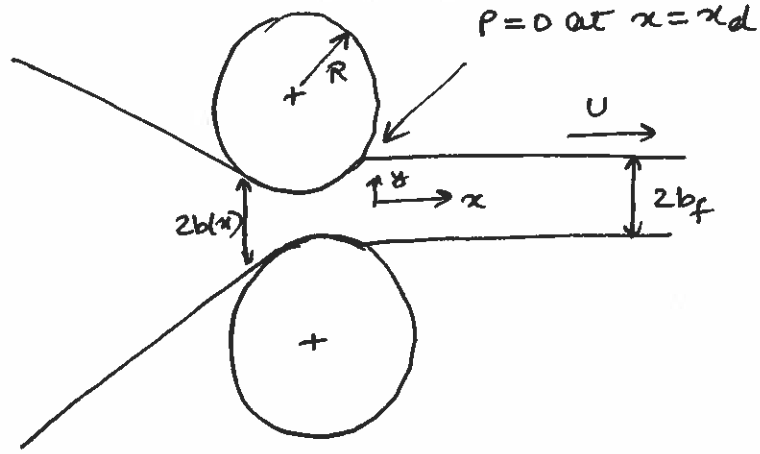


Within the gap, under lubrication limit:

$$-\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} = 0$$

$$\therefore \quad -\frac{\Delta P_c}{x_c}\frac{\partial p^*}{\partial x^*} + \frac{\mu U}{b_0^2}\frac{\partial^2 u^*}{\partial y^{*2}} = 0$$

ΔP_c – char. pressure

x_c – char. length scale.

$$\therefore \quad -\frac{\partial p^*}{\partial x^*} + \left(\frac{\mu U x_c}{b_0^2 \Delta P_c}\right)\frac{\partial^2 u^*}{\partial y^{*2}} = 0$$

$$\therefore \quad \Delta P_c = \frac{\mu U x_c}{b_0^2}$$

Given $b(x) = b_0 + \frac{1}{2}\frac{x^2}{R} \Rightarrow b^* = 1 + \frac{1}{2}\frac{x^{*2}\, x_c^2}{b_0 R}$

$\therefore$ for O(1) correction in b^*, $\boxed{x_c = \sqrt{b_0 R}}$

$$\therefore \quad \boxed{\Delta P_c = \frac{\mu U \sqrt{b_0 R}}{b_0^2}}$$

(a)

Force per unit width of rollers:

$$\frac{F}{W} = \int_{-\infty}^{x_d} p\, dx = \Delta P_c\, x_c \int_{-\infty}^{x_d^*} p^* dx^*$$

$$\frac{F}{W} = \frac{\mu U R}{b_0}\int_{-\infty}^{x_d^*} p^* dx^*$$

(b) Mass balance :

$$\int_{-b}^{b} u(y)\, w\, dy = 2U b_f W$$

$$\Rightarrow \quad \int_{-b}^{b} u\, dy = 2U b_f$$

Since $u(y)$ will be symmetric about $y=0$

$$\int_{0}^{b} u\, dy = U b_f$$

(c) $$\frac{d^2 u^*}{d y^{*2}} = \frac{d p^*}{d x^*}$$

B.C: $\left. u^* \right|_{y^* = \pm b^*} = 1$

$$\therefore \quad u^* = \frac{1}{2}\left(\frac{dp^*}{dx^*}\right)\left(y^{*2} - b^{*2}\right) + 1$$

and using $\int_{0}^{b^*} u^* dy^* = b_f^*$ (substitute u^* here to find $\frac{dp^*}{dx^*}$)

$$\Rightarrow \frac{dp^*}{dx^*} = \frac{-3\left(b_f^* - b^*\right)}{b^{*3}}$$

$$\Rightarrow \quad p^* = \int_{x_d}^{x} \frac{3\left(b_f^* - b^*\right)}{b^{*3}}\, dx^*$$

$$b^* = 1 + \frac{1}{2} x^{*2} \quad , \quad b_f^* = 1 + \frac{1}{2} x_{\cancel{f}}^{*\,2}$$

$$\frac{\text{Force}}{\text{width}} = \frac{F^*}{W} = \int_{-\infty}^{x_{\cancel{f}}^*} p^* \, dx^*$$

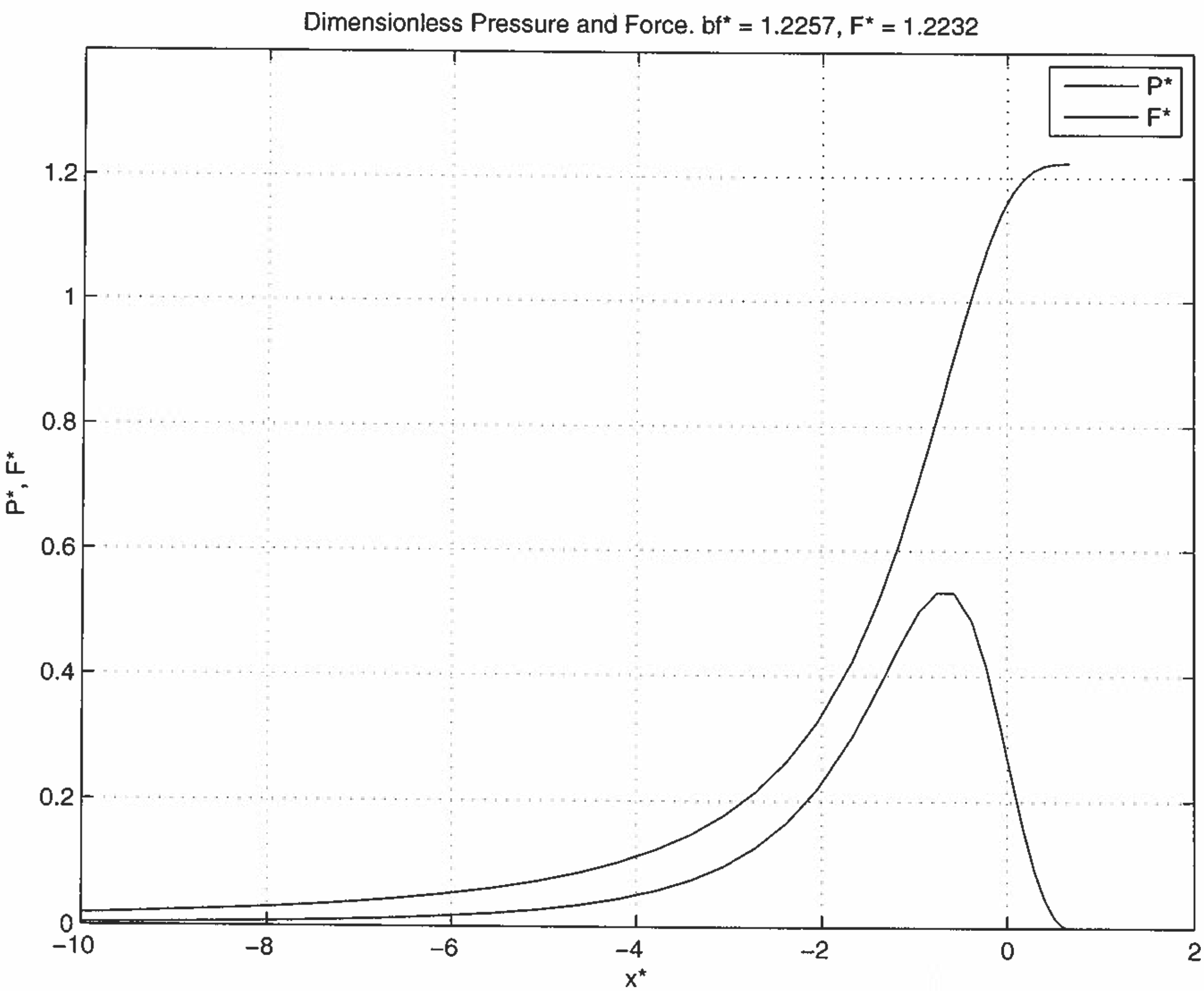
Now for numerical integration, find b_f^* s.t you get $p^* = 0$

Answer: at $b_f^* = 1.22$, $\left(\frac{F}{W}\right)^* = 1.22$

(4)

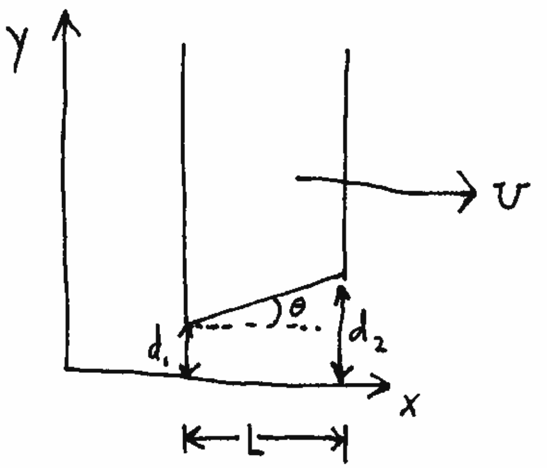

```
function out=pint(bf)
%This function calculates the pressure integral for some bf.
%It also calculates the force integral.
%To run type: bf = fzero('pint',1.5);

global bfpass
bfpass=bf; %passing bf into integrand

xf=(2*(bf-1)).^.5;

[xout,pout]=ode23('igrand',[-100,xf],[0;0]);

n=length(xout);
out=pout(n,1);

force=pout(n,2)

figure(1)
plot(xout,pout)
xlabel('x*')
ylabel('P*, F*')
legend('P*','F*')
title(['Dimensionless Pressure and Force. bf* = ',num2str(bf),', F* = ',num2str(force)])
grid on
axis([-10,2,0,1.4])
drawnow

function yout=igrand(x,y)
%this is the pressure integrand
global bfpass

yout=zeros(size(y));

b=1+0.5*x.^2;

yout(1)=3*(b-bfpass)./b.^3;
yout(2)=y(1);
```

(5)



Dimensionless Pressure and Force. bf* = 1.2257, F* = 1.2232
P*
F*
P*, F*
x*
1.2
1
0.8
0.6
0.4
0.2
0
−10
−8
−6
−4
−2
0
2

1) Sliding Block



$$F = \int_{\partial D} \underline{\underline{\sigma}} \cdot \underline{n} \, dA$$

$$F = \int_{\partial D} -P \underline{n} \, dA + \int_{\partial D} \underline{\underline{\tau}} \cdot \underline{n} \, dA$$

$$F = \int_{\partial D} -(P - P_0) \underline{n} \, dA + \int_{\partial D} \underline{\underline{\tau}} \cdot \underline{n} \, dA$$

$$f_x = \int_{\partial D} \underbrace{-(P - P_0) \, \hat{\underline{e}}_x \cdot \underline{n}}_{\text{zero, except inside the gap}} \, dA + \int_{\partial D} \hat{\underline{e}}_x \cdot \underline{\underline{\tau}} \cdot \underline{n} \, dA$$

$$f_x = \int_{A_b} -(P - P_0) \, \hat{\underline{e}}_x \cdot \underline{n} \, dA + \int_{A_b} \hat{\underline{e}}_x \cdot \underline{\underline{\tau}} \cdot \underline{n} \, dA$$

$\underline{n} = \sin\theta \, \hat{\underline{e}}_x - \cos\theta \, \hat{\underline{e}}_y$ = normal to the block.

$\tan\theta = \dfrac{d_2 - d_1}{L} = \dfrac{\Delta d}{L}$, and if θ is small $\theta \approx \dfrac{\Delta d}{L}$.

So:

$$\underline{n} = \frac{\Delta d}{L} \hat{\underline{e}}_x - \hat{\underline{e}}_y + O\left(\frac{H^2}{L^2}\right)$$

.) continued: 7 ②

Take the dot products:

$$f_x = \int_0^L -(p-p_0)\frac{\Delta d}{L} W dx + W\int_0^L \left(\tau_{xx}\frac{\Delta d}{L} - \tau_{yx}\right) dx$$

From Newton's Law of Viscosity:

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} \qquad \tau_{yx} = \mu\left(\cancel{\frac{\partial v}{\partial x}} + \frac{\partial u}{\partial y}\right) = \mu\frac{\partial u}{\partial y}$$

<u>Non-dimensionalize:</u>

$$f_x = \Delta P_c \frac{\Delta d}{H}\frac{H}{L} W L\int_0^1 -p^* dx^* + W L \mu \int_0^1 \Big\{ \underbrace{\frac{2u}{L}\frac{\Delta d}{L}\frac{\partial u^*}{\partial x^*}}_{o\left(\frac{H^2}{L^2}\right)} - \Big(\underbrace{\frac{U}{H}\frac{\partial u^*}{\partial y^*}}_{o\left(\frac{U}{H}\right)} + \underbrace{\frac{H}{L}\frac{U}{L}\frac{\partial v^*}{\partial x^*}}_{o\left(\frac{H^2}{L^2}\right)}\Big)\Big\} dx^*$$

<u>Cross out terms which are $o\left(\frac{H^2}{L^2}\right)$.</u>

$$f_x = \Delta P_c \frac{\Delta d}{H}\frac{H}{L} W L\int_0^1 - p^* dx^* - \frac{W L \mu U}{H}\int_0^1 \frac{\partial u^*}{\partial y^*} dx^*$$

$$\Delta P_c = \frac{U\mu L}{H^2}$$

$$f_x = -\frac{U\mu W L}{H}\left\{\int_0^1 \frac{\Delta d}{d_1} p^* dx^* + \int_0^1 \left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=h^*} dx^*\right\}$$

→

1. continued

$$\boxed{f_x = -\frac{U\mu WL}{H}\left\{\int_0^1 \left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=0} dx^*\right\}} \quad (b)$$

Now part (a):

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

x - Momentum:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial P}{\partial x}$$

small compared to

$$\frac{\partial^2 u^*}{\partial y^{*2}} = \frac{\partial p^*}{\partial y^*}$$

y - momentum

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{\mu}\frac{\partial P}{\partial y}$$

$$\frac{\partial p^*}{\partial y^*} = 0$$

B.C. $u^* = v^* = 0 \quad @ \; y^* = 0$

$$\left.\begin{array}{l} u^* = 1 \\ v^* = 0 \end{array}\right\} @ \; y^* = h^* = \frac{h}{H}$$

Now refer to course notes: p. 198

$$u^* = \frac{1}{2}\left(\frac{\partial p^*}{\partial x^*}\right) y^*(y^* - h^*) + \frac{y^*}{h^*}$$

$$\frac{\partial u^*}{\partial y^*} = \frac{1}{2}\left(\frac{\partial p^*}{\partial x^*}\right)(2y^* - h^*) + \frac{1}{h^*}$$

and at $y^* = h^*$:

$$\frac{\partial u^*}{\partial y^*} = \frac{1}{2}\frac{\partial p^*}{\partial x^*}h^* + \frac{1}{h^*}$$

$\longrightarrow$

... Continued

Also from the notes: p. 201

$$\frac{\partial p^*}{\partial x^*} = -6\,\frac{\Delta d}{d_1}\,\frac{x^*}{h^{*3}} + \frac{6}{h^{*3}}\left(\frac{\Delta d}{d_1 + d_2}\right)$$

$$\frac{\partial p^*}{\partial x^*} = \frac{6}{h^{*3}}\left[\frac{\Delta d/d_1}{1 + d_2/d_1} - x^*\left(\frac{\Delta d}{d_1}\right)\right]$$

$$\Rightarrow \frac{\partial u^*}{\partial y^*} = \frac{3}{h^{*3}}\left[\frac{\Delta d/d_1}{1 + d_2/d_1} - x^*\frac{\Delta d}{d_1}\right] + \frac{1}{h^*}$$

and, since $h^* = 1 + \dfrac{\Delta d}{d_1}$:

$$\frac{\partial u^*}{\partial y^*} = \frac{3}{\left(1 + \frac{\Delta d}{d_1}x^*\right)^2}\left[\frac{\Delta d/d_1}{1 + d_2/d_1} - x^*\frac{\Delta d}{d_1}\right] + \frac{1}{1 + \left(\frac{\Delta d}{d_1}\right)x^*}$$

$\downarrow$

Combine this with (b) to get the drag $f_x = g\left(\frac{\Delta d}{d_1}\right)$ [parameters]

a)
$$\boxed{f_x = \frac{U\mu WL}{H}\int_0^1 \left\{\frac{3}{\left(1 + \frac{\Delta d}{d_1}x^*\right)^3}\left[\frac{\Delta d/d_1}{1 + d_2/d_1} - x^*\frac{\Delta d}{d_1}\right] + \frac{1}{1 + \left(\frac{\Delta d}{d_1}\right)x^*}\right\} dx^*}$$

where $\Delta d = d_2 - d_1$

$\longrightarrow$

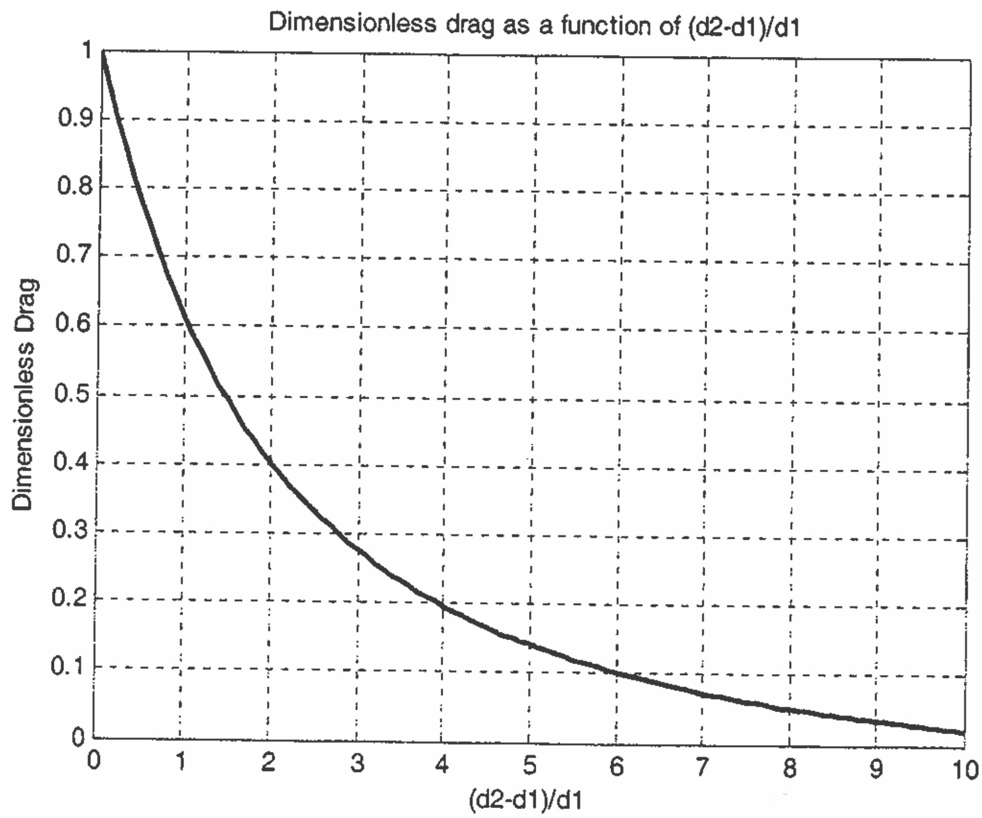
∴ continued

$$\text{dimensionless drag} = \frac{f_x}{\left(\frac{U\mu W L}{H}\right)} = f_x^*$$

$$f_x^* = \int_0^1 \left\{ \frac{3}{\left(1+\frac{\Delta d}{d_1}x^*\right)^3}\left[\frac{\Delta d/d_1}{1+d_2/d_1} - x^*\frac{\Delta d}{d_1}\right] + \frac{1}{1+\left(\frac{\Delta d}{d_1}\right)x^*}\right\} dx^*$$

(c)

Answer:



```
% CBE 30355 Homework 8, Problem 1(c)

% Using the solution for the pressure given in class,
% numerically integrate the expression in (b) to get
% the dinemsionless drag, and plot it up as a function
% of (d2-d1)/d1.  It should go to 1 for (d2-d1)/d1 = 0.

global D

% Define a vector Dmat, which is the values of (d2-d1)/d1.

Dmat = 0:0.1:10

% Use QUADL and the external function taoxy to integrate
% the expression for the dimensionless drag at each value
% of (d2-d1)/d1 in Dmat.

for i = 1:length(Dmat)
    D = Dmat(i);
    Fdrag(i) = quadl('taoxy',0,1);
end

% Plot the dimensionless drag vs. (d2-d1)/d1

plot(Dmat,Fdrag,'-r','LineWidth',2)
title('Dimensionless drag as a function of (d2-d1)/d1');
xlabel('(d2-d1)/d1');
ylabel('Dimensionless Drag');
grid on;
```

```
function y=taoxy(x)
global D
h = 1+D.*x;
dpdx = (-6.*D.*x + 6*(1+2.*D.^(-1)).^(-1))./h.^3;
y = 0.5.*dpdx.*h+(1./h);
```

1.) For a sliding block with the following:

$$\left.\begin{aligned} L &= 10\,cm \\ W &= 30\,cm \\ d_2 - d_1 &= 0.1\,cm \\ m &= 100\,kg \\ d_1 &= 0.01\,cm \\ \mu &= 0.5\,\text{poise} \end{aligned}\right\} \quad \text{determine the velocity } U$$

From the notes (p. 202)

$$F^* = \frac{F}{\left(\frac{\mu U L^2 W}{H^2}\right)} = 6\left(\frac{d_1}{\Delta d}\right)\left[\frac{d_1}{\Delta d}\ln\left(1+\frac{\Delta d}{d_1}\right) - \frac{1}{1+\frac{1}{2}\frac{\Delta d}{d_1}}\right]$$

$$= 0.044$$

<u>Solve for U.</u>

$$U = \frac{F}{\left(\frac{0.044\,\mu L^2 W}{H^2}\right)} = \frac{m g d_1^2}{0.044\,\mu L^2 W}$$

$$= \frac{(100\,kg)\left(\frac{1000\,g}{kg}\right)\left(980\,\frac{cm}{s^2}\right)0.01^2\,cm^2}{(0.044)\left(0.5\,\frac{g}{cm\cdot s}\right)10^2\,cm^2\;30\,cm}$$

$$\boxed{U = 148\,\frac{cm}{sec}}$$

4. Dimensional Analysis:

12

1 : 20 scale model of tank car.

$\mu_1 = 30$ poise

$\rho_1 = 1.4$ ~~poise~~ $\frac{g}{cm^3}$

(a) What should the fluid properties be in the scale model System?

– We need dynamic similarity.

$\underline{Re_1 = Re_2}$, $\nu_1 = \dfrac{30 \frac{g}{cm \cdot s}}{1.4 \frac{g}{cm^3}} = 21.4 \frac{cm^2}{sec}$

$$\frac{d_1 V_1}{\nu_1} = \frac{d_2 V_2}{\nu_2} \Rightarrow \nu_2 = \frac{d_2}{d_1} \frac{V_2}{V_1} \nu_1 \qquad (i)$$

$$Fr_1 = Fr_2$$

$$\frac{V_1^2}{d_1 \cancel{g}} = \frac{V_2^2}{d_2 \cancel{g}} \Rightarrow \frac{V_1}{V_2} = \left(\frac{d_1}{d_2}\right)^{1/2} \qquad (ii)$$

Combine eq. (i) and (ii):

$$\nu_2 = \nu_1 \left(\frac{d_2}{d_1}\right)^{3/2} = 21.4 \frac{cm^2}{sec} \left(\frac{1}{20}\right)^{3/2}$$

$$\boxed{\nu_2 = 0.24 \frac{cm^2}{sec}}$$

the model system should have a kinematic viscosity of 0.24 stokes.

$\longrightarrow$

t: continued

(b) How does the draw off rate scale?

$$Re_1 = Re_2 \quad \Rightarrow \quad \frac{V_1}{V_2} = \frac{\nu_1}{\nu_2}\frac{d_2}{d_1} \qquad (3)$$

$$Fr_1 = Fr_2 \quad \Rightarrow \quad \frac{V_1}{V_2} = \left(\frac{d_1}{d_2}\right)^{1/2} \qquad (4)$$

combine equations (3) and (4)

$$\frac{V_1}{V_2} = \left(\frac{\nu_1}{\nu_2}\right)^{1/3} = \left(\frac{\cancel{21.4}}{\cancel{21.4}\left(\frac{d_2}{d_1}\right)^{3/2}}\right)^{1/3}$$

So $$\frac{V_1}{V_2} = \left(\frac{d_1}{d_2}\right)^{1/2}$$

Recall: $Q = UA$

So the ratio of flow rates $$\frac{Q_2}{Q_1} = \frac{V_2 A_2}{V_1 A_1}$$

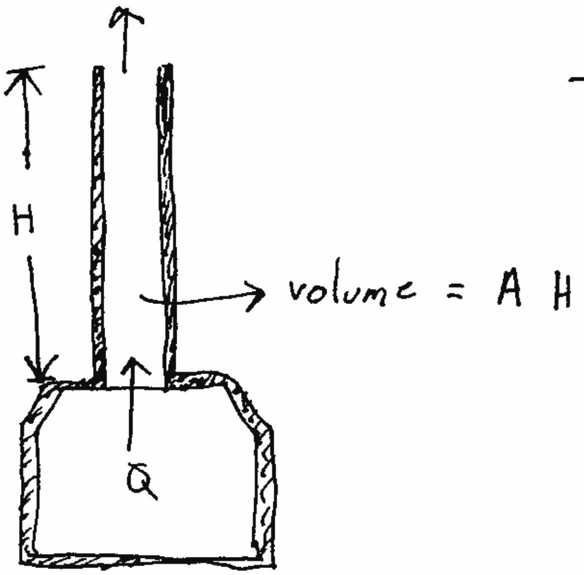
$$\frac{Q_2}{Q_1} = \left(\frac{d_2}{d_1}\right)^{1/2} \frac{d_2^2}{d_1^2}$$

$$\boxed{\frac{Q_2}{Q_1} = \left(\frac{d_2}{d_1}\right)^{5/2} = 5.59 \times 10^{-4}}$$

Scale model

So the draw off rate will be 5.59×10^{-4} times the full scale rate.

i. Thermodynamics and Scaling



– Temperature and Velocity estimation in a chimney

- Assume T and U are constant
- Neglect all frictional and heat losses.

Energy balance

Change in Enthalpy = heat input

$$\dot{m} C_p \Delta T = Q$$

$$U A \rho C_p \Delta T = Q \qquad (1)$$

Momentum Balance

$$\int_{\partial A} \rho \underline{u} (\underline{u} \cdot \underline{n}) \, dA = (\Delta \rho) g A H$$

Change in density due to temperature difference ΔT

$$\rho U^2 \cancel{A} = \Delta \rho g \cancel{A} H$$

$$\rho U^2 = \Delta \rho g H \qquad (2)$$

i. continued

We need $\Delta\rho$ as a function of ΔT

Use Ideal gas law:

$$PV = RT \quad \text{or, equivalently:} \quad \frac{P}{\rho} = RT$$

So: $$\rho = \frac{P}{RT} \quad \text{and} \quad \frac{d\rho}{dT} = -\frac{\overset{\text{Constant } P}{P}}{R}\left(\frac{1}{T^2}\right) \approx \frac{\Delta\rho}{\Delta T}$$

$$\Rightarrow \Delta\rho = \frac{-P}{R}\frac{\Delta T}{T^2}, \quad \text{and} \quad \frac{P}{R} = \frac{T}{V} = T$$

$$\text{So} \quad \Delta\rho = -\rho\frac{\Delta T}{T} \qquad (3)$$

From (1),

$$(4) \qquad \Delta T = \frac{Q}{U A \rho C_p}, \quad \text{plug into (3)} \Rightarrow$$

$$\Delta\rho = \frac{-\cancel{\rho} Q}{\cancel{\rho} C_p U A} = \frac{-Q}{C_p U A}$$

Now in eq. (2):

$$\rho U^2 = \Delta\rho g H = \frac{-Q g H}{C_p U A}$$

So

$$\text{(a)} \qquad \boxed{U = \left(\frac{-Q g H}{\rho C_p A T}\right)^{1/3}}$$

$\longrightarrow$

2. Continued

b)

From (4),

$$\Delta T = \frac{Q}{v A \rho C_p} = \frac{Q}{\rho C_p A}\left(\frac{-Q g H}{\rho C_p A T}\right)^{-1/3}$$

And, since $\Delta T = T_{chimney} - T$,

$$T_{chimney} = \left[\frac{Q^{\cancel{3}\,2}}{\rho^{\cancel{3}\,2} C_p^{\cancel{3}\,2} A^{\cancel{3}\,2}} \, \frac{\rho C_p A T}{Q g H}\right]^{1/3} + T$$

So:

$$\boxed{T_{chimney} = \left[\left(\frac{Q}{\rho C_p A}\right)^2 \frac{T}{g H}\right]^{1/3} + T}$$