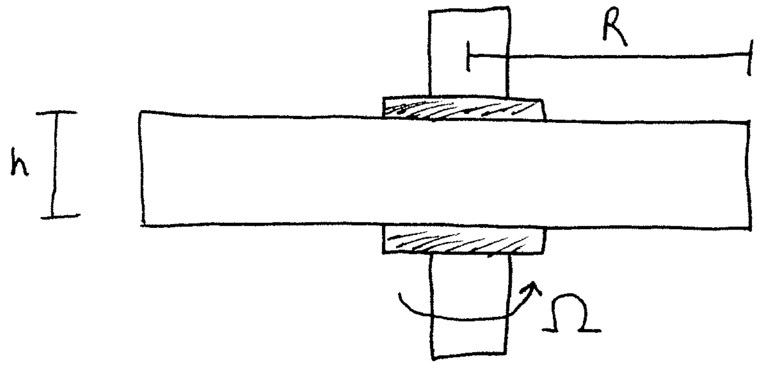
$\longrightarrow$

1) Continued

Entering in the kinematic viscosities,

$$\boxed{\begin{array}{l}\text{Lengths should be reduced by a factor of } 0.08 \\ \text{Velocities will be reduced by a factor of } 0.2836\end{array}}$$

2) Parallel plate Viscometer



$$\frac{h}{R} \ll 1$$

a.) Low-Re flow

Show that $V_\theta = f(r, z)$ is satified with $V_r = V_z = 0$.

<u>Start with N-S equation in θ-direction</u>

$$\rho\left(\frac{dV_\theta}{dt} + V_r\frac{dV_\theta}{dr} + \frac{V_\theta}{r}\frac{dV_\theta}{d\theta} + \frac{V_r V_\theta}{r} + V_z\frac{dV_\theta}{dz}\right) = -\frac{1}{r}\frac{dP}{d\theta}$$

$$+ \rho g_\theta + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial\theta^2} + \frac{\partial}{r^2}\frac{\partial V_r}{\partial\theta} + \frac{\partial^2 V_\theta}{\partial z^2}\right)$$

<u>Assumptions:</u>

- Steady state $\frac{\partial V_\theta}{\partial t} = 0$
- Continuity $\frac{\partial V_\theta}{\partial\theta} = 0$
- Unidirectional flow $V_r = V_z = 0$
- No pressure gradient, no gravity in θ-dir
- flow is at low Re, so $\cancel{\frac{\partial V_\theta}{\partial r} = 0}$.

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta)\right) = 0$$ (nonlinear inertial term)

2 (a) continued :

we are left with:

$$\frac{\partial^2 V_\theta}{\partial z^2} = 0 .$$

and B.C.s :

$$V_\theta\Big|_{z=0} = \Omega r \qquad V_\theta\Big|_{z=h} = 0$$

↳ lower disk rotates (i)

↳ upper disk is stationary. (ii)

Integrate twice to get:

$$V_\theta = C_1 z + C_2$$

Apply B.C.'s

(i) $\Omega r = C_1(0) + C_2$

$$\therefore \quad C_2 = \Omega r$$

(ii) $0 = C_1 h + \Omega r$

$$\therefore \quad C_1 = -\frac{\Omega r}{h}$$

$$\boxed{V_\theta = -\frac{\Omega r}{h} z + \Omega r}$$

Now for the torque.

$$T = \underline{F} \cdot \underline{r}$$

$\underline{F}$ = force on the upper plate

$\underline{r}$ = length of the moment arm.

→

2 (a) continued

$$F = \int_0^R -\tau_{\theta z} \, dA$$

and

$$\tau_{\theta z} = -\mu \frac{\partial U_\theta}{\partial z} = \mu \frac{\Omega r}{h} \qquad A = \pi r^2$$

$$dA = 2\pi r \, dr$$

so:

$$F = \int_0^R -\mu \frac{\Omega}{h} 2\pi r^2 \, dr$$

and

$$T = \int_0^R - r \cdot \mu \frac{\Omega \, 2\pi}{h} r^2 \, dr$$

$$= \int_0^R - r^3 \frac{2\pi \Omega \mu}{h} \, dr$$

$$\boxed{T = \frac{-\pi \mu \Omega R^4}{2h}}$$

2.(b)

Look at the N-S equation in the r-direction.

On the LHS,

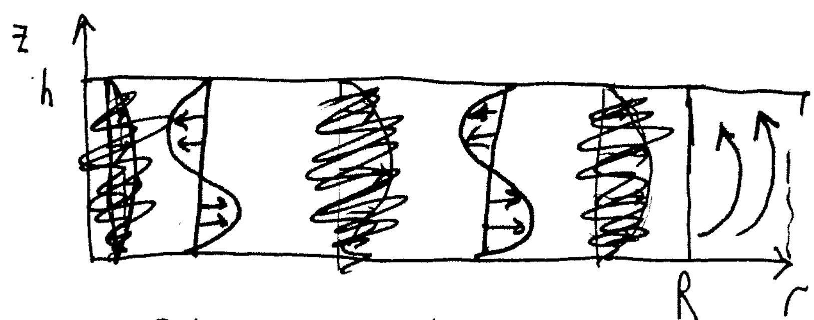
$$-\frac{V_\theta^2}{r} \neq 0, \quad \text{therefore } V_r \neq 0.$$

This velocity expression therefore does not satisfy the full N-S. equations.

Physical explanation:

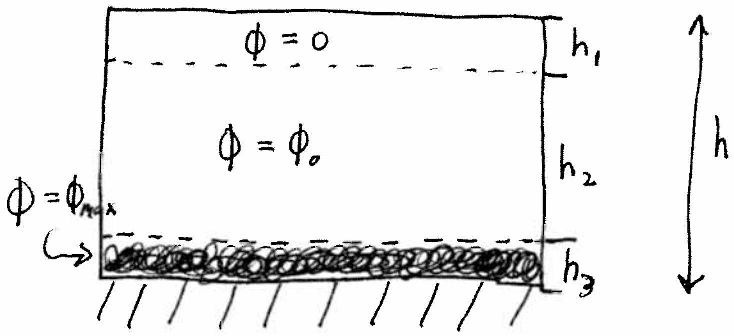
This term arrises from centrifugal force. This results in the non-zero r-direction velocity.

Profile



Backflow is the result of continuity-
Fluid is thrown outward at rotating plate, and returns inward near the stationary plate. There is a strong turning current at the outer edge - and if surface tension isn't strong enough to confine it, the lab gets sprayed!

3. Rheology of a settling suspension:



Mass balance on particles:

$$h\,\phi_0 = h_1 \cdot 0 + h_2\,\phi_0 + h_3\,\phi_{max} \qquad (i)$$

$$h = h_1 + h_2 + h_3 \qquad (ii)$$

Part 1

Determine the thickness of the settled layer and of the remaining suspension if the clear layer has a thickness h_1.

Rearrange eq. (i) and (ii) to solve for unknowns h_2 & h_3:

$$(i) \quad h = h_2 + h_3\,\frac{\phi_{max}}{\phi_0}$$

$\therefore$

$$\boxed{\begin{aligned} h_2 &= h - h_1\left(\frac{\phi_{max}}{\phi_{max} - \phi_0}\right) \\ h_3 &= h_1\left(\frac{\phi_0}{\phi_{max} - \phi_0}\right) \end{aligned}}$$

$\rightarrow$

3. <u>continued</u>

<u>Part 2</u>

<u>Three segments in the velocity profile:</u>

$$V_1 = Az + B$$

$$V_2 = Cz + D$$

$$V_3 = Ez + F$$

Note: since $\mu = \infty$ in the settled layer, $V_3 = \Omega r$.

<u>B.C.'s</u>

$V_1(h_2+h_3) = V_2(h_2+h_3)$, $V_1(h) = 0$

$V_2(h_3) = \Omega r$

$V_3(0) = \Omega r$ $\quad V_3(h_3) = \Omega r$

because $\mu = \infty$

<u>Three equations ~~from the~~ B.C.'s:</u>

1) $\Omega r = C h_3 + D$

2) $0 = A h + B$

3) $C h_2 + C h_3 + D = A h_2 + A h_3 + B$

<u>We have six unknowns</u>

A
B
C
D
h_2
h_3

- Must get 3 more equations.
- Two come from part (a):

4) $h_2 = h - h_1 \left(\dfrac{\phi_{max}}{\phi_{max} - \phi_0}\right)$

5) $h_3 = h_1 \left(\dfrac{\phi_0}{\phi_{max} - \phi_0}\right)$

3. Continued

To get the sixth equation, we remember that the torque must be a continuous function as well as the velocity.

$$\tau_{\theta z_1} = \tau_{\theta z_2}$$

$$\mu_0 \frac{\partial V_{\theta 1}}{\partial z} = \mu(\phi_0) \frac{\partial V_{\theta 2}}{\partial z}$$

this expression is between layers 1 & 2.
μ_0 = viscosity of clear layer
$\mu(\phi_0)$ = viscosity of suspension.

6) $\mu_0 A = \mu(\phi_0) C$

Solve equations (1) - (6) for the six unknowns.

$$\boxed{A = \frac{-\Omega r}{\left(\frac{\mu_0}{\mu(\phi_0)} h + h_1\left(1 - \frac{\mu_0 \phi_{max}}{\mu(\phi_0)(\phi_{max} - \phi_0)}\right)\right)}}$$

$$\boxed{B = \frac{-\Omega r}{\frac{\mu_0}{\mu(\phi_0)} + \frac{h_1}{h}\left(1 - \frac{\mu_0 \phi_{max}}{\mu(\phi_0)(\phi_{max} - \phi_0)}\right)}}$$

$$\boxed{C = \frac{-\Omega r}{h + h_1\left(\frac{\mu(\phi_0)}{\mu_0} - \frac{\phi_{max}}{\phi_{max} - \phi_0}\right)}}$$

→

3. Continued

$$\boxed{D = \Omega r \left(1 + \frac{\phi_0}{\left((\phi_{max} - \phi_0)\left(\frac{h}{h_1} + \frac{\mu(\phi_0)}{\mu_0} \right) - \phi_{max} \right)} \right)}$$

$$\boxed{E = 0}$$

$$\boxed{F = \Omega r}$$

h_2 & h_3 are given in eq. (4) & (5)

Now we know the velocity profiles.

Torque for settled system:

$$T_s = \int_0^R r \mu_0 \frac{\partial V_1}{\partial z} 2\pi r \, dr$$

$$= \int_0^R \mu_0 A 2\pi r^2 dr$$

$$= \frac{-2\pi \mu_0 \Omega}{\frac{\mu_0}{\mu(\phi_0)} h + h_1 \left(1 - \frac{\mu_0 \phi_{max}}{\mu(\phi_0)(\phi_{max} - \phi_0)} \right)} \int_0^R r^3 dr$$

$$\boxed{T_{settled} = \frac{-\pi \mu_0 \Omega R^4}{2 \left(\frac{\mu_0}{\mu(\phi_0)} h + h_1 \left(1 - \frac{\mu_0 \phi_{max}}{\mu(\phi_0)(\phi_{max} - \phi_0)} \right) \right)}}$$

$\rightarrow$

3. Continued

Now find the torque for the unsettled system:

$$V = C_1 z + C_2$$

$$V(0) = \Omega r = C_2$$

$$V(h) = 0 \implies C_1 = \frac{-\Omega r}{h}$$

$$\tau_{\theta z} = -\mu(\phi_o) \frac{-\Omega r}{h}$$

$$\boxed{T_{unsettled} = \frac{-\pi \mu(\phi_o) \Omega R^4}{2h}}$$

For the system given in the problem statement:

$$\phi_{max} = 0.62 \qquad \mu = 16.925 \mu_o$$

$$\phi_o = 0.45 \qquad \therefore \quad \frac{\mu_o}{\mu} = 0.05908$$

$$h_1 = (0.1) h$$

$$T_{us} = \frac{-16.925 \mu_o \pi \Omega R^4}{2h}$$

$\longrightarrow$

3. Continued

$$T_{settled} = \frac{-\pi \mu_0 \Omega R^4}{2\left(0.05908\,h + (0.1)h\left(1 - 0.5908(3.6471)\right)\right)}$$

$$= \frac{-\pi \mu_0 \Omega R^4}{2h \cdot 0.13753}$$

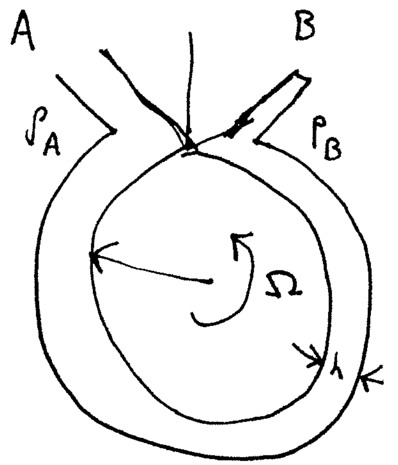
So:

$$\frac{T_{settled}}{T_{unsettled}} = \frac{\dfrac{-\pi \mu_0 \cancel{\Omega R^4}}{\cancel{2h} \cdot 0.13753}}{\dfrac{-16.925\, \mu_0 \pi \cancel{\Omega R^4}}{\cancel{2h}}}$$

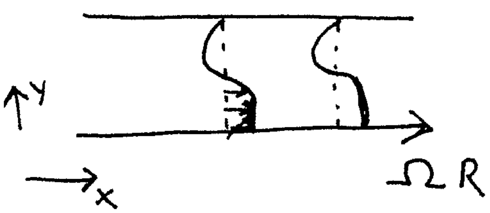
$$= \frac{\dfrac{1}{0.13753}}{16.925}$$

$$\boxed{\frac{T_s}{T_{us}} = 0.4296}$$

4. Viscosity Pump



Approximate the system in Cartesian coordinates



With assumptions of:

- S.S. $\frac{\partial}{\partial t} = 0$
- unidirectional $V_y = V_z = 0$
- continuity
- gravity is unimportant
- semi-infinite in z-dir

N-S equation reduces to:

$$0 = \frac{-\partial P}{\partial x} + \mu \frac{\partial^2 V_x}{\partial y^2}$$

From the problem statement, we can assume that $\frac{dP}{dx} = \frac{\Delta P}{L} = \frac{P_B - P_A}{L}$

Then:

$$\frac{\partial^2 V}{\partial y^2} = \frac{\Delta P}{\mu L}$$

with B.C.

$$V(y=0) = \Omega R$$

$$V(y=h) = 0$$

$$\boxed{V = \frac{1}{2}\frac{\Delta P}{\mu L} y^2 + Ay + B}$$

From the B.C., $\boxed{B = \Omega R}$ and $\boxed{A = -\left(\frac{\Delta P}{2\mu L} h + \frac{\Omega R}{h}\right)}$ $\rightarrow$

4. continued

So the velocity profile is:

$$\boxed{V = \frac{1}{2}\frac{\Delta P}{\mu L} y^2 - \left(\frac{\Delta P}{2\mu L}h + \frac{\Omega R}{h}\right) y + \Omega R}$$

Flow rate:

$$Q = \int_0^h V \, dA = \int_0^h V \, W \, dy$$

$$\frac{Q}{W} = \int_0^h \left[\frac{1}{2}\frac{\Delta P}{\mu L} y^2 - \left(\frac{\Delta P}{2\mu L}h - \frac{\Omega R}{h}\right) y + \Omega R\right] dy$$

$$\frac{Q}{W} = \left[\frac{1}{6}\frac{\Delta P}{\mu L} y^3 - \left(\frac{\Delta P h}{2\mu L} - \frac{\Omega R}{h}\right)\frac{1}{2} y^2 + \Omega R y\right]_0^h$$

$$\frac{Q}{W} = \frac{1}{6}\frac{\Delta P}{\mu L} h^3 - \left(\frac{\Delta P h}{2\mu L} - \frac{\Omega R}{h}\right)\frac{1}{2} h^2 + \Omega R h$$

$$\left.\begin{aligned} \Delta P &= P_B - P_A \\ L &= 2\pi R \end{aligned}\right\} \rightarrow \text{so} \quad \boxed{\frac{Q}{W} = \frac{\Omega R h}{2} - \frac{(P_B - P_A) h^3}{24 \pi \mu R}}$$

The maximum ΔP occurs when $\frac{Q}{W} = 0$:

$$0 = \frac{\Omega R h}{2} - \frac{(P_B - P_A) h^3}{24 \pi \mu R}$$

$$\boxed{P_B - P_A = \frac{12 \pi \mu \Omega R^2}{h^2}}$$

4. Continued

The torque on the shaft is found by:

$$T = 2\pi\, R W \cdot R \cdot \tau_{yx}\Big|_{y=0}$$

$$\tau_{yx} = -\mu\left(\frac{\partial v_x}{\partial y} + \cancel{\frac{\partial v_y}{\partial x}}\right)$$

$$\cancel{\frac{\partial v_y}{\partial x}} \rightarrow 0$$

$$= -\mu\left(\frac{\Delta P}{\mu L} y - \frac{\Delta P}{2\mu L} h - \frac{\Omega R}{h}\right)$$

$$\tau_{yx} = \frac{\Omega R \mu}{h} - \frac{\Delta P}{L} y - \frac{\Delta P}{2L} h$$

$$\boxed{T = \frac{2\pi \mu W \Omega R^3}{h} + \frac{(P_B - P_A) W h R}{2}}$$

Power = (Torque)(rotation speed)

$$= T \cdot \Omega$$

$$\boxed{\text{Power} = \frac{2\pi \mu W \Omega^2 R^3}{h} + \frac{(P_B - P_A) W h R \Omega}{2}}$$

→ This is the mechanical energy input.

To calculate the efficiency, we need the hydraulic energy out as well.

hydraulic power ~~[illegible]~~ $= Q \Delta P$

→

4. continued

$$Q\,\Delta P = \frac{\Omega R H \Delta P W}{2} - \frac{\Delta P^2 h^3 W}{24\pi\mu R}$$

$$\text{Power(in)} = \frac{4\pi\mu W \Omega^2 R^3 + \Delta P\, W h^2 R \Omega}{2h}$$

$$\epsilon = \text{efficiency} = \frac{Q\,\Delta P}{\text{Power(in)}}$$

$$\epsilon = \frac{\left(\dfrac{\overset{6}{\cancel{12}}\pi\mu \cancel{W}\Omega h \Delta P R^2 - \cancel{2}\Delta P^2 h^3 \cancel{W}}{\underset{12}{\cancel{48}}\pi\mu R}\right)}{\left(\dfrac{4\pi \cancel{W}\Omega^2 R^3 + \Delta P \cancel{W} h^2 R \Omega}{\cancel{2} h}\right)}$$

$$\boxed{\epsilon = \frac{6\pi\mu\,\Omega h \Delta P R^2 - \Delta P^2 h^3}{12\pi\mu R} \cdot \frac{h}{4\pi\Omega^2 R^3 + \Delta P h^2 R \Omega}}$$

Energy is lost due to friction and pressure-induced back flow.