CBE 30355   <u>Problem Set 7</u> $\boxed{\text{Solutions}}$

1) Dimensional Analysis of turkey cooking.

Cooking time t depends on: thermal diffusivity α

density ρ

mass m

Start by constructing a dimensional matrix

		t	$=f($ α	ρ	$m)$
mass	M	0	0	1	1
length	L	0	2	3	0
time	T	1	-1	0	0

We have 4 variables.

$$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$ is a largest nonzero determinant submatrix, so rank $= 3$

<u>Buckingham Π theorem:</u>

\# dimensionless groups = \# vars − \# rank of submatrix

$$1 = 4 - 3$$

To find the dimensionless group, solve the system:

$$Ax = b$$

1.) Continued

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Since $\Pi_1 = \alpha^a \rho^b m^c$

and $t = \Pi_1 (\text{constant})$.

Solve using Matlab or linear algebra for:

$$x = \begin{bmatrix} -1 \\ -2/3 \\ 2/3 \end{bmatrix}, \qquad \Pi_1 = \frac{m^{2/3}}{\alpha \cdot \rho^{2/3}} \cancel{(\text{constant})}$$

But α and ρ are (presumably) constant for all turkeys. So:

$$t = C_1 \, m^{2/3}$$

From the web, we can find a table of empirical data of cooking time vs. turkey mass.

Plot t vs. $m^{2/3}$, and solve using linear least-squares regression for the trend line. Then:

$$\boxed{t = 0.864 \left(m^{2/3}\right)}$$ (Results may vary)

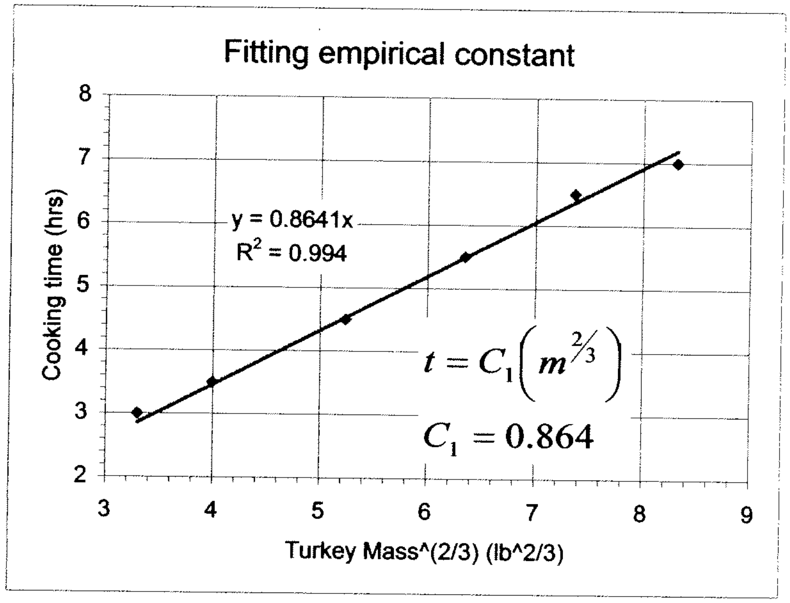
HW#7 Problem #1

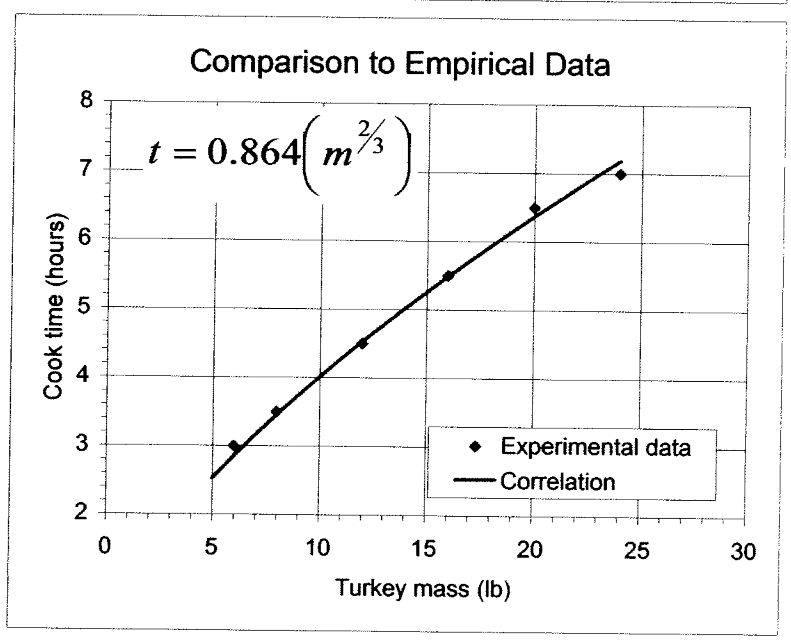
Experimental data

Mass (lbm)	Cooking time (hr)	weight^2/3
6	3	3.30
8	3.5	4.00
12	4.5	5.24
16	5.5	6.35
20	6.5	7.37
24	7	8.32

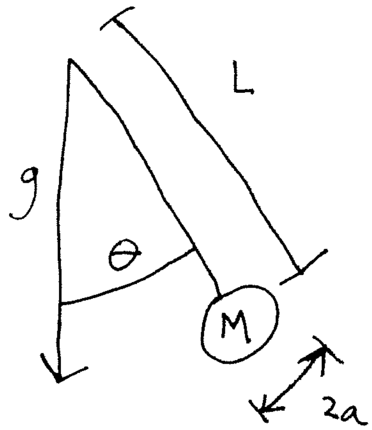
Correlation: $t = 0.864\left(m^{2/3}\right)$

Mass (lbm)	Correlated time (hr)
5	2.53
6	2.85
7	3.16
8	3.46
9	3.74
10	4.01
11	4.27
12	4.53
13	4.78
14	5.02
15	5.26
16	5.49
17	5.71
18	5.93
19	6.15
20	6.37
21	6.58
22	6.78
23	6.99
24	7.19





2. Damped pendulum in a viscous fluid



$$ML\frac{d^2\theta}{dt^2} = -Mg\sin(\theta) - 6\pi\mu a L\frac{d\theta}{dt}$$

$$\theta\Big|_{t=0} = \theta_0 \quad ; \quad \frac{d\theta}{dt}\Big|_{t=0} = 0$$

Start by rendering the equation dimensionless.

$$t^* = \sqrt{\frac{g}{L}}\,t \quad , \quad \underbrace{\theta^* = \frac{\theta}{\theta_0}}$$

Note: θ is already dimensionless, but θ^* ranges from $[0, 1]$.

$$Mg\theta_0\frac{d^2\theta^*}{dt^{*2}} = -Mg\sin(\theta^*\theta_0) - 6\pi\mu a\theta_0\sqrt{gL}\frac{d\theta^*}{dt^*}$$

divide through by $Mg\theta_0$

$$\frac{d^2\theta^*}{dt^{*2}} + \frac{\sin(\theta^*\theta_0)}{\theta_0} + \frac{6\pi\mu a\sqrt{gL}}{Mg}\frac{d\theta^*}{dt} = 0$$

two dimensionless groups are:

(a) $\Pi_1 = \frac{6\pi\mu a\sqrt{gL}}{Mg} = \boxed{\frac{6\pi\mu a}{M}\sqrt{\frac{L}{g}}} \rightarrow$ damping coefficient

$\Pi_2 = \boxed{\theta_0} \rightarrow$ initial angle

2. Continued

Physical meanings:

damping coefficient: describes the drag on the sphere. If $\Pi_1 \to 0$, the pendulum will oscillate infinitely. If $\Pi_1 \to$ large, the pendulum will settle very slowly.

At critical damping, $\Pi_1 = 1$. This corresponds to the pendulum just settling to $\Theta = 0$ with no oscillations, but without overdamping.

initial angle: self-explanatory.

(b) At critical damping:

$$\frac{6\pi\mu a}{M}\sqrt{\frac{L}{g}} = 1 \quad \Rightarrow \quad \boxed{\mu_{critical} = \frac{M}{6\pi a}\sqrt{\frac{g}{L}}}$$

To estimate the time to reach 25%, just use the time scaling.

$$\boxed{t_{25\%} = \sqrt{\frac{L}{g}}}$$

→

2. continued

(c) Make the approximation given; that

$\sin(\theta^* \theta_0) = \theta^* \theta_0$ when $\theta^* \theta_0$ is small.

This gives:

$$\frac{d^2\theta^*}{dt^{*2}} + \frac{6\mu\pi a}{M}\sqrt{\frac{L}{g}}\frac{d\theta^*}{dt^*} + \theta^* = 0$$

With B.C.:

$$\theta^*\Big|_{t^*=0} = 1 \qquad \frac{d\theta^*}{dt^*}\Big|_{t^*=0} = 0$$

the characteristic equation is of the form:

$$a\theta^{*''} + b\theta^{*'} + c\theta^* = 0$$

With eigenvalues:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Recall from differential equations that critical damping occurs when $\lambda_1 = \lambda_2$, or when $\sqrt{b^2-4ac} = 0$.

So:

$$\sqrt{\left(\frac{6\pi\mu a}{M}\sqrt{\frac{L}{g}}\right)^2 - 4} = 0, \quad \Rightarrow \boxed{\mu_{critical} = \frac{2M}{6\pi a}\sqrt{\frac{g}{L}}}$$

$\longrightarrow$

2. <u>Continued</u>

If we substitute this into the equation,

$$\theta^{*''} + 2\theta^{*'} + \theta^* = 0$$

and $\lambda = -1$.

<u>Solution:</u>

$$\theta^* = (c_1 + c_2 t^*) e^{\lambda t^*}$$

with boundary conditions:

$$\theta^*\Big|_{t^*=0} = 1 \quad \Rightarrow \quad c_1 = 1$$

$$\frac{d\theta^*}{dt^*}\Big|_{t^*=0} = 0 \quad \Rightarrow \quad \frac{d\theta^*}{dt^*}\Big|_{t^*=0} = \left\{C_1 \lambda e^{\lambda t} + C_2 e^{\lambda t} + C_2 \lambda t e^{\lambda t}\right\}\Big|_{t^*=0}$$

$$0 = \lambda + C_2 \quad \Rightarrow \quad C_2 = -\lambda$$

so: $$\theta^* = (1 + t^*) e^{-t^*}$$

Now find t^* for when $\theta^* = 0.25$

$$0.25 = (1 + t^*) e^{-t^*} \quad \Rightarrow \quad t^* = 2.69$$

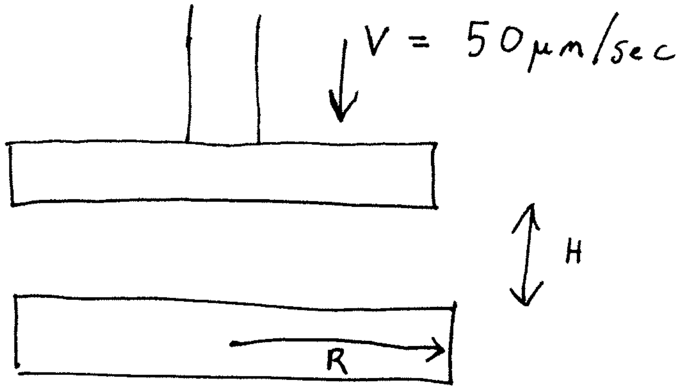
$$\Rightarrow \boxed{t_{25\%} = 2.69 \sqrt{\frac{L}{g}}}$$

Compare:

	dimensional Analysis	Exact
$t\big\vert_{25\%} =$	$\sqrt{\frac{L}{g}}$	$2.69 \sqrt{\frac{L}{g}}$
$\mu_{critical} =$	$\frac{M}{6\pi a}\sqrt{\frac{g}{L}}$	$\frac{2M}{6\pi a}\sqrt{\frac{g}{L}}$

Note: both $\mu_{critical}$ and $t_{25\%}$ were under-predicted by dim

3. Parallel - plate Viscometer



$F = 10^4$ dynes

$\mu_{air} = 1.8 \times 10^{-4}$ poise

$R = 3$ cm

(a) Find the error in the gap set zero.

- This is the height H when the force $F = 10^4$ dynes, for the given V and μ. (and R)

Start with the continuity equation in cylindrical coordinates.

$$\cancel{\frac{\partial \rho}{\partial t}}^{\,0} + \frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{\rho}\, r V_r\right) + \cancel{\frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho V_\theta\right)} + \frac{\partial}{\partial z}\left(\cancel{\rho} V_z\right) = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r V_r\right) + \frac{\partial}{\partial z}\left(V_z\right) = 0$$

<u>Non-dimensionalize:</u>

$$V_z^* = \frac{V_z}{V} \quad V_r^* = \frac{V_r}{U} \quad r^* = \frac{r}{R} \quad z^* = \frac{z}{H}$$

$$\frac{U}{R}\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* V_r^*\right) + \frac{V}{H}\frac{\partial V_z^*}{\partial z^*} = 0$$

divide through by $\frac{U}{R}$, find the dimensionless group $\frac{VR}{UH}$

So $U = \frac{R}{H} V$ $\longrightarrow$

3. continued

Since $\frac{R}{H}$ is small, $U << V$, and we can assume

quasi-parallel flow in r-direction.

Now look at the r-momentum equation:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} - \cancel{\frac{v_\theta^2}{r}} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} +$$

$$+ \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}} + \frac{\partial^2 v_r}{\partial z^2}\right] + \cancel{\rho g_r}$$

$$0 \qquad 0 \qquad 0 \qquad 0 \qquad 0$$

Use the non-dimensional substitutions: $t^* = \frac{V}{H}t$ $\quad p^* = \frac{p - p_0}{\Delta P_c}$

$$\rho\frac{U^2}{R}\left(\frac{\partial v_r^*}{\partial t^*} + v_r^*\frac{\partial v_r^*}{\partial r^*} + v_z^*\frac{\partial v_r^*}{\partial z^*}\right) = -\frac{\Delta P_c}{R}\frac{\partial p^*}{\partial r^*} +$$

$$+ \mu\frac{U}{H^2}\left[\frac{H^2}{R^2}\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^* v_r^*)\right) + \frac{\partial^2 v_r^*}{\partial z^{*2}}\right]$$

Since Re is low, use the viscous scaling. divide by $\mu\frac{U}{H^2}$:

$$\left(\frac{\rho U H}{\mu}\right)\left(\frac{H}{R}\right)\left(\frac{\partial v_r^*}{\partial t^*} + v_r^*\frac{\partial v_r^*}{\partial t^*} + v_z^*\frac{\partial v_r^*}{\partial z^*}\right) = -\left(\frac{\Delta P_c H^2}{R\mu U}\right)\frac{\partial p^*}{\partial r^*} +$$

$$+ \frac{H^2}{R^2}\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^* v_r^*)\right) + \frac{\partial^2 v_r^*}{\partial z^{*2}}$$

We can set $\frac{\Delta P_c H^2}{R\mu U} = 1$, so $\boxed{\Delta P_c = \frac{R\mu U}{H^2}}$

$\longrightarrow$

3. Continued

Since $H \ll R$, and Re is low, we can reduce the equation to:

$$\frac{\partial p^*}{\partial r^*} = \frac{\partial^2 V_r^*}{\partial z^{*2}}$$

Assuming pressure gradient is constant, we can integrate twice to get:

$$V_r^* = \frac{1}{2}\frac{\partial p^*}{\partial r^*} z^{*2} + C_1 z^* + C_2$$

Apply B.C.

$$V_r^*\Big|_{z^*=0} = 0 \Rightarrow C_2 = 0$$

$$V_r^*\Big|_{z^*=1} = 0 \Rightarrow C_1 = -\frac{1}{2}\frac{\partial p^*}{\partial r^*}$$

So $\underline{V_r^* = -\frac{1}{2}\frac{\partial p^*}{\partial r^*} z^*(1-z^*)}$

Now we can use a mass balance to find $\frac{\partial p^*}{\partial r^*}$

Mass in = mass out

$$V \pi r^{*2} R^2 = U H R 2\pi r^* \int_0^1 V_r^* \, dz^*$$

and since $U = \frac{R}{H} V$

$$\frac{1}{2} r^* = \int_0^1 V_r^* \, dz^*$$

$\longrightarrow$

3. Continued

$$\int_0^1 -\frac{1}{2}\frac{\partial p^*}{\partial r^*}\left(z^* - z^{*2}\right) dz^* = -\frac{1}{2}\frac{\partial p^*}{\partial r^*}\left(\frac{z^{*2}}{2}\Big|_0^1 - \frac{z^{*3}}{3}\Big|_0^1\right)$$

$$\frac{1}{2} r^* = -\frac{1}{2}\frac{\partial p^*}{\partial r^*}\frac{1}{6}$$

so: $$\frac{\partial p^*}{\partial r^*} = -6 r^*$$

Boundary condition: $$p^*\Big|_{r^*=1} = 0$$

$$p^* = 3\left(1 - r^{*2}\right)$$

From physics,

$$F = \int_0^R p\, 2\pi\, r\, dr = \Delta P_c R^2 2\pi \int_0^1 p^* r^* dr^*$$

Substitute in our expression for pressure:

$$F = 6\pi \Delta P_c R^3 \int_0^1 r^* - r^{*3} dr^* = \frac{6}{4}\pi \Delta P_c R^3 = \frac{3}{2}\pi \left[\frac{R\mu U}{H^2}\right] R^3$$

So $$\boxed{F = \frac{3\pi R^4 \mu V}{2H^3}}$$

3. continued

(a) what is the error in the gap set?

Solve for

$$\boxed{H = \left(\frac{3\pi R^4 \mu V}{2F}\right)^{1/3}} = \left(\frac{3\pi(3\text{ cm})^4\left(1.8\times10^{-4}\frac{g}{cm\cdot s}\right)\left(50\times10^{-4}\frac{cm}{sec}\right)}{2\left(10^4\ \frac{g\cdot cm}{s^2}\right)}\right)^{1/3}$$

$$\boxed{H = 0.00325\text{ cm}}$$

(b)

$$\boxed{H = 32.5\ \mu m} = \text{error}$$

in order to reduce this to 3.25 μm, V must be:

$$\boxed{V = \frac{F\,2\,H^3}{3\pi R^4 \mu} = 0.05\ \mu m/sec = 50\ nm/sec}$$

This is WAY to slow. Not practical.

(c) if $H_{actual} = 232.5\ \mu m$

$H = 200\ \mu m$, what is the error in viscosity?

from Problem Set 6,

$$T = \frac{-\pi\mu\Omega R^4}{2H_1}, \quad \text{so } \mu = H\left[\frac{2T}{\pi\Omega R^4}\right].$$

$$\boxed{\frac{\mu_{exp}}{\mu} = 1.16}$$

thus, $\frac{\mu_{expo}}{\mu} = \frac{232.5}{200} = 1.16$

4. Prove that $\nabla^2 p = 0$ for an incompressible fluid at $Re = 0$.

<u>N-S equation:</u>

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\underline{\nabla}\,\underline{u}\right) = -\underline{\nabla} P + \mu \nabla^2 \underline{u} + \rho \underline{g}$$

<u>Index notation:</u>

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} + \rho g_i$$

<u>Non-dimensionalize</u>

$$u_i^* = \frac{u_i}{V} \qquad x_i^* = \frac{x_i}{L} \qquad g_i^* = \frac{g_i}{g} \qquad P^* = \frac{P - P_0}{\left(\mu V/L\right)} \qquad t^* = \frac{t}{(L/V)}$$

Then:

$$\rho \frac{V^2}{L}\left[\frac{\partial u_i^*}{\partial t^*} + u_j^* \frac{\partial u_i^*}{\partial x_j^*}\right] = -\frac{\mu V}{L^2}\frac{\partial p^*}{\partial x_i^*} + \frac{\mu V}{L^2}\frac{\partial^2 u_i^*}{\partial x_j^{*2}} + g\rho g_i$$

We know $Re \to 0$, so we use the viscous scaling.

Divide through by $\mu \frac{V}{L^2}$:

$$\underbrace{\frac{\rho V L}{\mu}}_{Re}\left[\frac{\partial u_i^*}{\partial t^*} + u_j^* \frac{\partial u_i^*}{\partial x_j^*}\right] = -\frac{\partial p^*}{\partial x_i^*} + \frac{\partial^2 u_i^*}{\partial x_j^{*2}} + \underbrace{\left(\frac{\rho g L^2}{\mu V}\right)}_{\frac{Re}{Fr}} g_i^*$$

And with $Re \to 0$,

$$\frac{\partial^2 u_i^*}{\partial x_j^{*2}} = \frac{\partial p^*}{\partial x_i^*} \qquad \longrightarrow$$

4. Continued

Now take the divergence. The ODE is:

$$\frac{\partial^2 u_i^*}{\partial x_j^{*2}} = \frac{\partial p^*}{\partial x_i^*}$$

or $\nabla^{*2} \underline{u}^* = \underline{\nabla}^* p$ in Gauss.

Divergence:

$$\frac{\partial}{\partial x_i^*} \frac{\partial^2 u_i^*}{\partial x_j^{*2}} = \frac{\partial}{\partial x_i^*} \frac{\partial p^*}{\partial x_i^*}$$

or $\underline{\nabla}^* \cdot \nabla^{*2} \underline{u}^* = \underline{\nabla}^* \cdot \underline{\nabla}^* p$ in Gauss.

Note that $\underline{\nabla}^*$ and ∇^{*2} commute. So:

$$\frac{\partial^2}{\partial x_j^{*2}} \frac{\partial u_i^*}{\partial x_i^*} = \frac{\partial^2 p^*}{\partial x_i^{*2}}$$

or $\nabla^{*2}(\underline{\nabla}^* \cdot \underline{u}) = \nabla^{*2} p$ in Gauss.

We know from continuity that $\frac{\partial u_i^*}{\partial x_i^*} = 0$, or $\underline{\nabla} \cdot \underline{u} = 0$,

So: $\boxed{\frac{\partial^2 p^*}{\partial x_i^{*2}} = 0}$

or $\boxed{\nabla^2 p = 0}$