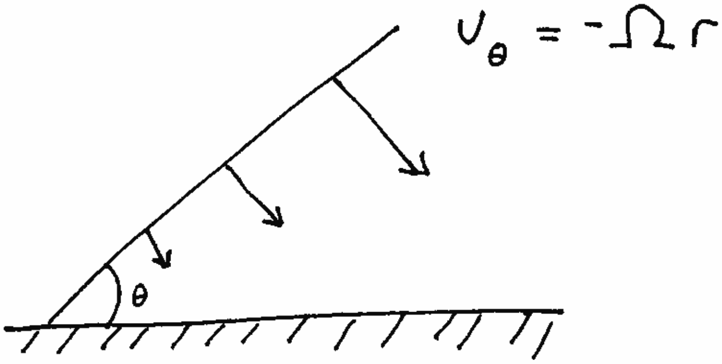
1. Flow inside a collapsing 45° wedge



$\bar{U} = 0$ @ $\theta = 0$ (no slip)

$$\left.\begin{aligned} U_r &= 0 \\ U_\theta &= -\Omega r \end{aligned}\right\} @ \theta = \frac{\pi}{4}$$

(a) $\psi = 0.5\, \Omega\, r^2 f(\theta)$ ← (from notes)

Biharmonic Equation (low Re)

$$\nabla^4 \psi = 0$$

Continuity

$$\frac{1}{r}\frac{\partial}{\partial r}(r U_r) + \frac{1}{r}\frac{\partial}{\partial \theta} U_\theta = 0$$

And:

$$U_\theta = \frac{-\partial \psi}{\partial r} \qquad U_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

B.C.

$$U_r\Big|_{\theta=0} = U_\theta\Big|_{\theta=0} = 0$$

$$U_r\Big|_{\theta=\theta_0} = 0 \qquad U_\theta\Big|_{\theta=\theta_0} = -\Omega r$$

(equivalently)

$$\left.\frac{\partial \psi}{\partial \theta}\right|_{\theta=0} = \left.\frac{\partial \psi}{\partial r}\right|_{\theta=0} = 0$$

$$\left.\frac{\partial \psi}{\partial \theta}\right|_{\theta=\theta_0} = 0 \qquad \left.\frac{\partial \psi}{\partial r}\right|_{\theta=\theta_0} = -(-\Omega r) = \Omega r$$

Since $\left.\frac{\partial \psi}{\partial r}\right|_{\theta=\theta_0} = \Omega r$, $\int \partial\psi = \int \Omega r\, \partial r$

$$\Rightarrow \underline{\psi = \frac{1}{2}\Omega r^2 f(\theta)}$$

→

1. Continued

$$\frac{\partial \psi}{\partial \theta} = \frac{1}{2}\Omega r^2 f'(\theta)$$

$$\frac{\partial \psi}{\partial r} = \Omega r f(\theta)$$

and at $\theta = 0$,

$$0 = \Omega r f(0) \Rightarrow f(0) = 0$$

$$0 = \frac{1}{2}\Omega r^2 f'(0) \Rightarrow f'(0) = 0$$

and at $\theta = \theta_0$

$$\frac{\partial \psi}{\partial r} = \Omega r = \Omega r f(\theta_0) \Rightarrow f(\theta_0) = 1 \quad , \ \theta_0 = \frac{\pi}{4}$$

$$\frac{\partial \psi}{\partial \theta} = 0 = \frac{1}{2}\Omega r^2 f'(\theta_0) \Rightarrow f'(\theta_0) = 0$$

Now:

$$\nabla^2 \psi = \nabla^2\left(\frac{1}{2}\Omega r^2 f(\theta)\right)$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \Omega f\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left(\frac{1}{2}\Omega r^2 f\right)$$

$$= 2\Omega f(\theta) + \frac{1}{2}\Omega f''(\theta)$$

$$\nabla^4 \psi = \nabla^2\left(2\Omega f(\theta) + \frac{1}{2}\Omega f''(\theta)\right)$$

$$= \underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(\quad\right)}_{0} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2}\left(2\Omega f(\theta) + \frac{1}{2}\Omega f''(\theta)\right)$$

$$= \frac{1}{r^2}\left[2\Omega f''(\theta) + \frac{1}{2}\Omega f^{IV}(\theta)\right] = 0$$

$\longrightarrow$

1. continued

So we have the 4th order ODE:

$$f^{IV} + 4f'' = 0$$

With four initial conditions: $f(0)=0$, $f'(0)=0$, $f\left(\frac{\pi}{4}\right)=1$, $f'\left(\frac{\pi}{4}\right)=0$

Find the roots: (i), (ii), (iii), (iv)

$$r^4 + 4r^2 = 0 \implies r_1 = r_2 = 0, \quad r_3 = 2i \quad r_4 = -2i$$

We are given the four homogeneous solutions:

$$f(\theta) = A + B\theta + C\sin(2\theta) + D\cos(2\theta)$$

$$f'(\theta) = B + 2C\cos(2\theta) - 2D\sin(2\theta)$$

Apply initial conditions to get four equations:

$$0 = A + D \quad \text{(i)}$$

$$0 = B + 2C \quad \text{(ii)}$$

$$0 = B - 2D \quad \text{(iii)}$$

$$1 = A + \frac{\pi}{4}B + C \quad \text{(iv)}$$

Solve for:

$$A = \frac{2}{4-\pi}$$

$$B = \frac{-4}{4-\pi}$$

$$C = \frac{2}{4-\pi}$$

$$D = \frac{-2}{4-\pi}$$

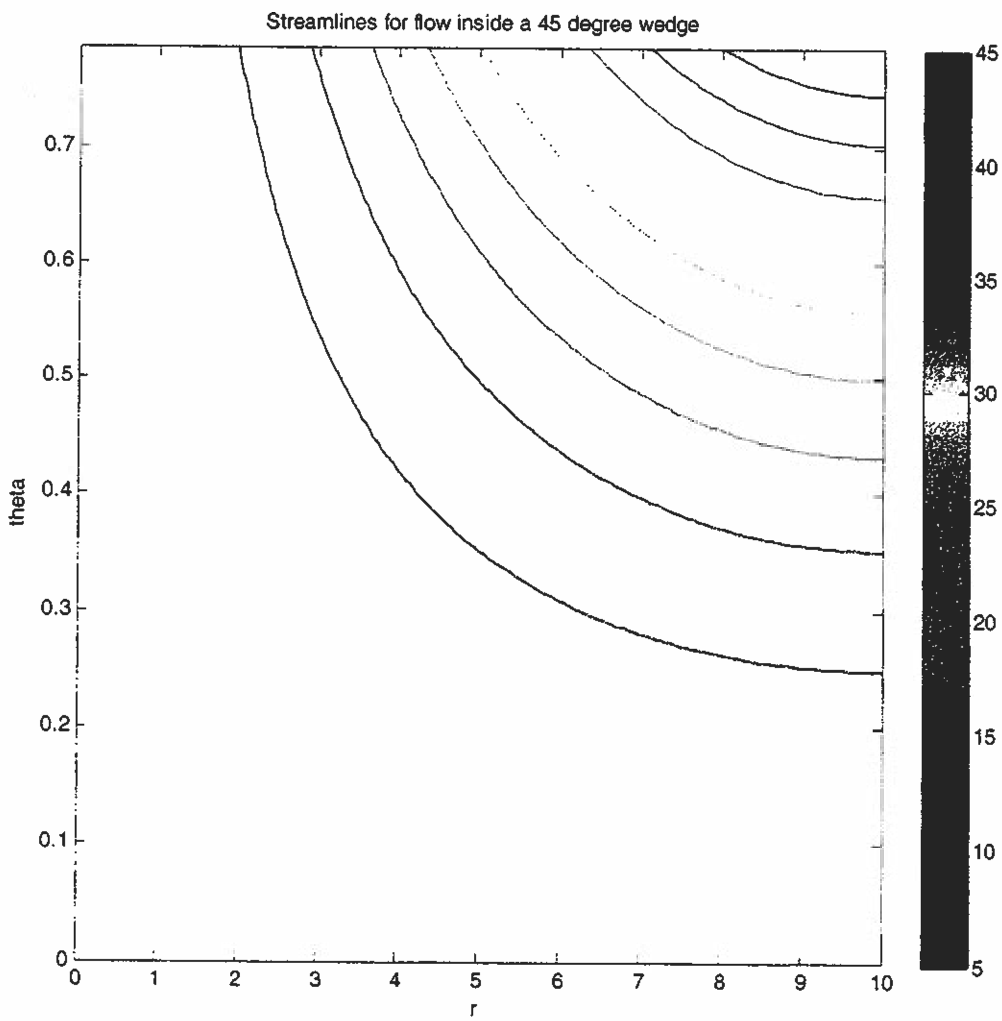
$$f(\theta) = \left(\frac{2}{4-\pi}\right)\left(1 - 2\theta + \sin 2\theta - \cos 2\theta\right)$$

Plug into the expression for ψ:

$$\boxed{\psi = \frac{\Omega r^2}{4-\pi}\left[1 - 2\theta + \sin(2\theta) - \cos(2\theta)\right]}$$

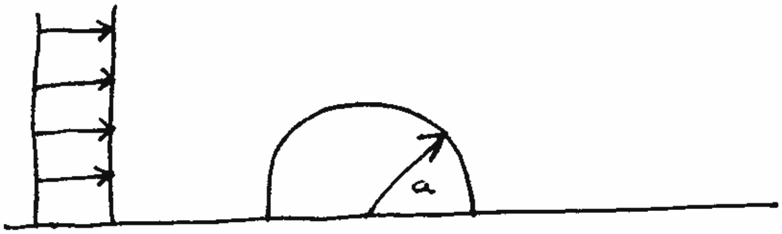
④②

```
% CBE 30355 Homework 8 Problem 3(b)

r = 0:0.1:10;
theta = 0:(pi/400):(pi/4);
y = zeros(100,100);

% assume omega = 1

for i = 1:101
    for j = 1:101
        y(i,j) = (r(i)^2/(4-pi))*(1-2*theta(j)+sin(2*theta(j))-
cos(2*theta(j)));
    end
end

contour(r,theta,y);
xlabel('r');
ylabel('theta');
title('Streamlines for flow inside a 45 degree wedge');
```



CBE 30355 Problem Set 9 Solutions

2b) Force on a cylinder imbedded in a plane



For irrotational, incompressible, inviscid flow:

$$\underline{\nabla} \times \underline{u} = 0 \quad \Rightarrow \quad \underline{u} = -\underline{\nabla}\phi \qquad \text{and } \underline{\nabla}\cdot\underline{u} = 0$$

(continuity)

So:

$$\underline{\nabla}\cdot\underline{u} = -\nabla^2\phi = 0$$

Now solve for ϕ. In cylindrical coordinates:

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

(semi-infinite in the z-direction)

Boundary Conditions

$$U_\theta\Big|_{r\to\infty} \equiv -\frac{1}{r}\frac{\partial\phi}{\partial\theta}\Big|_{r\to\infty} = U\sin\theta$$

$$U_r\Big|_{r\to\infty} \equiv -\frac{\partial\phi}{\partial r}\Big|_{r\to\infty} = -U\cos\theta$$

$$\underline{u}\cdot\underline{n}\Big|_{\partial D} = 0 \quad \Rightarrow \quad -\underline{\nabla}\phi\cdot\underline{n}\Big|_{r=a} = 0 \quad \therefore \quad U_r\Big|_{r=a} = 0$$

→

2) continued

We can make a guess at the form of ϕ in order to solve Laplace's equation:

$$\phi = f(r) \cos\theta$$

- This guess is based on the first two boundary conditions.

Now reevaluate the B.C.'s:

$$-\frac{1}{r}\frac{\partial \phi}{\partial \theta}\bigg|_{r\to\infty} = \frac{f(r)}{r}\sin\theta\bigg|_{r\to\infty} = U\sin\theta\bigg|_{r\to\infty}$$

$$\text{so } \frac{f(r)}{r}\bigg|_{r\to\infty} = U$$

$$-\frac{\partial \phi}{\partial r}\bigg|_{r\to\infty} = -U\cos\theta = -f'(r)\cos\theta$$

$$\text{so } f'(r)\bigg|_{r\to\infty} = U$$

Now back to Laplace's equation:

$$f''(r)\cancel{\cos\theta} + \frac{f'(r)}{r}\cancel{\cos\theta} - \frac{f(r)}{r^2}\cancel{\cos\theta} = 0$$

$$\text{So: } f''(r) + \frac{f'(r)}{r} + \frac{-f(r)}{r^2} = 0 \longrightarrow$$

From this equation, we can see that $f = r^n$ in order for the dimensions to remain consistent.

Then:

$$(n)(n-1)\cancel{(r^{n-2})} + (n)\cancel{(r^{n-2})} - \cancel{r^{n-2}} = 0$$

$$(n)(n-1) + n - 1 = 0$$

$$n^2 - 1 = 0$$

$$\text{So } n = \pm 1$$

$$f(r) = \frac{C_1}{r} + C_2 r$$

We know that as $r \to \infty$, $f(r) = Ur$, $\Rightarrow C_2 = U$

and $f'\big|_{r=a} = 0$, so $-\frac{C_1}{r^2} + U\Big|_{r=a} = 0 \Rightarrow C_1 = Ua^2$

$$f(r) = \frac{Ua^2}{r} + Ur$$

and $\phi = U\left(\frac{a^2}{r} + r\right)\cos\theta$

What we are asked to find is the force, and for that we need the pressure at the surface.

$\longrightarrow$

.. continued

At the surface, $U_r = 0$, so $U\big|_{r=a} = U_\theta\big|_{r=a}$.

$$U_\theta\Big|_{r=a} = -\frac{1}{r}\frac{\partial \phi}{\partial \theta}\Big|_{r=a} = \frac{U}{r}\left(\frac{a^2}{r} + r\right)\sin\theta\Big|_{r=a}$$

Bernoulli tells us that:

$$\frac{1}{2}\rho\left(U_\theta\Big|_{r=a}\right)^2 + P_{surface} = \frac{1}{2}\rho U^2 + P_0$$

$$\frac{1}{2}\rho\, 4U^2\sin^2\theta + P_{surface} = \frac{1}{2}\rho U^2 + P_0$$

So $$P_{surface} = P_0 + \frac{1}{2}\rho U^2\left(1 - 4\sin^2\theta\right)$$

Now from physics:

$$\underline{F} = -\int_{\partial D} P\,\underline{n}\,dA$$

$$F_x = L\int_0^\pi (-P)\left(\underline{n}\cdot\underline{\hat{e}}_x\right) a\,d\theta$$

$$= -L\int_0^\pi \tfrac{1}{2}\rho U^2\left(1 - 4\sin^2\theta\right)\cos\theta\,d\theta = 0$$

[zero because $\int_0^\pi A\cos\theta\,d\theta = 0$]

$$F_y = -L\int_0^\pi \frac{1}{2}\rho U^2\left(1-4\sin^2\theta\right)\sin\theta\,d\theta$$

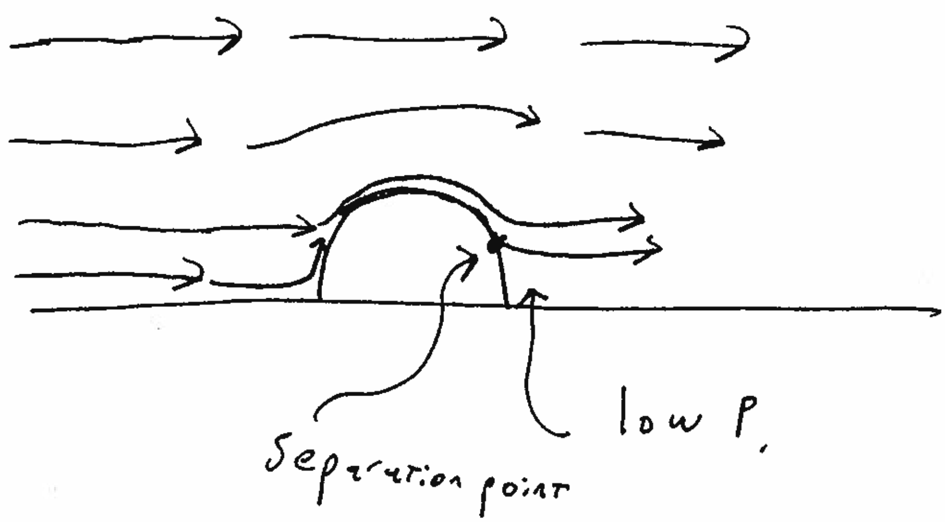
$$F_y = \frac{8}{3}\rho U^2 L$$

→

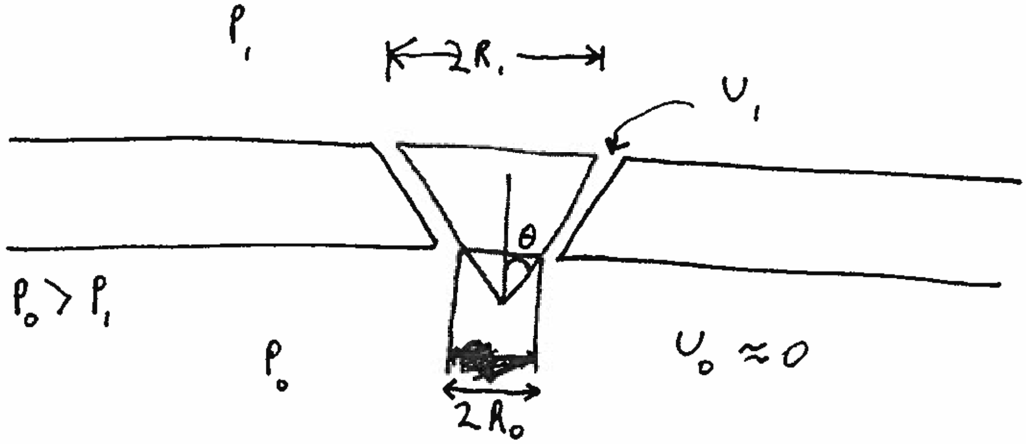
$$\boxed{F_x = 0 \qquad F_y = \frac{8}{3} \rho U^2 L}$$

Physically realizable?

It is impossible to have zero force in the x-dir because the stream doesn't follow the cylider all the way around, but separates somewhere along the back edge. This is called boundary layer separation, and creates a low-pressure region downstream.



3.) Conical cork in a conical hole.



Start with Bernoulli:

$$P_0 = \frac{1}{2}\rho U_1^2 + P_1$$

- U_1 is the velocity at the top of the gap.

(1) $$U_1 = \sqrt{\frac{2}{\rho}(P_0 - P_1)}$$

- Assume that below the cork the velocity $U_0 \approx 0$.

Using continuity:

$$A_1 U_1 = A U$$

- A is the area of the flow between cork & wall

$$A = \pi (R+g)^2 - \pi R^2$$

- A_1 is the area at the top of the cork
- g = gap width

Now apply Bernoulli's equation again:

$$\frac{1}{2}\rho U_1^2 + P_1 = \frac{1}{2}\rho U^2 + P$$

2) $$P = P_1 + \frac{\rho}{2} U_1^2 \left(1 - \left(\frac{U}{U_1}\right)^2\right)$$

$$\frac{U}{U_1} = \frac{A_1}{A} = \frac{\cancel{\pi R_1^2} + 2\pi R_1 g + g^2 - \cancel{\pi R_1^2}}{\cancel{\pi R^2} + 2\pi R g + g^2 - \cancel{\pi R^2}} = \frac{2R_1 + g}{2R + g} \rightarrow$$

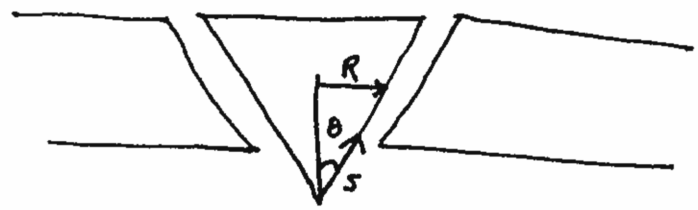
2. continued

make the assumption that $g << R$, so

$$\frac{U}{U_1} = \frac{R_1}{R}$$

Now define the coordinate system, with s = length of the cone side.

$$s = \frac{R}{\sin\theta}$$



So:

(3) $$\frac{U}{U_1} = \frac{R_1}{R} = \frac{s_1}{s}$$

Now we can plug this back into eq. (2), together with eq. (1) get an expression for the pressure in terms of s.

$$P = P_1 + \frac{\rho}{2}\left(\sqrt{\frac{2}{\rho}(P_0 - P_1)}\right)^2 \left(1 - \left(\frac{s_1}{s}\right)^2\right)$$

$$P = P_1 + (P_0 - P_1)\left(1 - \left(\frac{s_1}{s}\right)^2\right)$$

To find the force on the cork, just integrate over the sides of the cork:

$$F_{sides} = \int_{s_0}^{s_1} P \sin\theta \, ds \; 2\pi R$$

- The $\sin\theta$ arises from $\underline{n} \cdot \hat{e}_y$, to get the force in the y-direction only. forces on the sides will cancel out.

→

$$R = (S)\sin\theta \quad , \quad \text{so:}$$

$$F_{sides} = \int_{S_0}^{S_1} P\, 2\pi \sin^2\theta \, s\, ds$$

$$= \int_{S_0}^{S_1} \left(P_1 + (P_0 - P_1) - (P_0 - P_1)\left(\frac{S_1}{S}\right)^2 \right) 2\pi \sin^2\theta \; s\, ds$$

$$= 2\pi \sin^2\theta \left[\int_{S_0}^{S_1} P_1\, s\, ds + \int_{S_0}^{S_1} (P_0 - P_1)\, s\, ds - \int_{S_0}^{S_1} (P_0 - P_1)\left(\frac{S_1}{S}\right)^2 s\, ds \right]$$

$$= 2\pi \sin^2\theta \left[\frac{1}{2} P_1\, s^2 \Big|_{S_0}^{S_1} + \frac{1}{2}(P_0 - P_1)\, s^2 \Big|_{S_0}^{S_1} - (P_0 - P_1)\, S_1^2 \ln S \Big|_{S_0}^{S_1} \right]$$

$$= 2\pi \cancel{\sin^2}\theta \left[\frac{1}{2} P_1 \left(\frac{R_1^2 - R_0^2}{\cancel{\sin^2}\theta} \right) + \frac{1}{2}(P_0 - P_1)\left(\frac{R_1^2 - R_0^2}{\cancel{\sin^2}\theta} \right) - (P_0 - P_1)\frac{R_1^2}{\cancel{\sin^2}\theta} \ln\left(\frac{R_1}{R_0}\right) \right]$$

$$= \cancel{\pi P_1 (R_1^2 - R_0^2)} + \pi(P_0 - \cancel{P_1})(R_1^2 - R_0^2) - 2\pi (P_0 - P_1) \ln\left(\frac{R_1}{R_0}\right) R_1^2$$

$$= \pi P_0 (R_1^2 - R_0^2) - (P_0 - P_1) R_1^2\, 2\pi \ln\left(\frac{R_1}{R_0}\right)$$

$$F_{Total} = \pi P_0 \left(R_1^2 - \cancel{R_0^2}\right) + \cancel{\pi P_0 R_0^2} - \pi P_1 R_1^2 - (P_0 - P_1) R_1^2\, 2\pi \ln\left(\frac{R_1}{R_0}\right)$$

Set this equal to zero:

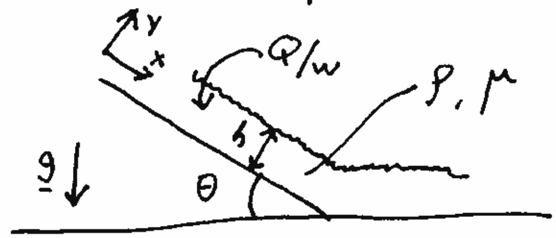
$$0 = \cancel{\pi R_1^2 (P_0 - P_1)} - \cancel{(P_0 - P_1)} R_1^2\, 2\cancel{\pi} \ln\left(\frac{R_1}{R_0}\right)$$

$$1 = 2 \ln\left(\frac{R_1}{R_0}\right)$$

$$\boxed{\frac{R_1}{R_0} = e^{1/2} = 1.65}$$

Answer: The cork will be ejected if $R_1 \leq R_0(1.65)$

1. Dimensional Analysis of Flow down an Inclined Plane



(a.) Form a dimensional matrix and prove that the problem involves just three dimensionless groups.

We wish to find the height $\{h\}$ as a function of $\begin{Bmatrix} Q/w \\ \nu \\ g \\ \theta \end{Bmatrix}$

$$h = f\left(\frac{Q}{w}, \nu, g, \theta\right)$$

	h	Q/w	ν	g	θ
M	0	0	0	0	0
L	1	2	2	1	0
T	0	-1	-1	-2	0

The top row contains all zeros, so rank = 2.

Buckingham Π theorem:

\# groups = \# parameters − \# rank of dimensional matrix

= 5 − 2 = 3 dimensionless groups.

$\Pi_1 = \theta$ by inspection

$\Pi_2 = Re = \dfrac{Q/w}{\nu}$ from problem statement, and from fluid mechanical intuition

Π_3:

We need to bring gravity into the picture. Solve the system:

$\underline{\underline{A}}\,\underline{x} = \underline{b}$

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \quad \underline{x} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \longrightarrow$$

This will give a dimensionless group of the form:

$$\pi_3 = h \left(Q/w\right)^{-a} g^{-b} . \qquad \underline{x} = \begin{bmatrix} 2/3 \\ -1/3 \end{bmatrix}$$

Then:

$$\pi_3 = \frac{h\, g^{1/3}}{\left(Q/w\right)^{2/3}} = f\left(\pi_1 , \pi_2\right)$$

So:

$$\boxed{\frac{h\, g^{1/3}}{\left(Q/w\right)^{2/3}} = f\left(\frac{Q}{w\nu}, \theta\right)}$$

(b) At low Re, we anticipate that $Q \propto g$. Use this to strengthen the result from (a), and determine the relationship between Q and h to within some unknown function of θ.

$$Q \propto g , \quad \text{or} \quad \frac{Q}{w} = \alpha g .$$

Substitute into the expression from (a), and ignore Re because it's small.

$$\frac{h \left(Q/w\right)^{1/3} \alpha'}{\left(Q/w\right)^{2/3}} = f(\theta)$$

or $$\boxed{h = \left(\frac{Q}{w}\right)^{1/3} \alpha'' f(\theta)}$$

Thickness is proportional to $\left(\frac{Q}{w}\right)^{1/3}$ power.

$\therefore$ Solve the momentum equations.

Continuity: $\underline{\nabla} \cdot \underline{v} = 0$

<u>X - momentum:</u>

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} +$$

$$+ \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x$$

<u>Y - Momentum:</u>

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial P}{\partial y} +$$

$$\mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y$$

<u>Assumptions:</u>

- low Re
- $v_y = 0$
- Semi-infinite in z-dir
- $\frac{\partial v_x}{\partial x} = 0$ (continuity)

<u>Reducing the momentum equations to:</u>

$$0 = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 v_x}{\partial y^2} + \rho g_x$$

$$0 = -\frac{\partial P}{\partial y} + \rho g_y$$

Now let's solve for $\frac{\partial P}{\partial y}$.

$\longrightarrow$

B.4 continued

$$\frac{\partial P}{\partial y} = -\rho \underbrace{g \cos\theta}_{\text{since } g_y = g\cos\theta} \implies P = f(x) - \rho g y \cos\theta$$

B.C.: $P\big|_{y=h} = P_0 \implies f(x) = P_0 + \rho g h \cos\theta$

$$\text{so} \quad P = P_0 + \rho g \cos\theta (h - y)$$

Now we see that $\frac{\partial P}{\partial x} = 0$, so x-momentum is:

$$0 = \mu \frac{\partial^2 U_x}{\partial y^2} + \rho g \sin(\theta), \quad \text{Solve for:}$$

$$U_x = -\frac{\rho g}{2\mu} \sin\theta \, y^2 + C_1 y + C_2$$

B.C.

$U_x\big|_{y=0} = 0$ (no slip) $\implies \underline{C_2 = 0}$

$\tau_{yx}\big|_{y=h} = 0 \implies \frac{\partial U_x}{\partial y}\Big|_{y=h} = 0 \implies -\frac{\rho g \sin\theta h}{\mu} + C_1 = 0$

$$\implies C_1 = \frac{\rho g \sin\theta h}{\mu}$$

$$U_x = \frac{\rho g \sin\theta}{2\mu}\left(2hy - y^2\right)$$

$$Q = W\int_0^h U_x \, dy = \frac{\rho g W \sin\theta}{2\mu}\left(2h \frac{y^2}{2}\Big|_0^h - \frac{y^3}{3}\Big|_0^h\right)$$

$$Q = \frac{\rho g W \sin\theta}{\cancel{2}\mu} \frac{\cancel{2}}{3} h^3$$

$\longrightarrow$

3. <u>Continued</u>

Solve for h :

$$\boxed{h = \left(\frac{Q}{w}\right)^{1/3} \left(\frac{3\mu}{\rho g}\right)^{1/3} \left(\frac{1}{\sin\theta}\right)^{1/3}}$$

α " $f(\theta)$

which is exactly what we got in part (b), from dimensional analysis alone.