Cheg 355 Transport I ①

This semester we will study fluid mechanics: the motion of fluids (and solids) in response to applied forces such as shear or pressure or body forces ranging from gravity to electrokinetic or magnetic forces.

We will use *conservation principles* to derive *mathematical* descriptions of simple & complex phenomena. Such *mathematical models* can be used to understand and *predict* phenomena, and solve problems in engineering.

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Admin details:

- weekly HW (15%)
- 2 hour exams (25% each)
- final exam (35%)

We'll also have a weekly tutorial: Mondays 6:00-7:00 PM,

The first tutorial will be a discussion of index notation

TA's:

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The notes, HW, etc. will be posted to the website:

www.nd.edu/~dtl/cheg355/cheg355.html

③

The first HW is already linked in - it's just a few practice problems to review vector calculus.

Texts:

1) Bird, Stewart, & Lightfoot, Transport Phenomena - the updated version of *the* classic text.

This should be available in the bookstore soon, and is a useful ref.

2) The course notes - we're still figuring out the best way of distributing these due to the new copyright regulations. Printed copies will be available soon, but on-line versions are up now!

Check the online version periodically, as the notes may be updated during the semester.

④

OK, why should we care about fluids??

⇒ Vital to the world around us!

- What causes a hurricane & determines its path? A tornado?
- How do you design a sprinkler system so that all areas are doused equally in case of fire?
- How can you design an artificial heart so that it pumps blood *without* tearing up blood cells?
- How can you mix fluids in a chip-based HTS system

⑤

All these questions are answered by applying fundamental conservation laws as well as material properties to complex systems!

What is conserved?

- Mass (neither created nor destroyed)
- Momentum ($F = ma$)
- Energy (we'll get there eventually ...)

we will apply these conservation laws to fluids, but they apply equally well to solids (or anything in between!)

⑥

What is a fluid?
fluid vs. solid

fluid: exhibits continuous deformati
⇒ doesn't snap back after stress is removed!
(thermodynamics: the state of mat'l depends on rate of shear!)

Solids: Elastic deformation – like a rubber band, snaps back after stress is removed!
(thermo: state depends on total deformation)

Virtually everything lies between these two states!

Examples: metal creep, elastic polymer flu

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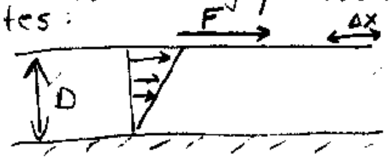
Properties of Fluids

If we characterize fluids by rate of deformation, most important prop. relates to resistance to deformation ⇒ viscosity!

We have a thought exp't.
put mat'l in a gap between plates:



If mat'l is elastic solid, we get some fixed displacement ΔX for a given force F at S.S.

If linearly elastic, relation

⑧

$$\underbrace{\frac{F}{A}}_{\text{force/area}} = \frac{\overset{\text{displacement}}{\Delta X}}{\underbrace{D}_{\text{gap}}}\ \underbrace{E}_{\text{Young's Modulus of Elasticity}}$$

What are units of E? ⇒ same as F/A! Usually given as psi, dyne/cm², etc!

What units to use?? – Depends on application, but you should know all of them! ⇒ know how to convert! I usually use cgs – most approp. for low Re flow (specialty). McCready would use MKS – high Re. Old systems in English units ⇒ all are the same physics!

⑨

OK, we fill it with a fluid — what happens? ⇒ will get continuous deformation!

Plate will move w/ some velocity

$U = \frac{\partial(\Delta x)}{\partial t}$

For a Newtonian fluid

$\frac{F}{A} = \frac{U}{D}\mu$ ← viscosity!

↳ $\frac{g}{cm s}$ (poise)

"poise" is short for Poiseuille, name assoc. w/ pipe flow.

$\frac{U}{D}$ is rate of strain ⇒ known as shear rate

Velocity field is known as plane Couette flow, simple shear flow

⑩

You should get to know the jargon!

What are the viscosities of some simple fluids?

Water ≅ 1 cp (centipoise, 10^{-2} poise)

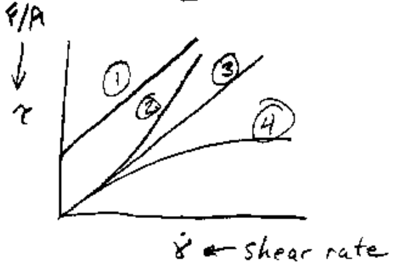
Karo Syrup ≅ 30 p (temp. dep.)

Air ≅ 0.02 cp

All these are Newtonian fluids!

What are ex. of non-Newtonian fluids.

⇒ One feature is stress-strain relation is not linear (or may not be!



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① Bingham plastic ⇒ a linear relation betw. τ & $\dot{\gamma}$, but there is a yield stress ⇒ no motion until critical strain exceeded!
Ex: frozen OJ, mayo

② Dilatant ⇒ μ increases w/ $\dot{\gamma}$
Not seen as often – some clay suspensions do this

③ Newtonian

④ Pseudoplastic ⇒ μ decreases w/ $\dot{\gamma}$
Also called shear thinning – very common in polymer melts!

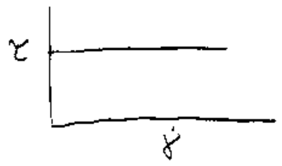
May be much more complicated than this! μ may be time dep., may

⑫

exhibit combination of phen.

Example: liquid chocolate – exhibits yield stress & shear thinning! Imp. if fabricating chocolate figurines!

Other examples: cytological fluid:



indeterminate shear rate for applied shear stress! Leads to complex patterns in cytological streaming!

Normal stresses ⇒ μ may not be a scalar! ⇒ If you shear fluid one way, may get stress in a different direction! Arises in fluids w/ structure.

⑰

Other properties:

speed of sound V_s - important in jet aircraft, high speed machinery

Related to compressibility of fluid:
Sound is a pressure wave travelling thru a fluid

$$V_s = \left(\frac{\partial P}{\partial \rho}\right)_T^{1/2}$$

For an ideal gas $P = \frac{\rho}{M} RT$

Thus $$\left(\frac{\partial P}{\partial \rho}\right)_T = \frac{RT}{M} = \frac{\left(8.3 \times 10^7 \frac{erg}{mol\,°K}\right)(300\,°K)}{(29\ g/mol)}$$

$$= 8.6 \times 10^8\ cm^2/s^2$$

Thus $V_s = 2.9 \times 10^4\ cm/s$
$= 655\ mph$

⑱

When $U/V_s \sim 1$ flow is compressible
⇒ this means that fluid density is affected by fluid motion.
Importance guaged by Mach #
$M = U/V_s$

For liquids $\left(\frac{\partial P}{\partial \rho}\right)_T$ is very large & U is usually smaller, so flow can be regarded as incompressible

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Surface Tension: usually denoted by σ (sometimes γ)
σ ⇒ energy required to create interfacial surface area

units = erg/cm^2

This causes bubbles to be spheres!
(minimize surface/volume)

⑲

Result is a "tension" along the surface → higher pressure within concave side of bubble like inside of a balloon!

$\Delta P \sim \frac{\sigma}{R}$ (inverse to radius)

surfactants (soap) are a material that likes both fluids, thus reduces σ

Coefficient of thermal expansion:

$$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P = \frac{1}{T}$$

for an ideal gas.
Important in natural convection problems, such as draft off window - will look at this in Sr. Lab.

⑳

OK, what types of flows are there?

Compressible vs. Incomp.
- Depends on $M = U/V_s$
- even in air, most flows are incompressible! Usually study compressible flows in Aero E.

Laminar vs. Turbulent
- Flow is laminar if layers of fluid slip smoothly over each other
- Laminar flow may be steady (unchanging in time) or unsteady
⇒ look at flow from tap. At low flows, looks like a glassy, steady stream.

⑬

Suspensions ⇒ area of research at ND. Example – wet sand – if you step on it, it dries out!

Study of stress-strain relationship is <u>rheology</u>

2nd property: Density

⇒ we are interested in transport of <u>momentum</u> which is velocity × <u>ma</u>

∴ Density is important!

Density of water = 1 g/cm³

air = 1.2×10^{-3} g/cm³

Hg = 18.6 g/cm³

Actually, we are interested in

⑭

momentum <u>diffusivity</u> ν

(better known as <u>kinematic</u> <u>viscos</u>

$\nu \equiv \frac{\mu}{\rho}$ – units L^2/T

in cgs $1\ \mathrm{cm^2/s} = 1$ stokes

(name associated w/ flow eq'ns)

Units of ν same as molecular diff. D_{AB}, thermal diff. α ⇒ governs <u>rate</u> w/ which mom. <u>diffuses</u>

material	ν
water	1 cs
air	15 cs
Hg	0.5 cs
Karo syrup	25 <u>stokes</u>

⑮

What do these numbers mean? Determine <u>time</u> to approach steady-state!

Thought exp't ⇒ take metal poker, stick one end in fire – eventually, your hand gets fried! How long? Controlled by <u>diffusivity</u>

Remember: $[\alpha] \equiv L^2/T$

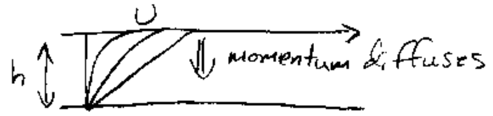
<u>Thus</u> $T \sim \frac{L^2}{\alpha}$

for a metal, $\alpha \sim 0.11\ \mathrm{cm^2/s}$ (steel)

Thus if poker is 2 ft long (60 cm) it takes O(10) hr for your end to get hot! Actually, more complicated as loses heat to air all along shaft

⑯

What about fluids? Look at diffⁿ of <u>momentum</u> ⇒ <u>same</u> thought exp't:



How long till lower plate feels motion?

$$T \sim \frac{h^2}{\nu}$$

If h = 1 cm

T = O(100 s) in water

= O(200 s) in Hg

= O(0.04 s) in Karo syrup!

Actually, this is only order of magnitude ⇒ sol'n of transient problem shows ~4x faster than these values.

(21)

What happens if we increase flow rate? => becomes rough, unsteady → transition to turbulenc

Turbulence is chaotic, time dep & very difficult to describe mathematically w/ precision – still, it occurs most of the time!

Both laminar & turbulent flow may occur in the same geometry => Famous expt in pipe flow by Osbourne Reynolds. Found transition from laminar to turbulent flow govern by dimensionless parameter Re:

$$Re = \frac{\text{inertial forces}}{\text{viscous forces}} = \frac{UD}{\nu} \approx 21$$

we'll look at this in detail later!

(22)

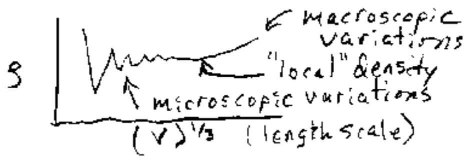
Continuum Hypothesis

we want to develop a mathematical descr. of fluid flow: this requires taking fluid to be a continuum.

Is this continuum hypothesis reasonable? => sometimes!

=> fluid is made up of molecules bouncing into each other. In a gas phase, molecules may go sig. dist. before hitting each other! Not a continuum on this length scale!

Suppose we have probe of arb. size – what would it see??



(23)

We would take value between "microscopic variation" length scale and "macroscopic variation" scale to be "local" density => same for "local" velocity, pressure, temp, etc! This may not work!

=> minimum length for continuum hyp. to hold is mean free path length – distance molecule travels before hitting another. In a gas $\lambda \sim \frac{1}{\sqrt{2}\pi d^2 n}$

where d is molecule dia. & n is number density (molecules/vol)

At 70 mi, $\lambda \sim 10$ cm, so will affect flow in boundary layer of a rocket, for ex.

(24)

At 1 atm & room temp, we have λ is just a few Å. For liquids it's even smaller!

Non-continuum effects are imp. even in liquids, though => the most imp. ex. is Brownian motion

→ In a liquid small particles are kicked around by molecules, thus they execute a random walk – gives rise to diffusion – usually imp. for particles 1 μm or less in dia.

We will assume continuum hyp. to apply, also leads to no-slip condition => at a solid surface in contact w/ fluid, velocity is continuous!

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Fluid layer adjacent to solid surface moves w/ velocity of surface

If $\lambda > l$ (char length of flow), may not be in contact, so would get a "slip" condition - modifies aerodynamics of returning shuttle, or flow in a vacuum pump. Also get breakdown of continuum hyp. in composite media (susp) - not valid on length scales of order particle size ⇒ leads to wall slip as well, makes working with suspensions tricky!

When we will describe motion, ρ, μ, etc. at a "point", really mean some avg over a volume large w.r.t. λ or molecule (particle) size!

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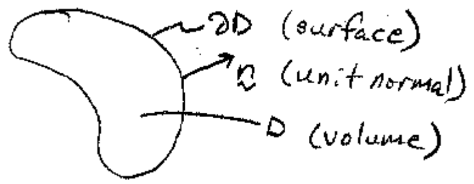
Forces on a Fluid Element

We need to apply $F = ma$ to a fluid ⇒ what are the forces?

Consider an arbitrary element:



What are the forces on the molecules in D? Divide into Surface Forces and Body Forces!

What is a body force? ⇒ They act on each molecule in D.

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Examples:

Gravity: $\underset{\sim}{F} = \rho \underset{\sim}{g}\, dV$ ↳ force on a differential volume!

Electric Field:

$\underset{\sim}{F} = \underset{\sim}{E}\, q\, dV$ ↳ charge/volume; ↑ electric field (volt/cm)

⇒ this force is critical in electro-osmosis & electrophoresis. We use this effect to separate proteins in our laboratory!

Magnetic field:

$\underset{\sim}{F} = \underset{\sim}{J} \times \underset{\sim}{B}$ ↙ magnetic field; ↑ current

⇒ Important in plasmadynamics (fusion reactors), field of MHD

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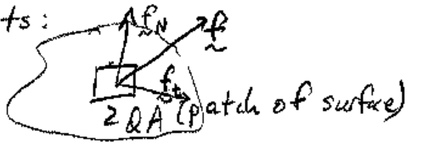
OK, what about Surface Forces? We divide these into shear forces and normal forces

⇒ Surface forces act on the surface of ∂D

⇒ shear forces act tangential to ∂D! The F/A in simple shear flow is a shear force!

⇒ normal forces act normal to the surface

Let the F/A of surface force be $\underset{\sim}{f}$ - a vector. We resolve into tangential & normal components:



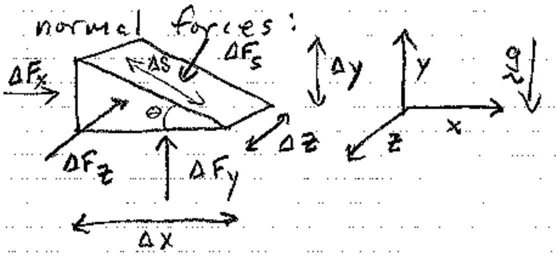
If the <u>unit normal</u> to a patch of surface dA is $\underset{\sim}{n}$

<u>Then</u> $\underset{\sim}{f}_N = (\underset{\sim}{f}\cdot\underset{\sim}{n})\underset{\sim}{n}$

We'll look at $\underset{\sim}{f}_t$ later, now focus on <u>normal</u> forces!

⇒ Consider an element at <u>rest</u>

If it's at rest, shear forces should be <u>zero</u>. Just have normal forces:



Let's do a <u>force balance</u>

⇒ Since element is at <u>rest</u>, the <u>net force</u> in <u>each direction</u> must be <u>zero</u>!

The force balance in the x-direction:

$$\sum F_x = \Delta F_x - \Delta F_s \sin\theta = 0$$

↳ component of ΔF_s in x-dir

Now $\sin\theta \equiv \frac{\Delta y}{\Delta s}$

Thus $\Delta F_x - \Delta F_s \frac{\Delta y}{\Delta s} = 0$

or, dividing by $\Delta z \Delta y$:

$$\frac{\Delta F_x}{\underbrace{\Delta z \Delta y}_{\text{area of x face}}} = \frac{\Delta F_s}{\underbrace{\Delta z \Delta s}_{\text{area of s face}}}$$

<u>Define</u> $\frac{\Delta F_x}{\Delta z \Delta y} = -\sigma_{xx}$ (normal stress)

Similarly,

$$\frac{\Delta F_s}{\Delta z \Delta S} = -\sigma_{ss}$$

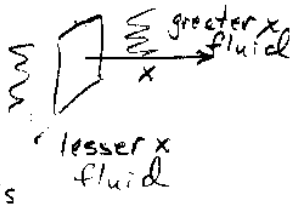
These are <u>normal</u> <u>stresses</u>

They rep. <u>diagonal</u> <u>elements</u> of the <u>stress</u> <u>tensor</u>!

✱ Stress tensor ≡ <u>momentum</u> <u>flux</u>

σ_{ij} ≡ Force/Area exerted <u>by</u> fluid of greater i <u>on</u> fluid of lesser i in j direction!

∴ σ_{xx} ⇒



Thus σ_{xx} is <u>negative</u> in <u>compression</u>.

<u>Note</u>: BS&L defines this <u>backwards</u> (ch 2) ⇒ doesn't change the physics, just the sign!

We'll use the conventional (most common, anyway) definition in this class!

OK, now look at y-direction:

$$\sum F_y = \Delta F_y - \Delta F_s \cos\theta - \underset{\nearrow\ \text{weight of fluid!}}{\rho g \frac{\Delta x \Delta y \Delta z}{2}}$$

$$\underline{\underline{= 0}}$$

Recall $\cos\theta = \frac{\Delta x}{\Delta S}$

Thus (dividing thru):

$$\underbrace{\frac{\Delta F_y}{\Delta x \Delta z}}_{-\sigma_{yy}} - \underbrace{\frac{\Delta F_s}{\Delta s \Delta z}}_{-\sigma_{ss}} = \underbrace{\frac{\rho g \Delta y}{2}}_{\text{Vanishes as } \Delta y \to 0!}$$

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Thus

$$\sigma_{xx} = \sigma_{yy} = \sigma_{ss}$$

* In a fluid at rest, normal stress is isotropic: same in all directions. This normal stress is just $\underset{\uparrow \text{ neg sign!}}{-}p$

$$p = -\sigma_{xx} = -\sigma_{yy} = -\sigma_{zz}$$

When not at rest, normal stress is, in general, not isotropic!

We define

$$p = -\frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$

average of the normal stresses!

equiv: $p = -\frac{1}{3}\, tr(\underset{\sim}{\sigma})$

↳ trace

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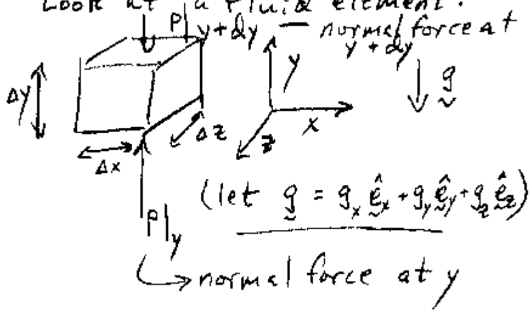
How does p vary in a fluid at rest??

Look at a fluid element:



(let $\underset{\sim}{g} = g_x \hat{\underset{\sim}{e}}_x + g_y \hat{\underset{\sim}{e}}_y + g_z \hat{\underset{\sim}{e}}_z$)

Let's do a force balance in the x-dir:

$$P|_x \Delta y \Delta z - p|_{x+\Delta x} \Delta y \Delta z + \rho g_x \Delta x \Delta y \Delta z = 0$$

Divide through:

$$-\frac{P|_{x+\Delta x} - P|_x}{\Delta x} + \rho g_x = 0$$

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Or, in limit $\Delta x \to 0$:

$$-\frac{\partial P}{\partial x} + \rho g_x = 0$$

Similarly, $\frac{\partial P}{\partial y} = \rho g_y$; $\frac{\partial P}{\partial z} = \rho g_z$

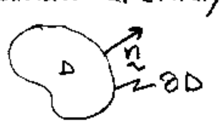
which yield 3 eq'ns!

In vector form:

$$\underset{\sim}{\nabla} P = \rho \underset{\sim}{g}$$

Last deriv. was done using shell balances. If you're good at vector notation, there's an easier (better) way!

Consider arbitrary fluid element:



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What are the forces acting on it?

Surface force: $\int_{\partial D} \underbrace{-p\underset{\sim}{n}\, dA}_{\uparrow \text{ force on each patch of surface}}$

Body Force: $\int_{D} \underbrace{\rho \underset{\sim}{g}\, dV}_{\hookrightarrow \text{ force on each fluid bit}}$

So: $\sum \underset{\sim}{F} = 0$

Thus $\int_{\partial D} -p\underset{\sim}{n}\, dA + \int_{D} \rho \underset{\sim}{g}\, dV = 0$

We now use the Divergence Theorem

$$\int_{\partial D} f \underset{\sim}{n}\, dA \equiv \int_{D} \underset{\sim}{\nabla} f\, dV$$

converts surface int. to vol. int!

So: $\int_D \{ \underset{\sim}{\nabla} P - \rho \underset{\sim}{g} \} dV = 0$

Now since D was completely arbitrary, it must be true at every point in fluid!

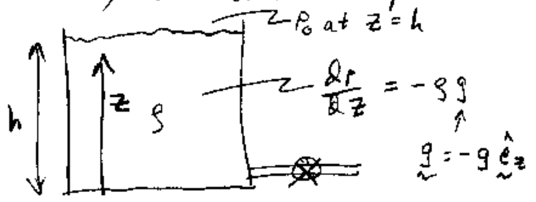
Thus

$$\underset{\sim}{\nabla} P - \rho \underset{\sim}{g} = 0$$

or $\underset{\sim}{\nabla} P = \rho \underset{\sim}{g}$

It will be alot easier to derive things this way when we get to fluids in motion!

Ok, let's solve some problems



Let's Integrate!

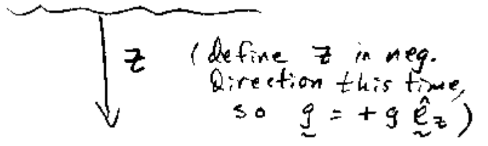
$P = -\rho g z + cst$

$P|_{z=h} = P_0$

Thus $P = P_0 + \rho g (h - z)$

This is just as true in an open body of water (diving):

How deep do you have to go to reach 1 atm guage (e.g. above the atmospheric pressure)?



(Define z in neg. direction this time, so $\underset{\sim}{g} = +g \hat{\underset{\sim}{e}}_z$)

$\frac{dP}{dz} = \rho g$ $\quad P = P_0 + \rho g z$

$= P_0 + \underline{1 \text{ atm}}$

So $z = \frac{1 \text{ atm}}{\rho g}$

Now $1 \text{ atm} = 1.01 \times 10^6 \frac{\text{dyne}}{\text{cm}^2}$ $\quad$ 1.01325

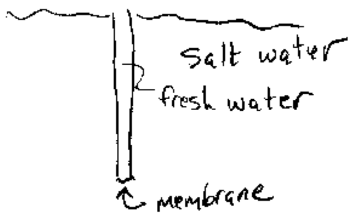
$\rho = 1 \text{ g/cm}^3$ (fresh water) $\rightarrow$ 3.98°C, no air!

$g = 980 \text{ cm/s}^2$ $\rightarrow$ 980.665

$\therefore z = 1033 \text{ cm} = 10.3 \text{m} \approx \underline{33.9 \text{ ft}}$

A bit less in salt water!

This suggests an interesting device! Use a semi-permeable membrane (Reverse Osmosis - RO-memb.) that just lets in water & keeps out salt! Stick it on the end of a -long- pipe & put it in the sea!



If the ΔP across the membrane exceeds the osmotic pressure water will flow through the membrane!

How deep must the pipe be to

(1) get water into the pipe

(2) get the lighter fresh water all the way to the surface?

(1) $g \rho_{SW} h_1 = \Delta P_{OSM}$

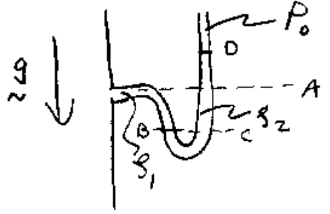
$\rho_{SW} = 1.04 \text{ g/cm}^3$, $\Delta P_{OSM} = \underline{28 \text{atm}}$

$\therefore h_1 = 275 \text{ m}!$

(2) $g \rho_{SW} h_2 - g \rho_{H_2O} h_2 = \Delta P_{OSM}$

$\therefore h_2 = \frac{\Delta P_{OSM}}{g(\rho_{SW} - \rho_{H_2O})} = 7 \text{ km}!$

How about a more practical example ?? ⇒ Manometer on a tank



What is the pressure in the tank at pt A ?

$$P_A = P_0 + (D-C)\rho_2 g - (A-B)\rho_1 g$$

(no pressure differential between pt B & C !)

Manometers are a simple & useful way to measure ΔP of O(1 atm) (Hg - not H_2O!) or O(1 psi) (H_2O)

provided you don't blow them out!

Use electronic or mechanical (spring based) sensors in industry!

Another example: Buoyancy

What is the force exerted by fluid on a submerged object?

$$\underset{\sim}{F} = -\int_{\partial D} p\,\underset{\sim}{n}\,dA$$

The pressure distrib in the fluid is the same as if the object were absent if it is at rest!

So :

$$\underset{\sim}{F} = -\int_{\partial D} P\,\underset{\sim}{n}\,dA = -\int_D \underset{\sim}{\nabla} \overset{\substack{\text{if object absent}\\ \downarrow}}{P}\,dV$$

$$= -\int_D \rho_f\,\underset{\sim}{g}\,dV = -\rho_f\,\underset{\sim}{g}\,V_f$$

So fluid exerts a force equal to the weight of displaced volume! (Archimedes, 3rd cent. B.C.)

OK, let's apply this:

What fraction of an iceberg is submerged?

$$\sum \underset{\sim}{F} = \underset{\substack{\nearrow\\ \text{Vol of iceberg}}}{V}\,\rho_i\,\underset{\sim}{g} - \underset{\substack{\uparrow\\ \text{vol submerged}}}{V_s}\,\rho_w\,\underset{\sim}{g} = \underline{\underline{0}}$$

$$\therefore\ \frac{V_s}{V} = \frac{\rho_i}{\rho_w} = \frac{0.917}{1.04} = 0.88$$

So only about 12% is exposed!

Question: If a glass with ice is filled to brim w/ water & ice projects over rim, will it spill when ice melts ?? ⇒ Nope!

Will it spill if we fill it w/ salt water? ⇒ Yep, as water has a lower density!

Fluids in Motion

Now that we've dealt w/ hydrostatics let's look at fluids in motion

What sort of questions ?? ⇒

If you have a fire hose w/ some pressure, what floor will it reach?

If you have viscous flow thru a tube, what is the velocity profile?

If you have flow over a wing, what is the lift? drag?

To answer these questions, we invoke Conservation Laws

What is conserved??

Mass: What goes in - what goes out = accumulation!

(45)

Momentum: Newton's 2nd law of motion! ($F = ma$)

Energy: First law of Thermo!

We'll use these conservation laws to derive eq^ns that govern fluid motion, then apply to problems!

To do this, need a mathematical framework to describe motion.

Two approaches: Lagrangian & Eulerian

1) Lagrangian: follow a fluid element as it moves thru flow:

$$\underset{\sim}{u} = \underset{\sim}{u}((a,b,c);t) \equiv \underset{\sim}{u}(\underset{\sim}{x}_0;t)$$

initial position of element   time

(46)

Also
$$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{x}_0;t)$$
$$= \underset{\sim}{x}_0 + \int_0^t \underset{\sim}{u}(\underset{\sim}{x}_0;t')\,dt'$$

which tracks the position of the fluid element starting at $\underset{\sim}{x}_0$ at $t=0$ for all time!

Lagrangian description isn't used much in fluids – a bit awkward! When would it be used? ⇒ celestial mechanics! Descr. positions of bodies (discrete) as f^n(t)

Also – study of suspensions (simulation) – track all the particles in a suspension!

⇒ Also important in pasteurization/related processes

(47)

2. Eulerian Approach: $\underset{\sim}{u} = \underset{\sim}{u}(\underset{\sim}{x}, t)$

Track velocity field at an instant of time relative to defined coord system.

Ex: If you take a snapshot of a highway at time, t, you could determine the velocity of all the cars, but you wouldn't know where they came from or where they wind up.

Both Eulerian & Lagrangian descr. can provide a complete descr. of the flow, but for most fluid problems Eulerian is more convenient – we'll focus on it!

---

other useful concepts:

streamline, pathline, streakline

(48)

Streamline: curve everywhere tangent to velocity vector at a given instant

⇒ a snapshot of the flow pattern!

→ this is what you get from Eulerian analysis

Pathline: Actual path traversed by a given fluid element – Lagrangian description!

⇒ What you would get from time-lapsed photograph of a marker in a flow field

Streakline: Locus of particles passing thru a given point

⇒ what is usually produced in flow visualization experiments: smoke is released continuously at a point, & pattern is photographed later!

For S.S. flow, all are identical!

Some unsteady flows may be made steady by shifting coords

Example: falling sphere in viscous fluid. It's moving w.r.t. laboratory reference frame, so flow is unsteady. If we shift coord system so it travels with sphere, it's steady

⇒ much more convenient mathematics as we eliminate time!

⇒ Note: we must use a constant velocity coord system! If we accelerate coord system, leads to non-inertial ref. frame ⇒ adds a term to the equations!

⇒ Also, flow past sphere may still be unsteady at higher Re due to vortex shedding, turbulence.

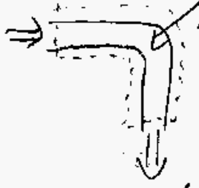
Another Concept: Control Volume

⇒ You used this in 255, etc.

– useful for deriving equations:

⇒ treat it as a "black box", keeping track of what goes in & what goes out!

For example: What is the force on a pipe elbow??



Just do a momentum balance!

Force = Momentum out – momentum in!

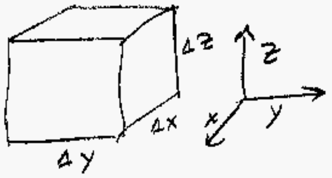
(remember – momentum & force are vectors!)

Exerts force diagonal to elbow – why elbows need bracing!

ok, now we derive eq'ns:

1) Conservation of mass (continuity eq'n)

We consider a fluid element (cube) as depicted below:



where $\underset{\sim}{u} \neq 0$!

We have the basic conservation law:

$$\left\{\begin{array}{c}\text{Rate of accumulation}\\ \text{of mass}\end{array}\right\} = \left\{\begin{array}{c}\text{Rate } \underline{\text{in}}\\ \text{by } \underline{\text{convection}}\end{array}\right\}$$

since it can't be created!

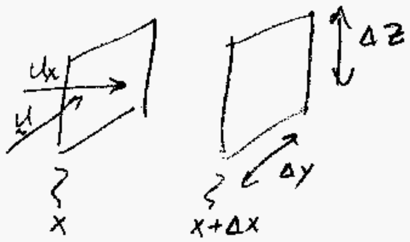
Since $\underset{\sim}{u} \neq 0$, fluid (& mass) may come in (or out) thru each face!

What is flux thru face $x = x_0$?

Volumetric flux $\equiv \underset{\sim}{u} \Rightarrow \frac{\text{Vol}}{\text{Area}\cdot\text{Time}}$

Mass flux $\equiv \rho\underset{\sim}{u} \Rightarrow \frac{\text{Mass}}{\text{Area}\cdot\text{Time}}$

Mass flux thru surface is proportion to component of $\rho\underset{\sim}{u}$ (a vector) normal to the surface!



So mass flow in thru these faces is:

$$(\rho u_x)\Big|_{x} \Delta y \Delta z - (\rho u_x)\Big|_{x+\Delta x} \Delta y \Delta z$$

And if we combine this with the other faces:

$$\text{Mass into cube} = \left[(\rho u_x)\Big|_{x} - (\rho u_x)\Big|_{x+\Delta x}\right]\Delta y \Delta z$$

$$+\left[(\rho u_y)\Big|_{y} - (\rho u_y)\Big|_{y+\Delta y}\right]\Delta x \Delta z$$

$$+\left[(\rho u_z)\Big|_{z} - (\rho u_z)\Big|_{z+\Delta z}\right]\Delta x \Delta y$$

$$= \frac{d}{dt}\underbrace{\left(\Delta x \Delta y \Delta z\, \rho\right)}_{\text{total mass!}}$$

Dividing by $\Delta x \Delta y \Delta z$ & taking the limit as they go to zero yields:

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial(\rho u_x)}{\partial x} + \frac{\partial(\rho u_y)}{\partial y} + \frac{\partial(\rho u_z)}{\partial z}\right)$$

or, $\dfrac{\partial \rho}{\partial t} = -\underset{\sim}{\nabla}\cdot(\rho \underset{\sim}{u})$

In words: The time rate of change of the density is the negative of the divergence of the mass flux vector!

We can rearrange this:

$$\frac{\partial \rho}{\partial t} = -\rho\, \underset{\sim}{\nabla}\cdot\underset{\sim}{u} - \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\rho$$

$$\text{or} \quad \underbrace{\frac{\partial \rho}{\partial t} + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\rho}_{} = -\rho\, \underset{\sim}{\nabla}\cdot\underset{\sim}{u}$$

This is known as the material derivative

$$\frac{D\phi}{Dt} \equiv \frac{\partial \phi}{\partial t} + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\phi$$

for any ϕ!

Remember the Lagrangian description; $\dfrac{D\phi}{Dt}$ is the time rate of change of any property ϕ experienced by a fluid element!

It has two components:

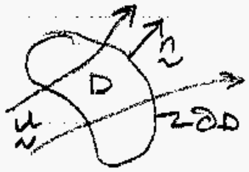
1) $\dfrac{\partial \phi}{\partial t} \Rightarrow$ local deriv. w.r.t. time
2) $\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\phi \Rightarrow$ change due to convection thru a field where ϕ varies with position

If a fluid is incompressible we have $\rho = cst$

Thus $\dfrac{D\rho}{Dt} \equiv 0$

and thus $\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 0$

An alternate derivation may be made using vector calculus. Consider an arbitrary control volume D:



What is the change in the total mass in D?

$$\frac{dM}{dt} = \frac{d}{dt}\int_D \rho\, dV \equiv \int_D \frac{\partial \rho}{\partial t}\, dV$$

$$= \int_{\partial D} \underbrace{-\rho \underset{\sim}{u}\cdot\underset{\sim}{n}}_{\text{mass flux in thru each patch of surface!}}\, dA$$

(57)

Thus:

$$\int_D \frac{\partial \rho}{\partial t}\, dV + \int_{\partial D} \rho \underset{\sim}{u}\cdot\underset{\sim}{n}\, dA = 0$$

Apply Divergence theorem:

$$\int_D \frac{\partial \rho}{\partial t}\, dV + \int_D \underset{\sim}{\nabla}\cdot(\rho \underset{\sim}{u})\, dV = 0$$

or $\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}\cdot(\rho \underset{\sim}{u}) = 0$

Which is the same equation!

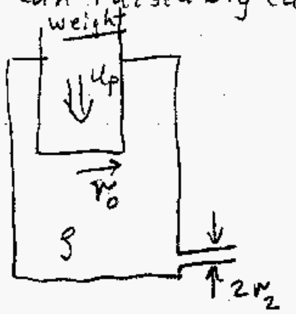
In index notation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0$$

(58)

An example:

Let's look at a hydraulic jack. This is an example of how a small pump can raise a big car!



For a given motion of the piston what is the flowrate thru the outlet pipe? For a given weight of car, what is the pressure in the pipe?

(59)

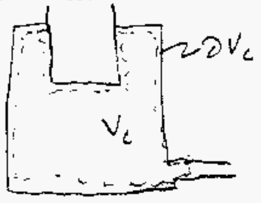
To get the flow rate we use the CE:

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}\cdot(\rho \underset{\sim}{u}) = 0$$

We take the fluid to be incompressible, so the density is cst

$\therefore$ $\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 0$

We draw a control volume:



$$\int_{V_c} \underset{\sim}{\nabla}\cdot\underset{\sim}{u}\, dV = \int_{\partial V_c} \underset{\sim}{u}\cdot\underset{\sim}{n}\, dA = 0$$

Fluid in = Fluid out!

(60)

$\underset{\sim}{u}\cdot\underset{\sim}{n} \neq 0$ only in exit pipe face & at piston face!

Let A_e be exit pipe, A_p be piston

$$\int_{A_p} \underset{\sim}{u}\cdot\underset{\sim}{n}\, dA + \int_{A_e} \underset{\sim}{u}\cdot\underset{\sim}{n}\, dA = 0$$

Now flow out through piston is

$-A_p u_p$ ($\underset{\sim}{u}\cdot\underset{\sim}{n}$ is negative here)

Flow out through exit pipe is

$+A_e\langle u_e\rangle$

(we define average velocity

$$\langle u_e\rangle = \frac{1}{A_e}\int_{A_e} \underset{\sim}{u}_e\cdot\underset{\sim}{n}\, dA$$

Thus the average velocity:

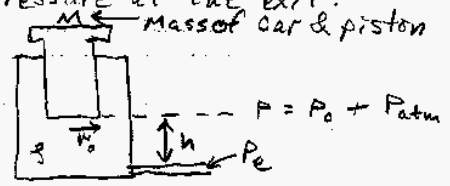
$$\langle u_e\rangle = \frac{A_p}{A_e} u_p$$

(61)

So the ratio of the average inlet velocity to the average outlet velocity is inverse of the ratios of the areas!

Note: the CE tells you about the average velocity normal to the exit, it doesn't tell you about the velocity distribution

If there's no flow, what is the pressure at the exit?



$$P_0 = F/A = \frac{Mg}{\pi r_0^2}$$

(62)

$$P_e = P_0 + P_{atm} + \rho g h$$

What is the force required to raise the piston?

$$F = (P_e - P_{atm}) A_e$$

$$= \left(\frac{Mg}{\pi r_0^2} + \rho g h\right) \pi r_e^2$$

$$= Mg \underbrace{\frac{r_e^2}{r_0^2}}_{} + \underbrace{\pi r_e^2 \rho g h}_{\text{small (usually)}}$$

↳ ratio reduces required force!

This is how hydraulics work! Examples: car brakes, wing elevators, hydraulic jacks, etc.

Note: energy expended to raise car is unchanged, but force is reduced!

(63)

Let's extend the CE to multi-component systems

Suppose we have m species, (e.g., salt sol'n H_2O, $NaCl$: $m=2$)

we can do a balance on each species

Let velocity of species i be given by $\underset{\sim}{u}_i$ (ok, not index notation here - subscript represents which species we're talking about)

Note: $\underset{\sim}{u}_i$ will, in general, be different from mass avg. velocity $\underset{\sim}{u}$ due to diffusion!

Let density of species i (mass/vol) be ρ_i ⇒ Note this is not the density of salt (say) but rather the mass/volume of salt in the solution! OK, we still have conservation for each species

(64)

$$\frac{d}{dt}\int_D \rho_i \, dV = -\int_{\partial D} \rho_i \, \underset{\sim}{u}_i \cdot \underset{\sim}{n} \, dA + \int_D R_i \, dV$$

R_i ⇒ Mass rate of production per unit volume of species i due to reaction!

We can apply divergence theorem to this:

$$\int_D \frac{\partial \rho_i}{\partial t}\, dV + \int_D \underset{\sim}{\nabla}\cdot(\rho_i \underset{\sim}{u}_i)\, dV = \int_D R_i\, dV$$

or the microscopic eq'n:

$$\frac{\partial \rho_i}{\partial t} + \underset{\sim}{\nabla}\cdot(\rho_i \underset{\sim}{u}_i) = R_i$$

The total density is just the sum of ρ_i:

$$\rho = \Sigma \rho_i$$

& mass avg velocity:

$$\rho \underset{\sim}{u} = \Sigma \rho_i \underset{\sim}{u}_i$$

Thus summing the equation ove all species:

$$\frac{\partial}{\partial t}\Sigma \rho_i + \underset{\sim}{\nabla}\cdot\left(\Sigma \rho_i \underset{\sim}{u}_i\right) = \Sigma R_i$$

or:

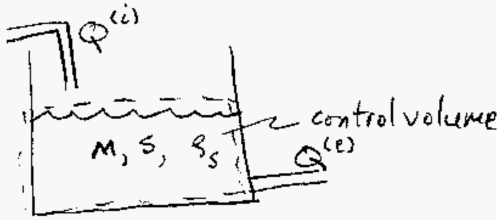
$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}\cdot(\rho \underset{\sim}{u}) = \Sigma R_i$$

Note that $\Sigma R_i = \underline{\underline{0}}$ since mass is conserved in reacting systems!

Next semester you will combine this equation with Fick's law to get the equation governing mass transfer!

Ok, let's work another example: Conservation of mass in a CSTR (Continuously Stirred Tank Reactor)

Suppose we have a well-mixed (stirred) tank:



We have a mass flow rate Q

$Q^{(i)}$ ⇒ inlet mass flow

$Q^{(e)}$ ⇒ exit mass flow

M = mass in tank = $\int_D \rho\, dV$

$\rho \equiv$ total density

$S \equiv$ salt in tank = $\int_D \rho_s\, dV$

$\rho_s \equiv$ density of salt

We wish to determine the fluid level & salt concentration as a function of time!

$$\{\text{Mass in}\} - \{\text{Mass out}\} = \{\text{Accum.}\}$$

Thus:

$$\frac{dM}{dt} = -\int_{\partial D} \rho\,(\underset{\sim}{u}\cdot\underset{\sim}{n})\, dA = Q^{(i)} - Q^{(e)}$$

$$\frac{dS}{dt} = -\int_{\partial D} \rho_s(\underset{\sim}{u}\cdot\underset{\sim}{n})\, dA = Q^{(i)}\underbrace{\frac{\rho_s^{(i)}}{\rho^{(i)}}}_{\omega_s^{(i)} \Rightarrow \text{mass fraction at inlet}} - Q^{(e)}\frac{\rho_s^{(e)}}{\rho^{(e)}}$$

⑥⑨

Now for a CSTR,

$$\frac{\rho_s^{(e)}}{\rho^{(e)}} = \frac{S}{M} \quad (\text{tank is well mixed})$$

Hence

$$\frac{dS}{dt} = Q^{(i)} \omega_s^{(i)} - \left(\frac{S}{M}\right) Q^{(e)}$$

$$\frac{dM}{dt} = Q^{(i)} - Q^{(e)} = \Delta Q$$

Solution: Solve for M first, then solve for S!

$$M = M_0 + \Delta Q t \quad (\text{linear change in time})$$

$$\frac{dS}{dt} = \frac{-Q^{(e)}}{M_0 + \Delta Q t} S + Q^{(i)} \omega_s^{(i)}$$

⑦⓪

or

$$\frac{dS}{dt} + \left\{ \frac{Q^{(e)}}{M_0 + \Delta Q t} \right\} S = Q^{(i)} \omega_s^{(i)}$$

w/ I.C. $S\Big|_{t=0} = S_0$

This is a first order linear ODE

We have the general solution

$$\frac{dy}{dx} + p(x) y = f(x)$$

Then:

$$y(x) = e^{-\int p(x) dx} \left[\int \left[f(x) e^{\int p(x') dx'} \right] dx + K \right]$$

where K is Determined from I.C.!

Let's apply this:

$x \equiv t$, $f(x) \equiv Q^{(i)} \omega_s^{(i)} = cst$

⑦①

$$p(x) \equiv \frac{Q^{(e)}}{M_0 + \Delta Q t}$$

So:

$$\int p(t) dt = \frac{Q^{(e)}}{\Delta Q} \ln \left(\frac{M_0}{\Delta Q} + t \right)$$

and thus:

$$S = e^{-\left[\frac{Q^{(e)}}{\Delta Q} \ln \left(\frac{M_0}{\Delta Q} + t \right)\right]} \cdot \left[\int Q^{(i)} \omega_s^{(i)} e^{\left[\frac{Q^{(e)}}{\Delta Q} \ln \left(\frac{M_0}{\Delta Q} + t \right)\right]} dt + K \right]$$

Now $e^{\left[\frac{Q^{(e)}}{\Delta Q} \ln \left(\frac{M_0}{\Delta Q} + t \right)\right]} \equiv \left(\frac{M_0}{\Delta Q} + t \right)^{\left(\frac{Q^{(e)}}{\Delta Q}\right)}$

Thus:

$$S = \left(\frac{M_0}{\Delta Q} + t \right)^{-\frac{Q^{(e)}}{\Delta Q}} \left[\int Q^{(i)} \omega_s^{(i)} \left(\frac{M_0}{\Delta Q} + t \right)^{\left(\frac{Q^{(e)}}{\Delta Q}\right)} dt + K \right]$$

$$= \left(\frac{M_0}{\Delta Q} + t \right)^{-\frac{Q^{(e)}}{\Delta Q}} \left[Q^{(i)} \omega_s^{(i)} \frac{\left(\frac{M_0}{\Delta Q} + t \right)^{\left(\frac{Q^{(e)}}{\Delta Q} + 1\right)}}{\left(\frac{Q^{(e)}}{\Delta Q} + 1 \right)} + K \right]$$

⑦②

$$= Q^{(i)} \omega_s^{(i)} \frac{\left(\frac{M_0}{\Delta Q} + t \right)}{\left(\frac{Q^{(e)}}{\Delta Q} + 1 \right)} + K \left(\frac{M_0}{\Delta Q} + t \right)^{-\frac{Q^{(e)}}{\Delta Q}}$$

We determine K from the I.C.

$$S\Big|_{t=0} = S_0$$

Thus:

$$S_0 = Q^{(i)} \omega_s^{(i)} \frac{\frac{M_0}{\Delta Q}}{\left(\frac{Q^{(e)}}{\Delta Q} + 1 \right)} + K \left(\frac{M_0}{\Delta Q} \right)^{-\frac{Q^{(e)}}{\Delta Q}}$$

So

$$K = S_0 \left(\frac{M_0}{\Delta Q} \right)^{\frac{Q^{(e)}}{\Delta Q}} - Q^{(i)} \omega_s^{(i)} \frac{\left(\frac{M_0}{\Delta Q} \right)^{\left(\frac{Q^{(e)}}{\Delta Q} + 1\right)}}{\left(\frac{Q^{(e)}}{\Delta Q} + 1 \right)}$$

Which yields:

$$S = S_0 \left(\frac{\frac{M_0}{\Delta Q}}{\frac{M_0}{\Delta Q} + t} \right)^{\frac{Q^{(e)}}{\Delta Q}} + \frac{Q^{(i)} \omega_s^{(i)}}{\left(\frac{Q^{(e)}}{\Delta Q} + 1 \right)} \times$$

$$\left[\left(\frac{M_0}{\Delta Q} + t \right) - \left(\frac{M_0}{\Delta Q} + t \right)^{-\frac{Q^{(e)}}{\Delta Q}} \left(\frac{M_0}{\Delta Q} \right)^{\left(\frac{Q^{(e)}}{\Delta Q} + 1\right)} \right]$$

(73)

$$= S_o \left(\frac{M_o}{M_o + \Delta Q t}\right)^{\frac{Q^{(e)}}{\Delta Q}} + \frac{Q^{(i)} \omega_s^{(i)}}{\left(\frac{Q^{(e)}}{\Delta Q} + 1\right)} \times$$

$$\left(\frac{M_o}{\Delta Q} + t\right)\left[1 - \left(\frac{\frac{M_o}{\Delta Q}}{\frac{M_o}{\Delta Q} + t}\right)^{\left(\frac{Q^{(e)}}{\Delta Q} + 1\right)}\right]$$

Note: $\frac{Q^{(e)}}{\Delta Q} + 1 = \frac{1}{\Delta Q}\left(\cancel{Q^{(e)}} + Q^{(i)} \cancel{- Q^{(e)}}\right)$

$= \frac{Q^{(i)}}{\Delta Q}$

So:

$$S = S_o\left(\frac{M_o}{M_o + \Delta Q t}\right)^{\frac{Q^{(e)}}{\Delta Q}}$$

$$+ \omega_s^{(i)} (M_o + \Delta Q t)\left(1 - \left(\frac{M_o}{M_o + \Delta Q t}\right)^{\frac{Q^{(i)}}{\Delta Q}}\right)$$

The first term results from the loss of the salt initially present in the tank. The second results from that added to the tank.

(74)

We can simplify a bit further if we recall:

$$M = M_o + \Delta Q t$$

Thus:

$$S = S_o \left(\frac{M_o}{M}\right)^{\frac{Q^{(e)}}{\Delta Q}}$$

$$+ \omega_s^{(i)} M \left(1 - \left(\frac{M_o}{M}\right)^{\frac{Q^{(i)}}{\Delta Q}}\right)$$

It is interesting to note that in the limit $\Delta Q \to 0$ (e.g., $Q^{(e)} = Q^{(i)}$) the power law form given here collapses to a pure exponential:

$M = M_o \quad Q = Q^{(e)} = Q^{(i)}$

$$S = M_o \omega_s^{(i)} + (S_o - M_o \omega_s^{(i)}) e^{-\frac{Q}{M_o} t}$$

The quantity M_o/Q is known as the Residence Time of the vessel!

(75)

Conservation of Momentum

Just as was the case for mass, momentum is also conserved.

For mass we had:

$$\left\{\begin{matrix}\text{accum of}\\ \text{mass}\end{matrix}\right\} = -\left\{\begin{matrix}\text{net rate out}\\ \text{by convection}\end{matrix}\right\}$$

or:

$$\int_D \frac{\partial \rho}{\partial t}\, dV = -\int_{\partial D} \rho\, \underset{\sim}{u} \cdot \underset{\sim}{n}\, dA$$

For momentum it's a bit messier:

$$\left\{\begin{matrix}\text{accum of}\\ \text{momentum}\end{matrix}\right\} = -\left\{\begin{matrix}\text{net rate momentum}\\ \text{out by convection}\end{matrix}\right\}$$

$$+ \left\{\begin{matrix}\text{sum of forces on}\\ D \text{ by surroundings}\end{matrix}\right\}$$

Force adds momentum via

$\underset{\sim}{F} = m\, \underset{\sim}{a}$ (rate of increase of momentum)

(76)

What do these terms look like?

$\rho \underset{\sim}{u} \equiv$ momentum per unit volume

Thus:

$$\left\{\begin{matrix}\text{Rate momentum out}\\ \text{by convection}\end{matrix}\right\} \equiv \int_{\partial D} (\rho \underset{\sim}{u})\, \underset{\sim}{u} \cdot \underset{\sim}{n}\, dA$$

$$\frac{\text{momentum}}{\text{volume}} \times \begin{matrix}\text{volumetric flux}\\ \text{normal to surface}\end{matrix}$$

= momentum flux!

What is the total momentum in D?

$\rho \underset{\sim}{u} \equiv$ momentum/volume

Thus accumulation is:

$$\int_D \frac{\partial}{\partial t} (\rho \underset{\sim}{u})\, dV$$

Combining these terms:

$$\int_D \frac{\partial(\rho \underset{\sim}{u})}{\partial t}\, dV + \int_{\partial D} (\rho \underset{\sim}{u})\, \underset{\sim}{u}\cdot\underset{\sim}{n}\, dA$$

$$= \sum \underset{\sim}{F}$$ (sum of forces <u>on</u> Control Volume)

Ok, what are the forces? We looked at these before!

<u>Body forces</u> (e.g., gravity)

$$\underset{\sim}{F}_g = \int_D \rho\, \underset{\sim}{g}\, dV$$

<u>Surface forces</u>

These include <u>normal forces</u> (e.g. pressure) and <u>shear forces</u>

The latter results from "dragging" <u>along</u> (tangential to) a surface!

Let $\underset{\sim}{f}$ be <u>all</u> surface forces at a point.

Thus:

$$\sum \underset{\sim}{F} = \int_D \rho \underset{\sim}{g}\, dV + \int_{\partial D} \underset{\sim}{f}\, dA$$

$$\underset{\sim}{f} \equiv \frac{\text{force}}{\text{area}} \equiv \text{surface stress}$$

Recall from our earlier examination of hydrostatics that:

$$\underset{\sim}{f} = \underset{\approx}{\sigma}\cdot\underset{\sim}{n}$$

where $\underset{\approx}{\sigma}$ is the <u>stress tensor</u>

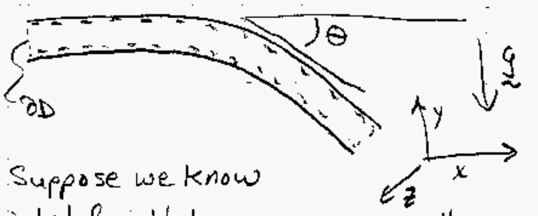
We'll use this in a bit.

For now we have:

$$\int_D \frac{\partial(\rho \underset{\sim}{u})}{\partial t}\, dV + \int_{\partial D} (\rho \underset{\sim}{u})\, \underset{\sim}{u}\cdot\underset{\sim}{n}\, dA =$$

$$\int_D \rho \underset{\sim}{g}\, dV + \int_{\partial D} \underset{\sim}{f}\, dA$$

How can we use this? ⇒ We can calculate the force on an elbow!



Suppose we know inlet & outlet pressures as well as the flow rate. We want to know the force exerted <u>by</u> the fluid <u>on</u> the bend (section of pipe) which is (−) force exerted <u>by</u> bend <u>on</u> fluid!

We have the momentum balance:

$$\int_D \rho \underset{\sim}{g}\, dV + \int_{\partial D} \underset{\sim}{f}\, dA = \int_{\partial D} (\rho \underset{\sim}{u})\, \underset{\sim}{u}\cdot\underset{\sim}{n}\, dA + \int_D \frac{\partial(\rho \underset{\sim}{u})}{\partial t}\, dV$$

We assume we are at <u>Steady State</u>, thus $\frac{\partial}{\partial t} \equiv 0$

If the fluid is <u>incompressible</u>

$$\int_D \rho \underset{\sim}{g}\, dV = \rho \underset{\sim}{g}\, V_D \equiv \text{weight of water!}$$

Now for the surface integrals:

We divide up ∂D into A_i, A_e, A_p,

e.g.,

(81)

Let's look at the convection term:

$$\int_{\partial D}(\rho\underset{\sim}{u})\underset{\sim}{u}\cdot\underset{\sim}{n}\,dA = \int_{A_i}(\rho\underset{\sim}{u})\underset{\sim}{u}\cdot\underset{\sim}{n}\,dA$$

$$+\int_{A_p}(\rho\underset{\sim}{u})\underset{\sim}{u}\cdot\underset{\sim}{n}\,dA + \int_{A_e}(\rho\underset{\sim}{u})\underset{\sim}{u}\cdot\underset{\sim}{n}\,dA$$

Over the pipe itself (A_p) $\underset{\sim}{u}\cdot\underset{\sim}{n} = \underline{\underline{0}}$ (no flow through the pipe), thus we just get integrals over inlet & exit!

⇒ Unlike mass conservation, we can't evaluate integrals exactly without knowing the velocity profile ($\underset{\sim}{u}(r)$) across the pipe in addition to the total flow rate Q

This is because the integral is non-linear in $\underset{\sim}{u}$!

(82)

To estimate the force we shall assume we have uniform flow

Let's take $\underset{\sim}{u}\Big|_{A_i} \approx \frac{Q}{A_i}\hat{\underset{\sim}{e}}_x$

Now at the inlet $\underset{\sim}{n}\Big|_{A_i} = -\hat{\underset{\sim}{e}}_x$

Thus:

$$\int_{A_i}(\rho\underset{\sim}{u})\underset{\sim}{u}\cdot\underset{\sim}{n}\,dA \cong \int_{A_i}\left(\rho\frac{Q}{A_i}\underset{\sim}{e}_x\right)\left(\frac{Q}{A_i}\hat{\underset{\sim}{e}}_x\cdot(-\hat{\underset{\sim}{e}}_x)\right)dA$$

$\hat{\underset{\sim}{e}}_x\cdot(-\hat{\underset{\sim}{e}}_x) = -1$

So we get:

$= -\rho\frac{Q^2}{A_i}\hat{\underset{\sim}{e}}_x$ which is negative because momentum is going into CV!

Note that this underestimates the momentum flux (in general).

(83)

This is because non-uniformities in $\underset{\sim}{u}$ increase the momentum flux over a uniform velocity!

⇒ The average of the square is always greater than or equal to the square of the average!

Let $\langle u\rangle = \frac{1}{A}\int_A u\,dA$

Let $\Delta u = u - \langle u\rangle$

So: $\int_A u^2\,dA = \int_A(\Delta u + \langle u\rangle)^2\,dA$

$$= \int_A\langle u\rangle^2\,dA + \int_A(\Delta u)^2\,dA + 2\langle u\rangle\underbrace{\cancel{\int_A\Delta u\,dA}}_{0}$$

(84)

Since $\int_A(\Delta u)^2\,dA \geq 0$

we have:

$$\int_A\rho u^2\,dA \geq \int_A\rho\left(\frac{Q}{A}\right)^2 dA$$

so we underestimate the momentum flux. For high Re (turbulence) the profile is nearly flat (uniform), so it's not a big error!

Over the exit we have the same integral:

$\underset{\sim}{u}\Big|_{A_e} \cong \frac{Q}{A_e}\hat{\underset{\sim}{e}}_\theta$, $\hat{\underset{\sim}{e}}_\theta = \cos\theta\,\hat{\underset{\sim}{e}}_x + \sin\theta\,\hat{\underset{\sim}{e}}_y$

The unit normal: $\underset{\sim}{n}\Big|_{A_e} = \hat{\underset{\sim}{e}}_\theta$

So:

$$\int_{A_e}\rho\underset{\sim}{u}(\underset{\sim}{u}\cdot\underset{\sim}{n})\,dA = \rho\frac{Q^2}{A_e}\hat{\underset{\sim}{e}}_\theta$$

(85)

Putting these together:

$$\int_{\partial D} (\rho \underset{\sim}{u})(\underset{\sim}{u}\cdot\underset{\sim}{n})\, dA \approx \rho Q^2\left(\frac{-\hat{\underset{\sim}{e}}_x}{A_i} + \frac{\hat{\underset{\sim}{e}}_\theta}{A_e}\right)$$

Note that since $\hat{\underset{\sim}{e}}_x \neq \hat{\underset{\sim}{e}}_\theta$ the force will be non-zero even if $A_i \neq A_e$

A force is required to deflect a stream!

Ok, now we look at the surface forces:

$$\int_{\partial D} \underset{\sim}{f}\, dA \equiv \int_{A_i} \underset{\sim}{f}\, dA + \int_{A_e} \underset{\sim}{f}\, dA + \int_{A_p} \underset{\sim}{f}\, dA$$

The last one is what we're after!

Let's do the first term:

$$\int_{A_i} \underset{\sim}{f}\, dA \equiv \text{Force exerted by fluid outside CV on CV integrated over } A_i$$

(86)

This force is complex, but we will approximate it by assuming it's just the normal force:

$$\left.\underset{\sim}{f}\right|_{A_i} \approx P_i\, \hat{\underset{\sim}{e}}_x$$

↳ pressure at A_i

So:

$$\int_{A_i} \underset{\sim}{f}\, dA \cong P_i A_i\, \hat{\underset{\sim}{e}}_x$$

$$\int_{A_e} \underset{\sim}{f}\, dA \cong -P_e A_e\, \hat{\underset{\sim}{e}}_\theta \qquad \text{(force is in } -\hat{\underset{\sim}{e}}_\theta \text{ direction)}$$

Finally, $\int_{A_p} \underset{\sim}{f}\, dA \equiv \underset{\sim}{F}_p$, force exerted by the pipe on the fluid!

(87)

Putting it all together:

$$\cancelto{0}{\frac{d}{dt}\int_D \rho \underset{\sim}{u}\, dV} + \int_{\partial D} (\rho \underset{\sim}{u})\, \underset{\sim}{u}\cdot\underset{\sim}{n}\, dA = \int_D \rho \underset{\sim}{g}\, dV + \int_{\partial D} \underset{\sim}{f}\, dA$$

Or:

$$\rho Q^2\left(\frac{-\hat{\underset{\sim}{e}}_x}{A_i} + \frac{\hat{\underset{\sim}{e}}_\theta}{A_e}\right) = -\rho g V_D\, \hat{\underset{\sim}{e}}_y + P_i A_i\, \hat{\underset{\sim}{e}}_x - P_e A_e\, \hat{\underset{\sim}{e}}_\theta + \underset{\sim}{F}_p$$

Or, rearranging,

$$\underset{\sim}{F}_p = \rho Q^2\left(\frac{-\hat{\underset{\sim}{e}}_x}{A_i} + \frac{\hat{\underset{\sim}{e}}_\theta}{A_e}\right) + \rho g V_D\, \hat{\underset{\sim}{e}}_y - P_i A_i\, \hat{\underset{\sim}{e}}_x + P_e A_e\, \hat{\underset{\sim}{e}}_\theta$$

This is a vector equation! We can look at the x component:

(88)

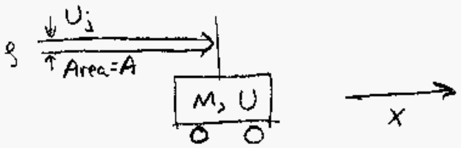
$$(F_p)_x = \underset{\sim}{F}_p \cdot \hat{\underset{\sim}{e}}_x = \rho Q^2\left(\frac{-1}{A_i} + \frac{\cos\theta}{A_e}\right) - P_i A_i + P_e A_e \cos\theta$$

or the y-component:

$$(F_p)_y = \underset{\sim}{F}_p \cdot \hat{\underset{\sim}{e}}_y = \rho Q^2\left(-\frac{\sin\theta}{A_e}\right) + \rho g V_D - P_e A_e \sin\theta$$

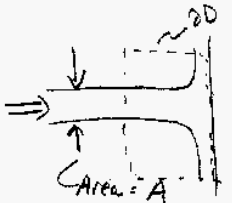
These forces could be used to determine the required bracing, for example!

Let's work through another *example*: Water jet pushing a car. Suppose we have a car with a plate sticking up as below:



A jet of water of diameter D & velocity U_j impinges on the plate. What is the force on the plate as a function of U? What is the velocity of the car as a function of time?

To solve, look at problem in a *reference frame* moving *with* the plate!



Water velocity in this frame is now $(U_j - U)$, *not* U_j!

We draw the CV as depicted. We have:

$$\sum \underset{\sim}{F} = \int_{\partial D} (\rho \underset{\sim}{u})\, \underset{\sim}{u}\cdot\underset{\sim}{n}\, dA$$

We are interested in the x-component of this force. Since the fluid *leaves* ∂D with a velocity only in the y-direction, we just worry about the *inlet*

Thus:

$$F_x = A\left(\rho(U_j - U)\right)\left(-(U_j - U)\right)$$

negative because fluid is *entering*

So the force *on* the fluid is just

$$F_x = -A\rho (U_j - U)^2$$

The force *on* the car is the *negative* of this!

Now since $F = M \frac{dU}{dt}$ we have:

$$\frac{dU}{dt} = \frac{A\rho}{M}(U - U_j)^2$$

we can solve this:

$$\frac{1}{(U-U_j)^2}\frac{dU}{dt} = \frac{A\rho}{M}$$

$$\frac{d}{dt}\left(\frac{1}{U - U_j}\right) = -\frac{A\rho}{M}$$

$$\frac{1}{U - U_j} = -\frac{A\rho}{M}t + C$$

Let $U\big|_{t=0} = 0$

$$\therefore\ C = -\frac{1}{U_j}$$

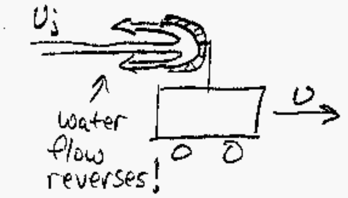
So $$\frac{1}{U - U_j} = -\frac{A\rho t}{M} - \frac{1}{U_j}$$

$$\frac{U}{U_j} = 1 - \frac{1}{\frac{A\rho}{M}U_j t + 1}$$

$$= \frac{\frac{A\rho U_j t}{M}}{1 + \frac{A\rho U_j t}{M}}$$

So U *asymptotically* approaches U_j as we would expect.

We can get a much higher force & acceleration if we modify the plate so it sends water back out in the reverse direction



In the moving reference frame we still have:

$$\sum \underset{\sim}{F} = \int_{\partial D} (\rho \underset{\sim}{u})\, \underset{\sim}{u} \cdot \underset{\sim}{n}\, dA$$

but now u_x is reversed for the fluid leaving ∂D rather than just zero. This doubles the momentum transfer!

$$F_x = -2A\rho\,(U_j - U)^2$$

(force on fluid)

so:

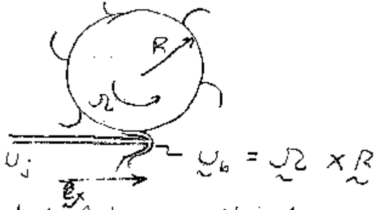
$$\frac{dU}{dt} = 2\,\frac{A\rho}{M}\,(U - U_j)^2$$

or

$$\frac{U}{U_j} = \frac{2\,\frac{A\rho U_j t}{M}}{1 + 2\,\frac{A\rho U_j t}{M}}$$

The asymptotic velocity is still U_j, it just gets there twice as fast! This effect is why in a Pelton wheel (a type of turbine) the buckets are curved - more efficient momentum & energy transfer

Let's analyze the Pelton Wheel:



We wish to determine the torque on the wheel, and the rate of work (Power) transfered to it!

First for the torque:

$$\underset{\sim}{M} = \underset{\sim}{F} \times \underset{\sim}{R}$$

The force is just the change in momentum of the stream! To get this, we need the exit velocity U_e. We have the two cases for different vanes: flat plate & reflection:

$$F_x = Q\left[(\rho \underset{\sim}{U}_j - \rho \underset{\sim}{U}_e)\right] \cdot \hat{\underset{\sim}{e}}_x$$

vol. flow rate
mom/vol in
mom/vol out
force on vane (neg. of force on fluid)

Only the x-component of the force contributes to the torque! (perp. to $\underset{\sim}{R}$)

Ok, for the flat plate we have:

$$\underset{\sim}{U}_e\Big|_x = \underset{\sim}{U}_b\Big|_x = (\underset{\sim}{\Omega} \times \underset{\sim}{R})\Big|_x$$

Thus for this case

$$F_x = Q\left[(\rho U_j - \rho \Omega R)\right]$$

The torque is $F_x R$.

What about the power?

$$P = \underset{\sim}{M} \cdot \underset{\sim}{\Omega} = F_x R \Omega$$

$$= Q R \Omega \left[(\rho U_j - \rho \Omega R)\right]$$

Note that the torque is maxed when $\Omega = 0$ <u>but</u> the power is zero!

What is the value of Ω for which the power is max?

$$\frac{\partial P}{\partial \Omega} = 0 = QR\left[\rho U_j - 2\rho\Omega_m R\right]$$

$$\therefore \quad \rho U_j = 2\rho\Omega_m R$$

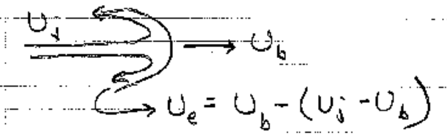
$$\text{or} \quad \Omega_m R = \frac{U_j}{2}$$

so the vanes move with half the velocity of the jet. The max power is:

$$P_m = Q\,\frac{U_j}{2}\left[\left(\rho U_j - \tfrac{1}{2}\rho U_j\right)\right]$$

$$= \tfrac{1}{2}\,Q\left(\tfrac{1}{2}\rho U_j^2\right)$$ which is <u>half</u> the total kinetic energy of the stream!

Now for <u>curved</u> <u>buckets</u>:



$$U_e = U_b - (U_j - U_b)$$

This yields a force:

$$F_x = Q\left[(\rho U_j - \rho U_e)\right]$$

$$= Q\left[\rho U_j + \rho U_j - 2\rho U_b\right]$$

$$= 2Q\left[\rho U_j - \rho\Omega R\right]$$

which is <u>twice</u> the force (and torque and power) of the flat vanes!

At the optimum (same) rotation rate, we have:

$$P_m = Q\left(\tfrac{1}{2}\rho U_j^2\right)$$

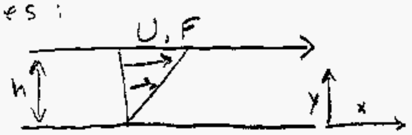
or <u>all</u> the kinetic energy of the jet is extracted. A real water wheel would lie between these values.

Microscopic Momentum <u>Balances</u>

So far we've done our calculations by <u>assuming</u> velocity profiles were <u>flat</u> (uniform). This, in general, <u>is not correct</u>! To get it right, we need to <u>calculate</u> the <u>velocity profile</u>. We need to develop the equation which governs the velocity everywhere in the fluid.

To do this, we need to reexamine the stress tensor $\underset{\approx}{\tau}$

Look at the flow between parallel plates:



Fluid <u>resists</u> deformation so a force F is required to keep the plate in motion!

The <u>magnitude</u> of the force is proportional to the <u>Area</u>, thus we look at F/A => <u>shear stress at the wall</u>

Shear stress is transmitted through the fluid to the lower plate!

Shear stress ≡ <u>momentum</u> <u>flux</u>

For <u>this</u> <u>geometry</u> each layer of fluid exerts the <u>same</u> force on the layer below it! The shear stress is <u>constant</u>, otherwise momentum would <u>accumulate</u> in the interior!

Recall the definition of σ_{ij}:

$\sigma_{ij} \equiv F/A$ exerted *by* fluid of greater i *on* fluid of lesser i in j direction!

In this case we have

$$\sigma_{yx} = F/A$$

which, for *this* geometry, is constant!

What are the properties of σ_{ij}?

=> The stress tensor is symmetric!

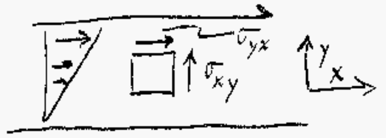
$$\sigma_{ij} = \sigma_{ji}$$

or

$$\underset{\approx}{\sigma} = \underset{\approx}{\sigma}^T$$

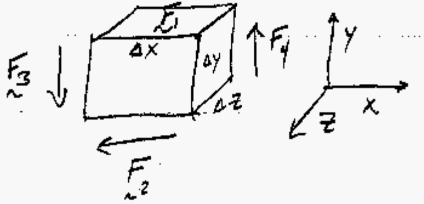
This is *really* counter intuitive!

In this flow



$\sigma_{yx} = \sigma_{xy}$??

Let's *prove* this! Consider a fluid *element*:



$$\underset{\sim}{F_1} \equiv \sigma_{yx}(\Delta z \Delta x)\,\hat{\underset{\sim}{e}}_x$$

$$\underset{\sim}{F_2} \equiv -\sigma_{yx}(\Delta z \Delta x)\,\hat{\underset{\sim}{e}}_x$$

$$\underset{\sim}{F_3} \equiv -\sigma_{xy}(\Delta z \Delta y)\,\hat{\underset{\sim}{e}}_y$$

$$\underset{\sim}{F_4} \equiv \sigma_{xy}(\Delta z \Delta y)\,\hat{\underset{\sim}{e}}_y$$

Now we have $\sum \underset{\sim}{F} = 0$ because element isn't *accelerating*

What about the *Torque*??

$$\underset{\sim}{M} = \sum \underset{\sim}{r} \times \underset{\sim}{F} = -\frac{\Delta y}{2}\sigma_{yx}(\Delta z \Delta x)\hat{\underset{\sim}{e}}_z - \frac{\Delta y}{2}\sigma_{yx}(\Delta z \Delta x)\hat{\underset{\sim}{e}}_z + \frac{\Delta x}{2}\sigma_{xy}(\Delta x \Delta y)\hat{\underset{\sim}{e}}_z + \frac{\Delta x}{2}\sigma_{xy}\Delta z \Delta y\,\hat{\underset{\sim}{e}}_z$$

$$= \Delta x \Delta y \Delta z\,\hat{\underset{\sim}{e}}_z(\sigma_{xy} - \sigma_{yx})$$

We have, just like F = ma, a relation for the *angular acceleration* of any object:

$$\frac{d\underset{\sim}{\Omega}}{dt} = \frac{\underset{\sim}{M}}{I} \leftarrow \text{moment of inertia}$$

$$I = \int_D \rho\, s^2\, dV = \frac{\Delta x \Delta y \Delta z}{12}\rho(\Delta x^2 + \Delta y^2)$$

↳ distance from axis of rotation

Thus:

$$\frac{d\underset{\sim}{\Omega}}{dt} = \frac{\underset{\sim}{M}}{I} = \frac{12\hat{\underset{\sim}{e}}_z}{\rho}\frac{(\sigma_{xy} - \sigma_{yx})}{(\Delta x^2 + \Delta y^2)}$$

as Δx, Δy → 0 *any* angular acceleration *must be finite*, *thus* we conclude $\sigma_{xy} = \sigma_{yx}$!

There *is* an exception to this: For *very* weird systems you can get a *body torque*: torque applied uniformly through a fluid. This would make the stress tensor assymetric! How can you do this? If you have an ER (electro rheological) fluid in a *rotating* electric or magnetic field you get this effect. *Don't Worry About It*: For all

← a fluid w/ a bunch of dipoles

normal systems, the stress tensor <u>is symmetric</u> !!

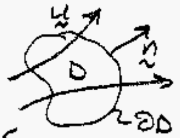
Another useful property:

For <u>any</u> surface w/ normal $\underset{\sim}{n}$, the stress (force/area) exerted <u>by</u> surroundings <u>on</u> fluid is just:

$$\underset{\sim}{f} = \underset{\approx}{\sigma} \cdot \underset{\sim}{n}$$

We can use this in our momentum balance equation!

Recall:



$$\begin{Bmatrix}\text{net momentum } \underline{\text{out}} \\ \text{by convection}\end{Bmatrix} + \begin{Bmatrix}\text{Accumulation}\end{Bmatrix} = \begin{Bmatrix}\text{body forces}\end{Bmatrix} + \begin{Bmatrix}\text{surface forces}\end{Bmatrix}$$

So:

$$\int_{\partial D} \rho \underset{\sim}{u} (\underset{\sim}{u} \cdot \underset{\sim}{n})\, dA + \int_D \frac{\partial}{\partial t} (\rho \underset{\sim}{u})\, dV = \int_D \rho \underset{\sim}{g}\, dV + \int_{\partial D} \underset{\approx}{\sigma} \cdot \underset{\sim}{n}\, dA$$

We apply the <u>Divergence theorem</u>:

$$\int_D \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{u} \underset{\sim}{u})\, dV + \int_D \frac{\partial (\rho \underset{\sim}{u})}{\partial t}\, dV = \int_D \rho \underset{\sim}{g}\, dV + \int_D \underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma}\, dV$$

Or, since D is <u>arbitrary</u>:

$$\underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{u} \underset{\sim}{u}) + \frac{\partial (\rho \underset{\sim}{u})}{\partial t} = \rho \underset{\sim}{g} + \underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma}$$

We can simplify this by differentiating by parts:

$$\underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{u} \underset{\sim}{u}) \equiv \rho \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \underset{\sim}{u} + \underset{\sim}{u}\, \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{u})$$

$$\frac{\partial (\rho \underset{\sim}{u})}{\partial t} = \underset{\sim}{u} \frac{\partial \rho}{\partial t} + \rho \frac{\partial \underset{\sim}{u}}{\partial t}$$

Substituting in:

$$\rho \frac{\partial \underset{\sim}{u}}{\partial t} + \rho \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \underset{\sim}{u} + \underset{\sim}{u} \left[\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{u}) \right] = \underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma} + \rho \underset{\sim}{g}$$

Now from conservation of mass:

$$\frac{\partial \rho}{\partial t} = - \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{u}) \quad !$$

thus the term in brackets is <u>zero</u>

So: $\rho \left[\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \underset{\sim}{u} \right] = \underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma} + \rho \underset{\sim}{g}$

Or $\rho \frac{D \underset{\sim}{u}}{D t} = \underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma} + \rho \underset{\sim}{g}$

We can also write this in <u>index notation</u>:

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i$$

Note that each term has only one unrepeated index, and that they are <u>all the same</u>!

To proceed, we look at the total stress σ_{ij}. We <u>define</u>:

$$\sigma_{ij} = -p\, \delta_{ij} + \tau_{ij}$$

where $p = -\frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33})$ is the <u>pressure</u> – the average of the normal stresses in the three orthogonal directions (well, the negative of this anyway)

Other ways of saying this:

$p = -\frac{1}{3}\sigma_{ij}\delta_{ij} = -\frac{1}{3}\sigma_{ii}$

where σ_{ii} = trace ($\underset{\approx}{\sigma}$)

τ_{ij} is known as the deviatoric stress and arises due to fluid motion. It is identically zero for isotropic fluids at rest (e.g., hydrostatics)

What are the properties of τ_{ij}??

=> Since σ_{ij} is symmetric, so is τ_{ij}

=> By definition τ_{ij} is traceless

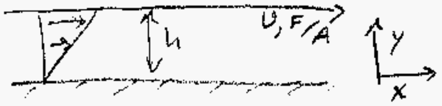
e.g., $\tau_{ij} = \sigma_{ij} + p\delta_{ij}$

$\therefore \tau_{ij}\delta_{ij} = \sigma_{ij}\delta_{ij} + p\delta_{ij}\delta_{ij}$

$\tau_{ii} = \sigma_{ii} + 3p = 0$

⇒ τ_{ij} arises from the deformation of a fluid!

As an example, consider flow between two parallel plates:



In this geometry, $\tau_{yx} = F/A$

Experimentally, we find:

$$F/A = \mu \frac{U}{h}$$

where μ is the fluid viscosity!

Now we also have:

$$\frac{U}{h} = \frac{\partial u_x}{\partial y}$$ (linear profile)

Thus we get Newton's Law of Viscosity:

$$\tau_{yx} = \mu \frac{\partial u_x}{\partial y}$$

We can generalize this a bit!

Remember that $\underset{\approx}{\tau}$ is symmetric!

Thus $\tau_{yx} = \tau_{xy} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$

Actually, we can generalize this still further. If τ_{ij} is proportional to the rate of strain tensor $\frac{\partial u_i}{\partial x_j}$, we have the general relation:

$$\tau_{ij} = A_{ijkl}\frac{\partial u_k}{\partial x_l}$$

where A_{ijkl} is a fourth order tensor. We have three restrictions on A_{ijkl}. First, if the fluid is isotropic, then A_{ijkl} must also be isotropic (it's a material property).

Thus:

$$A_{ijkl} = \lambda_1\delta_{ij}\delta_{kl} + \lambda_2\delta_{ik}\delta_{jl} + \lambda_3\delta_{il}\delta_{jk}$$

Second we know that $\underset{\approx}{\tau}$ is symmetric, e.g. that $\tau_{ij} = \tau_{ji}$

This requires $A_{ijkl} = A_{jikl}$

or that $\lambda_2 = \lambda_3$

Finally, we know that $\underset{\approx}{\tau}$ is traceless, e.g. that $\tau_{ij}\delta_{ij} = 0$

This requires $\delta_{ij}A_{ijkl} = 0$

Plugging this in, we get $\lambda_1 = -\frac{2}{3}\lambda_2$

Thus:

$$A_{ijkl} \equiv \lambda_2\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right]$$

or, as it's usually written:

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\,\underset{\sim}{\nabla}\cdot\underset{\sim}{u}\right)$$

So we see that this complex expression for the shear stress arises naturally from the assumptions of linearity, isotropy, and the definition of the pressure ($\tau_{ii}=0$).

For more complex fluids the stress-strain relation is alot messier! The study of such relations is the field of rheology.

For an incompressible fluid $\underset{\sim}{\nabla}\cdot\underset{\sim}{u} \equiv 0$, thus:

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

we can plug this in:

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i$$

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

Thus:

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i$$

but: (for incompressible fluids)

$$\frac{\partial \tau_{ij}}{\partial x_j} = \mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$= \mu\left[\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial}{\partial x_i}\underbrace{\cancel{\frac{\partial u_j}{\partial x_j}}}_{0}\right]$$

assume μ is constant

So:

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial P}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j^2} + \rho g_i$$

or

$$\rho\left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{u}\right) = -\underset{\sim}{\nabla}P + \mu\nabla^2\underset{\sim}{u} + \rho\underset{\sim}{g}$$

which are known as the Navier-Stokes equations. They are valid for

incompressible Newtonian fluids with constant viscosity (or at least not a function of position!). If any of these assumptions are not valid, the equations need to be modified! Fortunately, they work for most chem. eng. problems!

Let's look at the equation term by term:

$\rho\frac{\partial \underset{\sim}{u}}{\partial t}$ ⇒ time dependent accumulation of momentum

$\rho\,\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\,u$ ⇒ Convection of momentum, associated with fluid inertia

$-\underset{\sim}{\nabla}P$ ⇒ Gradients in the pressure act as a source or sink of momentum

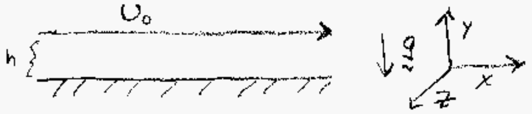
$\mu\nabla^2\underset{\sim}{u}$ ⇒ Viscous diffusion of momentum

$\rho\underset{\sim}{g}$ ⇒ Gravitational (body force) source of momentum

Try to build up a physical picture of each of the physical mechanisms behind these terms! Such an understanding will help you determine which terms are important in any physical problem!

Ok, now let's apply these equations to the simplest flow problem:

Plane Couette Flow



we assume an incompressible, Newtonian fluid with constant viscosity, thus

(113)

we have the equations:

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u}=0$$

$$\rho\left(\frac{\partial \underset{\sim}{u}}{\partial t}+\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{u}\right)=-\underset{\sim}{\nabla}P+\mu\nabla^2\underset{\sim}{u}+\rho\underset{\sim}{g}$$

We also need boundary conditions

$\underset{\sim}{u}\Big|_{y=0}=0$ (all 3 components)

$\underset{\sim}{u}\Big|_{y=h}=U_0\,\underset{\sim}{\hat{e}}_x$ (y & z components are zero)

Now we start throwing out terms.

We anticipate that the flow is only in the x-direction, thus

$u_y=u_z=0$

From continuity:

$$\frac{\partial u_x}{\partial x}+\underset{0}{\cancel{\frac{\partial u_y}{\partial y}}}+\underset{0}{\cancel{\frac{\partial u_z}{\partial z}}}=0$$

(114)

Thus $\frac{\partial u_x}{\partial x}=0$

⇒ There is no change in the velocity in the flow direction for unidirectional flow.

⇒ The converse: If the velocity changes in the flow direction, then it cannot be uni-directional! (e.g., if $\frac{\partial u_x}{\partial x}\neq 0$ then u_y or u_z must be non-zero somewhere!)

We assume that the flow is 2-D (no change in z-direction), thus $\frac{\partial}{\partial z}=0$

We assume that there are no applied pressure gradients, thus $\frac{\partial P}{\partial x}=0$

(115)

we take $\underset{\sim}{g}=-g\,\underset{\sim}{\hat{e}}_y$ (not in x-direction)

We assume flow is at steady-state, so $\frac{\partial}{\partial t}=0$

Ok, what's left??

C.E.: $$\frac{\partial u_x}{\partial x}+\underset{0}{\cancel{\frac{\partial u_y}{\partial y}}}+\underset{0}{\cancel{\frac{\partial u_z}{\partial z}}}=0$$

so $\frac{\partial u_x}{\partial x}=0$

z-momentum:

$$\rho\left(\overset{0}{\cancel{\frac{\partial u_z}{\partial t}}}+\overset{0}{\cancel{u_x\frac{\partial u_z}{\partial x}}}+\overset{0}{\cancel{u_y\frac{\partial u_z}{\partial y}}}+\overset{0}{\cancel{u_z\frac{\partial u_z}{\partial z}}}\right)=-\frac{\partial P}{\partial z}+\mu\nabla^2\underset{0}{\cancel{u_z}}+\underset{0}{\cancel{\rho g_z}}$$

So $\frac{\partial P}{\partial z}=0$ (no pressure gradient in z-direction)

(116)

y-momentum:

$$\rho\left[\overset{0}{\cancel{\frac{\partial u_y}{\partial t}}}+\overset{0}{\cancel{u_x\frac{\partial u_y}{\partial x}}}+\overset{0}{\cancel{u_y\frac{\partial u_y}{\partial y}}}+\overset{0}{\cancel{u_z\frac{\partial u_y}{\partial z}}}\right]$$

$$=-\frac{\partial P}{\partial y}+\mu\nabla^2\underset{0}{\cancel{u_y}}+\rho g_y$$

So $\frac{\partial P}{\partial y}=\rho g_y\equiv-\rho g$

Hence $P=f(x)-\rho g y$

↳ actually, will be a cst since no gradient is applied in x-direction

Just hydrostatic pressure variation!

Now for x-momentum (this is the important one, because the flow is in the x-direction!)

(117)

$$\rho\left[\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}\right]$$
$$= -\frac{\partial P}{\partial x} + \mu\left[\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right] + \rho g_x$$

What survives? u_y & $u_z = \underline{0}$, so those convective terms are zero!

System is at S.S., so $\frac{\partial u_x}{\partial t} = 0$

From C.E. $\frac{\partial u_x}{\partial x} = 0$, so $u_x\frac{\partial u_x}{\partial x} = 0$, likewise $\frac{\partial^2 u_x}{\partial x^2} = 0$ (in RHS)

No variation in z-direction, so

$$\frac{\partial^2 u_x}{\partial z^2} = 0$$

No gravity in x-direction, so

$$\rho g_x = 0$$

(118)

No pressure gradient (applied) in x-direction, so

$$-\frac{\partial P}{\partial x} = 0$$

What's left?

$$\frac{\partial^2 u_x}{\partial y^2} = 0, \quad u_x\Big|_{y=0} = 0$$
$$u_x\Big|_{y=h} = U_0$$

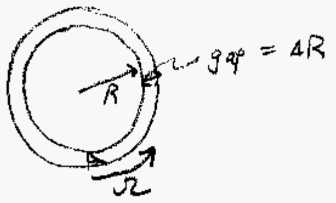
This is easily solved - just integrate twice!

$$u_x = Ay + B$$
$$u_x\Big|_{y=0} = 0 \quad \therefore \quad B = 0$$
$$u_x\Big|_{y=h} = U_0 \quad \therefore \quad A = \frac{U_0}{h}$$

and $u_x = U_0 \frac{y}{h}$

(119)

This is called simple shear flow or plane Couette flow. It's used to study the rheology of fluids, and is usually produced in the narrow gap between concentric rotating cylinders:



By rotating the outer cylinder you deform the fluid in the gap and exert a torque on the inner cylinder. This torque is used to calculate the viscosity!

(120)

What's this relationship??
$\Rightarrow$ if $\frac{\Delta R}{R} \ll 1$ we can ignore curvature effects:

Thus $u_x \approx U_0 \frac{y}{\Delta R}$

The stress on the inner cylinder is:

$$\tau_{yx} = \mu\frac{\partial u_x}{\partial y} \cong \frac{\mu U_0}{\Delta R}$$

The torque is:

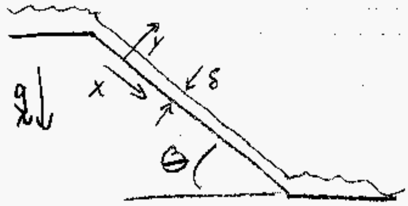
$$M \cong (\tau_{yx})(R)(2\pi R H)$$

where H is the height in the z-direction. Thus:

$$M \cong \mu\, 2\pi \frac{\Omega R^3 H}{\Delta R} \quad \text{for } \frac{\Delta R}{R} \ll 1$$

which can be used to estimate μ

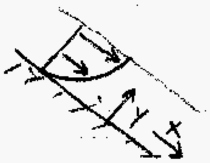
Another example: Flow down an inclined plane



If the fluid is viscous, it will rapidly reach some constant thickness δ, and some steady velocity profile. What is the relationship between Q/w, u, δ, ρ, μ, g, and θ?? Just apply the Navier-Stokes equations!

First we choose a coordinate system aligned with the geometry!

Let x be the direction along the plate, and y be normal to the plate w/ $y=0$ at the plate:



Thus $\underset{\sim}{g} = -g\cos\theta\,\underset{\sim}{\hat{e}}_y + g\sin\theta\,\underset{\sim}{\hat{e}}_x$

Again, we have unidirectional flow in the x-direction. We expect there will be no flow in the y-direction – just a hydrostatic pressure variation.

Note: to solve these sorts of problems, look at it physically & keep those terms which appear to

important, crossing out those you expect to be zero. If you can satisfy all B.C.'s with the simplified equation, you got it right! This is strictly true only for linear problems, as non-linear equations often have multiple solutions! Even there, it's a good way to start.

Know Each Term Physically

OK, we expect unidirectional flow. Thus:

$$0 = -\frac{\partial P}{\partial y} + \rho g_y$$

$$0 = -\frac{\partial P}{\partial x} + \mu \nabla^2 u_x + \rho g_x$$

Recall that ~~for unidirectional~~ incompressible flow

$$\frac{\partial u_x}{\partial x} = 0$$

There is no variation in the z-direction (2-D flow), thus

$$\nabla^2 u_x = \frac{\partial^2 u_x}{\partial y^2}$$

Now to solve: First we get the pressure distribution.

$$g_y = -g\cos\theta$$

$$\therefore \quad P = f(x) - \rho g y \cos\theta$$

but $P\big|_{y=\delta} = P_0$ (atmospheric)

Thus $P = P_0 + \rho g(\delta - y)\cos\theta$

Note that $\underline{\frac{\partial p}{\partial x} = 0}$ so it Disappears from the x-momentum equation!

$$0 = -\underset{0}{\cancel{\frac{\partial p}{\partial x}}} + \mu \frac{\partial^2 u_x}{\partial y^2} + \rho g \sin\theta$$

so $\frac{\partial^2 u_x}{\partial y^2} = -\frac{\rho g}{\mu} \sin\theta$

Integrating:

$$u_x = -\frac{1}{2} y^2 \frac{\rho g}{\mu} \sin\theta + Ay + B$$

We now Determine the unknown constants from the B.C.'s. What are they??

1) No-slip condition at $y=0$! plate isn't moving at $y=0$, so neither is the fluid!

$$u_x\big|_{y=0} = 0$$

Thus $\underline{B \equiv 0}$

2) at $y=\delta$ the shear stress is zero! The gas (air) over the fluid doesn't exert any stress on it, so

$$\tau_{yx}\Big|_{y=\delta} = \mu \frac{\partial u_x}{\partial y}\Big|_{y=\delta} = 0$$

So $A = +\frac{\rho g \delta}{\mu} \sin\theta$

Thus:

$$u_x = \frac{\rho g \delta^2}{\mu} \sin\theta \left(\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^2\right)$$

From this we see that u_x varies as δ^2, and at $y=\delta$ we have a maximum $(u_x)_{max} = \frac{1}{2} \frac{\rho g \delta^2}{\mu} \sin\theta$

We also want to look at the total flow rate:

$$Q = \int_A \underset{\sim}{u} \cdot \underset{\sim}{n}\, dA$$

$$= \underline{W} \int_0^\delta u_x\, dy$$

$$= \frac{W \rho g \delta^3}{\mu} \sin\theta \left[\frac{1}{2} - \frac{1}{6}\right]$$

$$= \frac{1}{3} \frac{W \rho g \delta^3}{\mu} \sin\theta$$

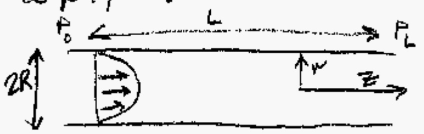
So Q varies as δ^3. If we __knew__ Q (or Q/W), this relation would give us δ.

This equation __will__ __not__ __hold__ if δ is too large (or μ too small). What happens is the flow field becomes __unstable__ and ripples form!

This is an example of the effect of non-linearities! There are __multiple__ __solutions__ to the full equations where $u_y \neq 0$, and where u_x and u_y are functions of __time__. Such waves have been extensively studied over the past 30 years! In our department Chang is perhaps the leading expert on falling films, while McCready is the leading expert on instabilities in cocurrent gas-liquid flows where the gas exerts some stress on the interface ($\tau_{yx}|_\delta \neq 0$). These two areas are important in coating __flows__ and pipeline flows.

(129)

Another example: Flow through a pipe!



Suppose we have an <u>axial</u> <u>pressure</u> <u>gradient</u> (e.g., $\frac{\partial P}{\partial z} \neq 0$)

What is the flow profile?

For a given μ, $\frac{\Delta P}{L}$, R what is the flow rate? Again we choose a coordinate system <u>aligned</u> <u>with</u> the <u>boundary</u>: Cylindrical coordinates!

Let's solve this: We begin with the C.E.:

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = 0 \equiv \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

(130)

For <u>uni-directional</u> flow in the z-direction, $\underline{u_r = u_\theta = 0}$

Thus $\frac{\partial u_z}{\partial z} = 0$

The <u>assumption</u> of unidirectional flow will limit the applicability of our solution! We'll see how this works later!

Ok, now we solve for the velocity distribution. We focus on the z-momentum eq'n in <u>cylindrical</u> coord:

$$\rho\Big(\underbrace{\cancel{\frac{\partial u_z}{\partial t}}}_{0\,(SS)} + \underbrace{\cancel{u_r}}_{0}\frac{\partial u_z}{\partial r} + \underbrace{\cancel{\frac{u_\theta}{r}}}_{0}\frac{\partial u_z}{\partial \theta} + u_z\underbrace{\cancel{\frac{\partial u_z}{\partial z}}}_{0\,(CE)}\Big)$$

$$= -\frac{\partial P}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\underbrace{\cancel{\frac{\partial^2 u_z}{\partial \theta^2}}}_{0\,(\text{symmetry})} + \underbrace{\cancel{\frac{\partial^2 u_z}{\partial z^2}}}_{0\,(CE)}\right]$$

$$+ \rho g_z$$

(131)

So:

$$\mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = \frac{\partial P}{\partial z} - \rho g_z$$

Note that there are <u>two</u> possible sources for momentum: pressure gradients <u>or</u> gravity. Both act in <u>exactly</u> the same way! Both (if constant) are <u>uniform</u> <u>sources</u> (or sinks) <u>of momentum</u> in the fluid! Here we take $g_z = 0$ and look at the <u>pressure</u> <u>gradient</u>

Let $\frac{\partial P}{\partial z} \equiv \frac{\Delta P}{L}$ (pressure drop/length) (note: this is <u>negative</u>!)

So $\frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right) = \frac{1}{\mu}\frac{\Delta P}{L} \equiv \underline{cst}$

We integrate once:

(132)

(mult. both sides by r before integ.):

$$r\frac{du_z}{dr} = \frac{1}{2}r^2\frac{1}{\mu}\frac{\Delta P}{L} + A$$

$$\frac{du_z}{dr} = \frac{1}{2}r\frac{1}{\mu}\frac{\Delta P}{L} + \frac{A}{r}$$

$$u_z = \frac{1}{4}r^2\frac{1}{\mu}\frac{\Delta P}{L} + A\ln r + B$$

Now at $r = 0$ u_z must be <u>finite</u>

thus: $\underline{A \equiv 0}$

At $r = R$ $u_z = 0$ (no-slip),

thus:

$$B = -\frac{1}{4}R^2\frac{1}{\mu}\frac{\Delta P}{L}$$

or $u_z = -\frac{1}{4}\frac{R^2}{\mu}\frac{\Delta P}{L}\left(1 - \frac{r^2}{R^2}\right)$

which is a parabola again!

Gravity would yield the same result, just replace $-\frac{\Delta P}{L}$ with ρg_z!

What is the total flow rate

$$Q = \int_A u_z\, dA = \int_0^R 2\pi u_z r\, dr$$

since it's not a f$^{\underline{n}}$ (θ)

Integrating:

$$Q = 2\pi\left(\frac{-1}{4}\frac{\Delta P}{L}\frac{R^2}{\mu}\right)R^2\int_0^1 (1-r^{*2})r^* dr^*$$

where $r^* \equiv r/R$

so: $$Q = -\frac{\pi}{8}\frac{\Delta P}{L}\frac{R^4}{\mu}$$

which is known as Poiseuille's Law & flow thru a tube is also called Poiseuille flow.

Ok, what is it good for? It is the basis of the capillary viscometer

$$\mu = \left(\frac{-\Delta P}{L}\right)\frac{\pi}{8}\frac{R^4}{Q}$$

Usually, the ΔP is provided by hydrostatic pressure variation: just measure time for fluid to fall between two lines! It's an easily calibrated instrument.

What are the limitations on Poseuille's Law? => Assumption of unidirectional flow!

There are two ways this is violated: entrance effects & turbulence

Look at turbulence first: If flow is too fast, becomes unstable! Reynolds showed that for a tube the transition is governed by a Dimensionless Number

$$Re = \frac{UD}{\nu}$$

Physically, this represents the ratio of inertial forces to viscous forces ⇒ An alternative interpretation is in terms of characteristic times: Recall that momentum can move either by convection or diffusion (e.g., the kinematic viscosity). Then Re is the ratio of the diffusion time to the convection time:

$$Re = \frac{(D^2/\nu)}{(D/U)} = \frac{UD}{\nu}$$

Reynolds found flow to be unstable for $\frac{UD}{\nu} \gtrsim 2100$ for tubes. You get different values of Re_{cr} for different geometries.

Ok, what about entrance length effects? => Initially, entering flow profile is (more or less) flat, & must evolve to parabolic shape. How far down the tube does this take?

The flow evolves due to diffusion of momentum, so:

$$t_D \sim \frac{R^2}{\nu} \quad \leftarrow \text{distance over which diffusion takes place}$$

How far does it move during t_D?

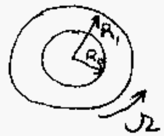
$$L \sim t_D U \sim \frac{UR^2}{\nu} \equiv \frac{1}{4} D \frac{UD}{\nu}$$

Actually, the entrance length is usually given as: $L_e = 0.035\, D \frac{UD}{\nu}$

which is just a bit numerically smaller!

Let's look at another problem in Cylindrical Coordinates:

Couette flow!



we again use the r, θ, z coord. system. This time, however, the velocity is in the θ direction!

$$\nabla \cdot \underset{\sim}{u} = 0 = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

Thus if $u_r = u_z = 0$ then $\underline{\underline{\frac{\partial u_\theta}{\partial \theta} = 0}}$

(no variation in θ direction)

Now for the momentum equations:

$\Rightarrow$ we looked at z-momentum last time, now look at r & θ components!

r-momentum:

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z\frac{\partial u_r}{\partial z}\right)$$

$$= -\frac{\partial P}{\partial r} + \rho g_r + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_r}{\partial z^2}\right]$$

Now if $g_r = 0$, $u_r = u_z = 0$ and $\frac{\partial u_\theta}{\partial \theta} = 0$ we're left with:

$$\underset{\uparrow}{-\rho\frac{u_\theta^2}{r}} = -\frac{\partial P}{\partial r}$$

centrifugal force term! It is a "pseudo force" which arises from the coordinate transformation!

Thus $P = f(\theta, z) + \int \rho \frac{u_\theta^2}{r}\, dr$ which can be integrated $\underline{\underline{\text{if}}}$ you know $u_\theta(r)$!

Ok, let's look at the θ component (where the action is!)

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + u_z\frac{\partial u_\theta}{\partial z}\right)$$

$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \rho g_\theta + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial z^2}\right]$$

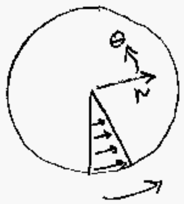
Ok, most of these terms are zero too! Let's look at one that pops up due to the coordinate transformation:

$$\rho\frac{u_r u_\theta}{r}$$

This is the coriolis force. It is very important in large scale (e.g., high Re) rotating systems! The most important example is the weather! It's why the wind direction

is perpendicular to pressure gradients!

To see why this occurs, consider a disk undergoing solid body rotation:



Now $u_\theta = \Omega r$ for solid body rotation. The local angular velocity is constant. If fluid is displaced inwards, then if u_θ is conserved (say, conservation of kinetic energy) the local rate of rotation $\Omega' = \frac{u_\theta}{r - \Delta r} > \Omega$. In the rotating reference frame, it looks like it's going faster!

(141)

On the earth, rotational velocities are much higher than wind velocities, at least on large length scales, thus the Coriolis force is dominant

$$\Omega R \sim \frac{2\pi}{24\,hr}\cdot 4{,}000\,mi \sim \underline{10^3\,mph}!$$

On lab length scales it's small (at least due to earth rotation) ⇒ the bathtub vortex is due to some initial swirling motion!

Ok, how about Couette flow? $u_r = 0$ so coriolis force doesn't matter!

$\frac{\partial P}{\partial \theta} = 0$ from symmetry, so:

$$0 = \mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right)$$

(142)

we integrate this once:

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) = C_1$$

And a second time:

$$r u_\theta = \frac{1}{2} C_1 r^2 + C_2$$

or: $$u_\theta = \frac{1}{2} C_1 r + \frac{C_2}{r}$$

We have the no-slip B.C.'s:

$$u_\theta = \begin{cases} 0 & r = R_0 \\ \Omega R_1 & r = R_1 \end{cases}$$

Thus: $$\frac{1}{2} C_1 R_0 + \frac{C_2}{R_0} = 0$$

$$\frac{1}{2} C_1 R_1 + \frac{C_2}{R_1} = \Omega R_1$$

So: $$C_1 = \frac{-2C_2}{R_0^2}\ ; \quad C_2 = -\Omega\left(\frac{R_1^2 R_0^2}{R_1^2 - R_0^2}\right)$$

and: $$u_\theta = \Omega R_1 \left(\frac{R_1 R_0}{R_1^2 - R_0^2}\right)\left(\frac{r^2 - R_0^2}{R_0 r}\right)$$

(143)

We wish to calculate the torque on the inner cylinder. We have:

$$\underset{\sim}{M} = \underset{\sim}{r} \times \underset{\sim}{F}$$

Now the force F is just the shear stress $\tau_{r\theta}$ times the area of the cylinder. Recall $\tau_{r\theta} \equiv F/A$ exerted by fluid of greater r on fluid of lesser r in the θ direction!

So:

$$\underset{\sim}{M} = \overset{\text{lever arm}}{r} \cdot \underbrace{2\pi r h_1}_{\text{Area}}\ \tau_{r\theta}\ \underset{\sim}{\hat{e}}_z$$

In cylindrical coordinates:

$$\tau_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]$$

(144)

Now $\underline{u_r = 0}$. and u_θ is given by:

$$\frac{u_\theta}{r} = \frac{\Omega R_1^2}{R_1^2 - R_0^2}\left(1 - \frac{R_0^2}{r^2}\right)$$

So:

$$\tau_{r\theta} = 2\mu \frac{\Omega R_1^2}{R_1^2 - R_0^2}\ \frac{R_0^2}{r^2}$$

and hence the torque:

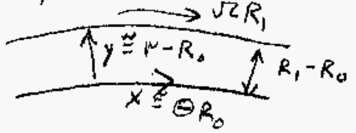
$$\underset{\sim}{M} = 4\pi\mu h \Omega \frac{R_1^2 R_0^2}{R_1^2 - R_0^2}\ \underset{\sim}{\hat{e}}_z$$

Note that this is independent of r! This makes sense: the torque exerted by the outer cylinder is the same as that exerted on the inner cylinder, and every cylindrical surface in between. Otherwise the flow would be accelerating (not at steady-state)!

OK, what about the thin-gap approximation? Just as the earth looks flat when viewed on a human length scale, so fluid mechanics problems may be simplified when characteristic lengths (e.g. the gap width between cylinders) is much smaller than the radius of curvature!

We take $\frac{R_1 - R_0}{R_1} << 1$

Locally, we define coordinates:



The force F is approximately:

$$F \approx \tau_{yx} \cdot 2\pi R_0 h$$

where: $\tau_{yx} \approx \mu \frac{\Omega R_1}{R_1 - R_0}$

So:

$$(\underset{\sim}{M})_{approx} = \mu \frac{\Omega R_1}{R_1 - R_0} R_0 \, 2\pi R_0 h \, \underset{\sim}{\hat{e}}_z$$

We can compare this to the exact result:

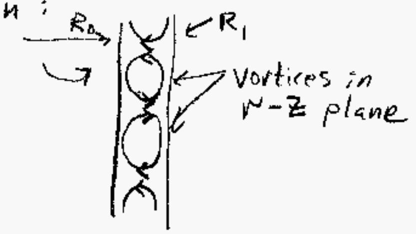
$$\frac{(\underset{\sim}{M})_{approx}}{(\underset{\sim}{M})_{exact}} = \frac{1}{2} \frac{R_1^2 - R_0^2}{R_1(R_1 - R_0)} = 1 - \frac{1}{2} \frac{R_1 - R_0}{R_1}$$

So if R_0 is 1" and $R_1 - R_0 = 0.02$" (about 500μm), then the error is only around 1%!

In this derivation we have assumed that $u_r = u_z = 0$. This will be valid provided the rotation rate is sufficiently small. At higher rotation rates the flow becomes unstable, yielding what are called Taylor-Couette vortices.

To see why, remember the centrifugal force term in the r-momentum eq'n: $\rho \frac{u_\theta^2}{r}$

Because u_θ is higher inside (smaller r) than outside (larger r), the fluid inside "wants" to flow out while that outside "wants" to flow in. This produces the vortex pattern:



The critical rotation speed at which vortices appear is given by:

$$Ta_{cr} = \frac{\Omega_{cr}^2 \bar{R} \, \Delta R^3}{\nu^2} = 1712 \quad \text{for } \frac{\Delta R}{R} << 1$$

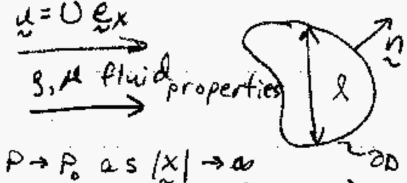
This phenomenon was first demonstrated by G.I. Taylor in 1923.

Note that if Ω is further increased, these vortices will themselves become unstable to other secondary flows – they become wavy in the Θ direction. Eventually the entire flow becomes turbulent.

Taylor-Couette flow is still actively studied today!

Dimensional Analysis

Now that we're familiar w/ the Navier-Stokes equations, let's use them to look at a more complex, general problem: Uniform flow past an arbitrary shape:



$p \to p_0$ as $|\underset{\sim}{x}| \to \infty$
(far away)

What is the drag (force) on the object??

The force exerted by the fluid on the object is:

$$\underset{\sim}{F} = \int_{\partial D} \underset{\approx}{\tau} \cdot \underset{\sim}{n} \, dA$$

where

$$\tau_{ij} \equiv -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

for an incompressible Newtonian fluid!

Thus, to calculate the force, we need the stress, and to get that we need $\underset{\sim}{u}$ and p! We thus have to solve the N-S eq'ns, which is very difficult for a complex geometry!

⇒ Suppose that, instead, we wish to do it experimentally, using a scale model system. How would this work? Effectively we are "solving" the equations using a

"mechanical computer" - if the assumptions used in deriving the equations are valid, the experiment should match the solution to the N-S eq'ns!

To work with a scale model (& interpret the results), we have to render the problem dimensionless w/ appropriate length & time scales.

✱ All dimensionless variables should be $O(1)$ in the region of interest!

Ok, let's see how this works: We have the Continuity Eq'n & the N-S eq'ns:

$\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = 0$ (incompressible)

$$\rho\left[\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla}\underset{\sim}{u}\right] = -\underset{\sim}{\nabla} p + \mu\nabla^2\underset{\sim}{u} + \rho\underset{\sim}{g}$$

(Newtonian fluid)

Let's choose U as the velocity scaling, ℓ as the length scale, ℓ/U as the time scale, and ΔP_c as the pressure scale (to be determined)

Thus:

$$\underset{\sim}{u}^* = \frac{\underset{\sim}{u}}{U}, \quad \underset{\sim}{x}^* = \frac{\underset{\sim}{x}}{\ell}, \quad \underset{\sim}{\nabla}^* = \ell\underset{\sim}{\nabla},$$

$$p^* = \frac{p - p_0}{\Delta P_c} \leftarrow \text{subtract off far-field pressure}$$

$$g^* = \frac{\underset{\sim}{g}}{g} \quad \text{(vector in gravity direction)}$$

OK, now we render this problem dimensionless:

$$\frac{U}{\ell}\,\underset{\sim}{\nabla}^*\cdot\underset{\sim}{u}^* = 0$$

$$\therefore\ \underset{\sim}{\nabla}^*\cdot\underset{\sim}{u}^* = 0 \quad \text{(unchanged)}$$

$$\rho\left[\frac{U}{\ell/U}\frac{\partial \underset{\sim}{u}^*}{\partial t^*} + \frac{U^2}{\ell}\,\underset{\sim}{u}^*\cdot\underset{\sim}{\nabla}^*\underset{\sim}{u}^*\right] = -\frac{\Delta P_c}{\ell}\,\underset{\sim}{\nabla}^* p^* + \mu\frac{U}{\ell^2}\,\nabla^{*2}\underset{\sim}{u}^* + \rho g\,\underset{\sim}{g}^*$$

Now we divide through by one of these terms to make the problem dimensionless. Which one? Pick a term representing a physical mechanism you're pretty sure is important!

⇒ At high velocities the inertial terms are likely to be important so:

Divide by $\frac{\rho U^2}{\ell}$:

$$\frac{\partial \underset{\sim}{u}^*}{\partial t^*} + \underset{\sim}{u}^*\cdot\underset{\sim}{\nabla}^*\underset{\sim}{u}^* = \frac{-\Delta P_c}{\rho U^2}\,\underset{\sim}{\nabla}^* p^* + \frac{\mu}{\rho U\ell}\,\nabla^{*2}\underset{\sim}{u}^* + \frac{g\ell}{U^2}\,\underset{\sim}{g}^*$$

At high velocities pressure gradients arise from inertial effects (e.g., convection of momentum), so we choose:

$$\frac{\Delta P_c}{\rho U^2} = 1$$

or $\Delta P_c = \rho U^2$ as the characteristic pressure differential!

Note that we have two dimensionless groups of parameters in the equations! The magnitude of these groups determine the relative importance of the dimensionless terms they multiply and the corresponding physical mechanisms!

What are they?

$$\frac{\mu}{\rho U\ell} \equiv \frac{1}{Re} = \frac{\text{viscous forces}}{\text{inertial forces}}$$

If $Re \gg 1$ then viscous forces are unimportant on a length scale ℓ comparable to the size of the body! We'll see later that they are important in boundary layers next to the body of thickness δ because without viscosity, you can't satisfy the no-slip condition!

$$\frac{g\ell}{U^2} = \frac{1}{Fr}$$

$Fr \equiv$ Froude # $= \frac{\text{Inertia}}{\text{Gravity}}$

This plays a role in free surface flows, such as the wake behind a ship (or a bow wave).

We also render the boundary conditions dimensionless:

$$\underset{\sim}{u}\Big|_{|\underset{\sim}{x}|\to\infty} = U\,\hat{\underset{\sim}{e}}_x$$

$$\text{so}\quad \underset{\sim}{u}^*\Big|_{|\underset{\sim}{x}^*|\to\infty} = \hat{\underset{\sim}{e}}_x$$

$$\underset{\sim}{u}\Big|_{\underset{\sim}{x}\in\partial D} = 0 \quad \text{so} \quad \underset{\sim}{u}^*\Big|_{\underset{\sim}{x}^*\in\partial D^*} = 0$$

Sometimes you get additional dimensionless groups from B.C.'s, say if object is rotating.

Here there are only *two* dimensionless groups which contain *all* the dimensional information! If these are held *constant* between the model & the full size system, the dimensionless flow will be *exactly the same*!!

This is known as *dynamic similarity*!

Ok, how could we use this? Suppose we want to model a submarine with a 1/100 scale model, preserving dynamic similarity.

If there's no free surface, Fr doesn't matter, so we just have to keep Re cst. Let L_1 = length of sub, L_2 = length of model

For dynamic similarity, $Re_1 = Re_2$

so: $$\frac{U_1 L_1}{\nu_1} = \frac{U_2 L_2}{\nu_2}$$

If both experiments are in water, then $\nu_1 = \nu_2$

so: $$\frac{U_2}{U_1} = \frac{L_1}{L_2}$$

or $$U_2 = U_1\left(\frac{L_1}{L_2}\right) \equiv U_1\left(\frac{100}{1}\right)$$

Note that this is *really* hard! if $U_1 = 40$ mph, $U_2 = 4{,}000$ mph! Note that even if we *could* get

the model up to this speed, it *still* wouldn't achieve similarity! Our *assumptions* break down because $U_2/U_s \not\ll 1$ (e.g., the Mach # isn't small) and so the fluid is *compressible*.

It *can* work well, however – suppose we want to look at the flow patterns in a big tank of Karo syrup. We model this with a small tank of water.

$\nu_1 = 25$ stokes $\nu_2 = 0.01$ stokes

$L_1 = 20$ ft $L_2 = 2''$

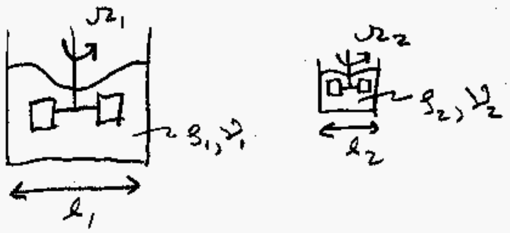
Ok, what's U_2?

$$U_2 = \frac{\nu_2}{\nu_1}\frac{L_1}{L_2}U_1 = \left(\frac{0.01}{25}\right)\left(\frac{240}{2}\right)U_1$$

so $$U_2 = 0.048\ U_1$$

if $U_1 = 1$ ft/s, $U_2 = 1.46$ cm/s which is a reasonable value!

If there is a free surface we have to preserve *both* Re & Fr! As an example, consider a vortex in an agitated tank:



To preserve dynamic similarity we require:

$$Re_1 = Re_2 \ ; \ Fr_1 = Fr_2$$

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where $Re = \frac{U\ell}{\nu}$, $Fr = \frac{U^2}{\ell g}$

Note: $U \sim \Omega \ell$ since all <u>geometric ratios</u> must be preserved as well! So:

$$\frac{\Omega_1 \ell_1^2}{\nu_1} = \frac{\Omega_2 \ell_2^2}{\nu_2}$$

$$\frac{\Omega_1^2 \ell_1}{g} = \frac{\Omega_2^2 \ell_2}{g}$$

Suppose we are modeling a tank of glycerin w/ one of water. This <u>fixes</u> the ratio ν_1/ν_2

so: $\frac{\Omega_1^2}{\Omega_2^2} = \frac{\ell_2}{\ell_1}$; $\frac{\Omega_1 \ell_1^2}{\Omega_2 \ell_2^2} = \frac{\nu_1}{\nu_2}$

(162)

So $\left(\frac{\ell_1}{\ell_2}\right)^{3/2} = \frac{\nu_1}{\nu_2}$

or $\ell_2 = \ell_1 \left(\frac{\nu_2}{\nu_1}\right)^{2/3}$

If $\nu_1/\nu_2 = 1000$ (about right)

we get $\ell_2 = \frac{\ell_1}{100}$

The <u>angular</u> velocity of the impeller is <u>increased</u>:

$$\Omega_2 = \Omega_1 \left(\frac{\ell_1}{\ell_2}\right)^{1/2} = \Omega_1 \left(\frac{\nu_1}{\nu_2}\right)^{1/3}$$

What would be the <u>power</u> <u>input</u>?

$$\text{Power} \sim \underset{\substack{\uparrow \\ \text{angular} \\ \text{velocity}}}{\Omega} \cdot \Big(\underbrace{\underset{\substack{\uparrow \\ \text{char} \\ \text{velocity}}}{(\Omega \ell)^2} \rho}_{\text{product} \sim \text{pressure}} \cdot \overset{\text{area}}{\ell^2} \cdot \underset{\text{lever arm}}{\ell} \Big) f(Re,$$

$\sim$ (ang. velocity) $\cdot$ (Pressure)(Area)(lever arm)

(163)

$\sim \Omega^3 \ell^5 \rho \, f(Re, Fr)$

<u>But</u> if Re, Fr are <u>constant</u> between model system and original, $f(Re, Fr)$ (unknown Re & Fr dependence) will <u>also</u> be constant!

Thus:

$$\frac{(\text{Power})_1}{(\text{Power})_2} = \frac{\Omega_1^3 \ell_1^5 \rho_1}{\Omega_2^3 \ell_2^5 \rho_2}$$

$$= \left(\frac{\nu_2}{\nu_1}\right) \left(\frac{\nu_1}{\nu_2}\right)^{10/3} \frac{\rho_1}{\rho_2}$$

$$= \left(\frac{\nu_1}{\nu_2}\right)^{7/3} \frac{\rho_1}{\rho_2}$$

which allows us to estimate the power requirements of the full-scale system!

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While <u>strict</u> Dynamic Similarity is often <u>very</u> difficult (or impossible) to achieve, a more approximate form is much easier and more practical. An excellent example is in hull design for surface ships. <u>Strict</u> similarity requires <u>both</u> Re & Fr to be preserved between model and full scale, which isn't really possible. If Re is "high" for both ship & model, however, we may be at some "high Re limit" where viscous effects are unimportant. That would mean that only Fr would have to be kept constant - <u>much</u> easier!

(165)

Let's see how this works:

We wish to model the behavior of the Enterprise (CVN 65) with a 1/100 scale model. In this case $U_1 \approx$ 40 mph = 1,800 cm/s, $L_1 \cong 1000$ ft = 3.0×10^4 cm, $\nu_1 = 0.01$ stokes

Thus: $Re_1 = 5.4\times10^9$, $Fr_1 = 0.11$

We give up on Re, but try to match Fr:

$$\frac{U_1^2}{L_1 g} = \frac{U_2^2}{L_2 g} \quad \therefore \; U_2 = U_1\left(\frac{L_2}{L_1}\right)^{1/2}$$

or, since $L_2/L_1 = \frac{1}{100}$,

$$U_2 = \frac{1}{10}U_1 = 4 \text{ mph} - \text{not bad!}$$

We also have:

$$Re_2 = \frac{U_2 L_2}{\nu_2} = 10^{-3} Re_1 = 5.4\times10^6$$

(166)

which is still pretty large!

What about the relation between the force on the model and the force on the ship? If viscosity is unimportant we get:

$$\frac{F_1}{\rho_1 U_1^2 L_1^2} \cong \frac{F_2}{\rho_2 U_2^2 L_2^2}$$

or

$$\frac{F_1}{F_2} \approx \frac{U_1^2 L_1^2}{U_2^2 L_2^2} = 10^6$$

Provided that Re_2 is large enough that the flow around the model is fully turbulent ($Re_2 >> 10^5$ or so) this actually works pretty well! This has been the basis for testing ship designs over the past century!

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So far we've scaled the N-S equations using the inertial terms (convection of momentum). This is appropriate for $Re >> 1$. What about low Re?? Here we use the viscous scalings! Recall:

$$\rho\frac{U^2}{\ell}\left[\frac{\partial \underset{\sim}{u}^*}{\partial t^*} + \underset{\sim}{u}^*\cdot\underset{\sim}{\nabla}^*\underset{\sim}{u}^*\right] = -\frac{\Delta P_c}{\ell}\underset{\sim}{\nabla}^* p^* + \frac{\mu U}{\ell^2}\nabla^{*2}\underset{\sim}{u}^* + \rho g\,\underset{\sim}{g}^*$$

This time we divide thru by viscous scaling $\frac{\mu U}{\ell^2}$:

$$\left(\frac{\rho U \ell}{\mu}\right)\left[\frac{\partial \underset{\sim}{u}^*}{\partial t^*} + \underset{\sim}{u}^*\cdot\underset{\sim}{\nabla}^*\underset{\sim}{u}^*\right] = -\left(\frac{\Delta P_c \ell}{U\mu}\right)\underset{\sim}{\nabla}^* p^* + \nabla^{*2}\underset{\sim}{u}^* + \frac{\rho g \ell^2}{\mu U}\,\underset{\sim}{g}^*$$

(168)

Now we choose ΔP_c s.t.

$$\frac{\Delta P_c\,\ell}{U\mu} = 1$$

or $\Delta P_c = \mu\frac{U}{\ell}$ (scaling for shear stress)

Thus:

$$Re\left[\frac{\partial \underset{\sim}{u}^*}{\partial t^*} + \underset{\sim}{u}^*\cdot\underset{\sim}{\nabla}^*\underset{\sim}{u}^*\right] = -\underset{\sim}{\nabla}^* p^* + \nabla^{*2}\underset{\sim}{u}^* + \frac{Re}{Fr}\,\underset{\sim}{g}^*$$

Now if $Re << 1$ we neglect terms which are of O(Re):

$$-\underset{\sim}{\nabla}^* p^* + \nabla^{*2}\underset{\sim}{u}^* = O\left(Re, \frac{Re}{Fr}\right)$$

or $\nabla^{*2}\underset{\sim}{u}^* \cong \underset{\sim}{\nabla}^* p^*$

and the CE: $\underset{\sim}{\nabla}^*\cdot\underset{\sim}{u}^* = 0$

These are the Creeping Flow Eq'ns: Starting point for low Re flow!

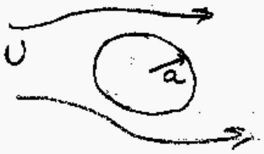
(169)

So far we've used our knowledge of the flow equations to determine conditions where flows will be dynamically similar. This wasn't really necessary => all that we really had to know was what <u>physical parameters</u> a problem depends on! This is known as <u>Dimensional Analysis</u>

The <u>key</u> is that <u>Nature Knows No Units</u>: A "foot" or a "meter" has <u>no physical significance</u>. Thus, any physical relationship <u>must</u> be expressible as a relationship between <u>dimensionless quantities</u>!

(170)

Let's see how this works - Consider drag on a sphere:



The <u>force</u> is a function of U, a, μ, ρ, but all these are <u>dimensional quantities</u>. How many <u>dimensionless groups</u> can be formed?

Buckingham π theorem:

\# dimensionless groups = \# parameters − <u>rank</u> of dimensional matrix (this is the number of <u>independent</u> fundamental units involved in the problem)

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The dimensional matrix is given by:

$$\begin{array}{c|ccccc} & F & a & \rho & \mu & U \\ \hline M & 1 & 0 & 1 & 1 & 0 \\ L & 1 & 1 & -3 & -1 & 1 \\ T & -2 & 0 & 0 & -1 & -1 \end{array}$$

Rank = dimension of largest sub-matrix w/ non-zero determinant!

⇒ In this case, we take the first three columns:

$$\left| \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -3 \\ -2 & 0 & 0 \end{bmatrix} \right| = 2 \neq 0 \quad \text{so rank} = 3$$

By the π theorem:

\# dimensionless groups = 5 − 3 = 2

and thus

$$\pi_1 = f(\pi_2)$$

(172)

We may <u>choose</u> π_1 & π_2 ~~any~~ way we wish <u>provided</u> they are 1) dimensionless and 2) independent (this means that if there are N π groups, the N^{th} <u>can't</u> be formed by any combination of the other N−1 groups!)

We usually choose groups so that <u>one</u> involves the <u>dependent parameter</u> of interest, and all the others involve combinations of the <u>independent parameters</u>.

One choice:

$$\frac{F}{\rho U^2 a^2} = f^{\underline{n}}\left(\frac{U a \rho}{\mu}\right) \equiv f^{\underline{n}}(Re)$$

or, in words, the *Dimensionless Drag* is only a function of the Reynolds Number! This is exactly what we got from scaling the N-S eq^ns!

Often we can *strengthen* results if we have additional physical insight. Suppose we have $Re << 1$. In this case we expect inertia (& hence ρ) is unimportant:

$$\begin{array}{c|cccc} & F = & f^n(U & \mu & a) \\ \hline M & 1 & 0 & 1 & 0 \\ L & 1 & 1 & -1 & 1 \\ T & -2 & -1 & -1 & 0 \end{array}$$

$\hookrightarrow$ rank $= 3$

$\therefore\ 4-3=\underline{\underline{1}}$ group

This is the *strongest possible result*:

$$\frac{F}{\mu U a} = \underline{\underline{cst}} = 6\pi$$

where the constant is determined by solving Stokes flow eq^ns (or from *one* experiment).

It is *extremely important* to have a *complete* list of the applicable parameters. Otherwise the result will be incorrect. As an example, look at flow past an *infinitely long* cylinder of radius a:

$$\underset{\overset{\uparrow}{\text{force/length}}}{F/L} = f^n(U, a, \mu, \rho)$$

Again, there are $5-3=2$ Dimensionless groups:

$$\frac{F/L}{\rho U^2 a} = f^n\left(\frac{U a \rho}{\mu}\right)$$

So this works fine! What about low Re?? We anticipate that ρ (inertia) doesn't matter, so we have:

$$F/L = f^n(U, a, \mu)$$

or $4-3=1$ Dimensionless groups.

Thus:

$$\frac{F/L}{U\mu} = \underline{cst}$$

But this suggests that the *Drag* is *independent* of a! This *can't* be correct! This reflects the

fact that inertia is *always* important: there is *no solution* to the Stokes Eq^ns for 2-D flow past a cylinder! This is known as *Stokes' Paradox*

An approximate solution for $Re << 1$ is given by Lamb:

$$F/L \approx 4\pi \frac{U\mu}{\ln(4/Re) - \underset{\overset{\uparrow}{\text{Euler's Const}}}{\gamma} + \frac{1}{2}}$$

which depends on Re even as $Re \to 0$!

The complete reduction of a problem to a single Dimensionless group sometimes happens even for functions. The best example of this is the

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expanding shockwave due to a point source explosion studied by GI Taylor during WWII. The radius R of the shock will be a function of time t, the density of the gas (before the explosion) ρ_0, the energy E, the adiabatic exponent $\gamma = 7/5$ for a diatomic gas, and the initial atmospheric pressure P_0.

Thus:

$$\begin{array}{c|cccccc} & R = f^n(t ; & \rho_0, & E, & P_0, & \overset{\text{Dimensionless}}{\gamma}) \\ \hline M & 0 & 0 & 1 & 1 & 1 & 0 \\ L & 1 & 0 & -3 & 2 & -1 & 0 \\ T & 0 & 1 & 0 & -2 & -2 & 0 \end{array}$$

Thus 6-3 = 3 groups!

One is obviously γ, but this won't change if we keep using air!

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We <u>can</u> construct a reference length and time:

$$\frac{R}{(E/P_0)^{1/3}} = f^n\left(\frac{t}{\left(\frac{E^2\rho_0^3}{P_0^5}\right)^{1/6}} , \gamma \right)$$

Which <u>isn't</u> particularly useful. Still, we could plug this into the shock eq'ns & try to solve the problem.

<u>Instead</u> we look at the case of <u>strong</u> explosions such that

$$\frac{P_0 R^3}{E} << 1$$

In this case the pressure inside the shock is <u>far</u> greater than that due to the atmosphere P_0. Thus, we shall assume that P_0 <u>doesn't</u> <u>matter</u>!

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Thus we have:

$$R = f^n(t ; \rho_0 , E , \gamma)$$

or 5-3 = 2 groups!

Since one is still γ, the other is:

$$\frac{R}{\left(\frac{Et^2}{\rho_0}\right)^{1/5}} = f(\gamma) = \text{cst for diatomic gases!}$$

It turns out that this constant is 1.033 from solution of the flow equations! Thus $R \sim t^{2/5}$ and with knowledge of R & t you can calculate E. This was done by Taylor from images of the NM atom bomb tests - while the yields were still classified Top Secret!

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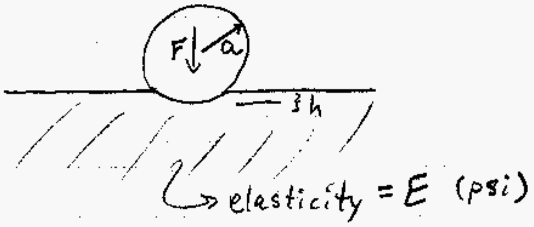
As a last point, while # indep. fudamental units = # fundamental units, this <u>isn't</u> <u>always</u> <u>true</u>. As an example, consider the deflection produced by a ball sitting on an elastic solid (e.g. a ball bearing on a block of rubber):



$$h = f^n(F, a, E)$$

these involve M, L, & T, so we <u>might</u> expect 4-3 = 1 dimensionless groups! Thus $\frac{h}{a}$ = cst ??

This can't be correct, since elasticity E must matter! The problem is in the rank of the dimensional matrix!

$$
\begin{array}{c|cccc}
 & h & F & a & E \\
\hline
M & 0 & 1 & 0 & 1 \\
L & 1 & 1 & 1 & -1 \\
T & 0 & -2 & 0 & -2
\end{array}
$$

$$h = f^{n}(F, a, E)$$

There exists no 3x3 matrix w/ non-zero determinant, thus rank = 2

So:

$$\frac{h}{a} = f^{n}\left(\frac{F}{Ea^2}\right)$$

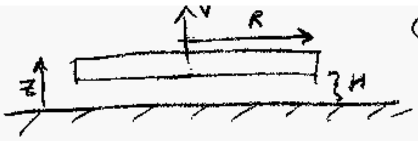
which makes more sense!

Dimensional Analysis is powerfull, but be careful and always check to see if your results make sense!

Lubrication Flows

An important problem in fluid mechanics is lubrication theory: the study of the flow in thin films, where hydrodynamic forces keep solid surfaces out of contact, reducing wear. These problems are actually quite simple to solve due to a separation of length scales (one dimension >> another) which leads to the quasi parallel flow approximation.

Let's see how this works. Suppose we look at the squeeze flow between a disk and a plane:



The flow in the narrow gap $H/R \ll 1$ will resist the upward motion of the disk. We want to calculate this force. The flow is three dimensional, but $u_\theta = 0$ & we can neglect θ derivatives! Let's start with the C.E.:

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \underset{0}{\cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}}} + \frac{\partial u_z}{\partial z} = 0$$

Because $H/R \ll 1$ we expect that u_r & u_z will need different scales! Let $u_z^* = \frac{u_z}{V}$, $u_r^* = \frac{u_r}{U}$

We also take:

$$r^* = r/R \quad , \quad z^* = z/H$$

So: $$\frac{U}{R}\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^* u_r^*) + \frac{V}{H}\frac{\partial u_z^*}{\partial z^*} = 0$$

or, dividing through:

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^* u_r^*) + \frac{RV}{UH}\frac{\partial u_z^*}{\partial z^*} = 0$$

Both terms of the CE must be of the same order for any 2-D problem! Thus we take:

$$\frac{RV}{UH} = 1 \quad \text{or} \quad U = \frac{R}{H}V \gg V$$

Thus the velocity along the gap is much higher (by $O(R/H)$) than the velocity perpendicular to the gap! This means we have quasi-parallel flow in the radial direction

Now for the momentum equations:

Let $t^* = \frac{Vt}{H}$, $p^* = \frac{p - p_0}{\Delta P_c}$

$\hookrightarrow$ char. time scale $= H/V$

r-momentum:

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial p}{\partial r}$$

$$+ \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right) + \frac{\partial^2 u_r}{\partial z^2}\right]$$

where we have ignored θ terms

Scaling:

$$\rho\frac{U^2}{R}\left(\frac{\partial u_r^*}{\partial t^*} + u_r^*\frac{\partial u_r^*}{\partial r^*} + u_z^*\frac{\partial u_r^*}{\partial z^*}\right)$$

$$= -\frac{\Delta P_c}{R}\frac{\partial p^*}{\partial r^*} + \mu\frac{U}{H^2}\left[\frac{H^2}{R^2}\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^* u_r^*)\right) + \frac{\partial^2 u_r^*}{\partial z^{*2}}\right]$$

In lubrication flows we expect viscous terms to dominate, so

divide through by $\mu\frac{U}{H^2}$ (leading term)

$$\left(\frac{\rho U H}{\mu}\right)\left(\frac{H}{R}\right)\left(\frac{\partial u_r^*}{\partial t^*} + u_r^*\frac{\partial u_r^*}{\partial r^*} + u_z^*\frac{\partial u_r^*}{\partial z^*}\right)$$

$$= -\left(\frac{\Delta P_c H^2}{R\mu U}\right)\frac{\partial p^*}{\partial r^*} + \frac{H^2}{R^2}\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^* u_r^*)\right)$$

$$+ \frac{\partial^2 u_r^*}{\partial z^{*2}}$$

We choose the viscous scaling for the pressure:

$$\left(\frac{\Delta P_c H^2}{R\mu U}\right) = 1 \quad \therefore \quad \Delta P_c = \frac{R\mu U}{H^2}$$

and thus, putting all terms of O(1) on the LHS:

$$\frac{\partial^2 u_r^*}{\partial z^{*2}} - \frac{\partial p^*}{\partial r^*} = -\frac{H^2}{R^2}\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^* u_r^*)\right)$$

$$+\left(\frac{\rho V H}{\mu}\right)\left(\frac{\partial u_r^*}{\partial t^*} + u_r^*\frac{\partial u_r^*}{\partial r^*} + u_z^*\frac{\partial u_r^*}{\partial z^*}\right)$$

Provided that $H/R \ll 1$ we can neglect the r-diffusion terms, and provided $\left(\frac{\rho V H}{\mu}\right) \ll 1$ we can ignore the inertial terms.

Thus: $$\frac{\partial^2 u_r^*}{\partial z^{*2}} = \frac{\partial p^*}{\partial r^*}$$

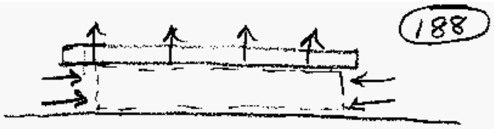
which is just channel flow! ($p^* \neq f^*(z^*)$)

with boundary conditions:

$u_r^*\big|_{z^*=0,1} = 0$ we get:

$$u_r^* = -\frac{\partial p^*}{\partial r^*}\,\frac{1}{2}\,z^*(1 - z^*)$$

Now we still need to figure out the pressure gradient. We do this from a mass balance



Flow out thru top $= V\pi r^2$

Flow in thru sides $= -2\pi r\int_0^H u_r\,dz$

These must balance!

$$V\pi r^2 = -2\pi r\int_0^H u_r\,dz$$

or $$\int_0^1 u_r^*\,dz^* = -\frac{1}{2}r^*$$

So: $$\int_0^1 -\frac{1}{2}\frac{\partial p^*}{\partial r^*}\,z^*(1 - z^*)\,dz^* = -\frac{1}{2}r^*$$

$$\frac{\partial p^*}{\partial r^*} = \frac{r^*}{\int_0^1 z^*(1-z^*)\,dz^*} = \underline{\underline{6r^*}}$$

Now since $p^*\big|_{r^*=1} = 0$, we get

$$p^* = -3(1 - r^{*2})\,!$$

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The force is just the integral of the pressure (normal force)

$$F = \int_0^R P \, 2\pi r \, dr = \Delta P_c R^2 \, 2\pi \int_0^1 P^* r^* dr^*$$

$$= -\frac{3\pi}{2}\left(\frac{R\mu U}{H^2}\right)R^2$$

or, since $U = V\frac{R}{H}$,

$$F = -\frac{3\pi}{2}\frac{\mu V R^4}{H^3}$$

Note the force blows up as $H \to 0$! This is characteristic of lubrication flows!

How long does it take to detatch from the plane? It spends all the time travelling the first little bit!

For a constant force F:

(190)

$$\underset{\uparrow}{F} = +\frac{3\pi}{2}\frac{\mu R^4}{H^3}\underset{\uparrow}{\frac{dH}{dt}}$$

applied force - balances resistance   = V

$$\therefore \; F = -\frac{3\pi}{4}\mu R^4 \frac{d(H^{-2})}{dt}$$

$$\text{So} \quad H^{-2} = \underset{\uparrow}{H_0^{-2}} + \frac{-4}{3\pi}\frac{F}{\mu R^4}t$$

initial separation

Thus we have fallen away when $H^{-2} \cong 0$! This occurs when:

$$t_{\infty} = \frac{3\pi}{4}\frac{\mu R^4}{H_0^2}\frac{1}{F}$$

which approaches infinity as $H_0 \to 0$

I developed a technique based on this "fall time" concept to measure the roughness of spheres. Basically, the surface imperfections control the initial separation.

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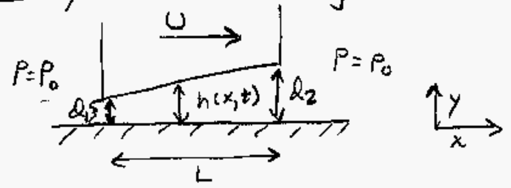
An important problem in lubrication theory is the sliding block:



If $H \equiv d_1 << L$ we can use lubrication theory to calculate the upward force on the block for some U, d_1, d_2, L, etc.

We have the equations:

$$\rho\left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\underset{\sim}{u}\right) = -\underset{\sim}{\nabla}P + \mu\nabla^2\underset{\sim}{u}$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 0$$

The flow is two-dimensional, so we take $u \equiv u_x$, $v \equiv u_y$ and $u_z = \frac{\partial}{\partial z} = 0$ (no z-dependence)

(192)

We have the C.E.:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

As before, we scale u w/ U; x w/ L; and y w/ H:

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{H}, \quad u^* = \frac{u}{U}$$

where all *'d variables are $O(1)$ in the region of interest. To preserve both terms in the C.E. we require:

$$v^* = \frac{v}{U\frac{H}{L}}$$

$$\text{Thus} \quad \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

We shall define $\varepsilon \equiv H/L << 1$

Thus v is $O(\varepsilon U)$.

OK, now for x-momentum:

(193)

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Let $t^*=\frac{Ut}{L}$ (e.g., $t_c\equiv\frac{L}{U}$)

and $P^*=\frac{P-P_0}{\Delta P_c}$

Plugging in,

$$\rho\frac{U^2}{L}\left(\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right)=$$

$$-\frac{\Delta P_c}{L}\frac{\partial P^*}{\partial x^*}+\mu\left(\frac{U}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{U}{H^2}\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

We anticipate that the dominant mechanism for momentum transport is viscous shear stresses in the narrow gap. Thus we divide by $\mu\frac{U}{H^2}$, its scaling!

(194)

So:

$$\left(\frac{\rho UH}{\mu}\right)\left(\frac{H}{L}\right)\left(\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right)=$$

$$-\frac{\Delta P_c}{\left(\frac{U\mu L}{H^2}\right)}\frac{\partial P^*}{\partial x^*}+\frac{\partial^2 u^*}{\partial y^{*2}}+\frac{H^2}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}}$$

Thus we have the pressure scale

$$\Delta P_c=\frac{U\mu L}{H^2}$$

and:

$$\frac{\partial^2 u^*}{\partial y^{*2}}=\frac{\partial P^*}{\partial x^*}-\varepsilon^2\frac{\partial^2 u^*}{\partial x^{*2}}+\varepsilon Re\left(\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right)$$

We shall ignore terms of $O(\varepsilon^2)$ and $O(\varepsilon Re)$. Thus:

$$\frac{\partial^2 u^*}{\partial y^{*2}}=\frac{\partial P^*}{\partial x^*}$$

(195)

Note that we can determine the scale of the force on the block with no further work! The upward force is just:

$$\frac{F}{W}=\int_0^L (P-P_0)\,dx$$

or $\frac{F}{W}=\Delta P_c L\int_0^1 P^*\,dx^*$

so $F=\frac{U\mu L^2 W}{H^2}\cdot\text{cst}$

where to get the cst we have to solve the problem!

Now for the y-momentum eq'n:

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

(196)

Plugging in our scalings we get:

$$\frac{\partial P^*}{\partial y^*}=\varepsilon^2\frac{\partial^2 v^*}{\partial y^{*2}}+\varepsilon^4\frac{\partial^2 v^*}{\partial x^{*2}}+\varepsilon^3 Re\left(\frac{\partial v^*}{\partial t^*}+u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}\right)$$

Thus, if we ignore terms of $O(\varepsilon^2, \varepsilon^4, \varepsilon^3 Re)$ we get:

$$\frac{\partial P^*}{\partial y^*}=0 \quad !$$

This is generically true for problems with separations of length scales $H/L \ll 1$, which also occurs in boundary layer flows we'll study later. Basically, you don't get variations in pressure across the thin dimension, in this case the gap!

OK, we have the scaled eq'ns:

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0$$

$$\frac{\partial^2 u^*}{\partial y^{*2}}=\frac{\partial P^*}{\partial x^*} \quad ; \quad \frac{\partial P^*}{\partial y^*}=0$$

Now for the B.C.'s:

$$u^*\Big|_{y^*=0}=v^*\Big|_{y^*=0}=0 \quad \left(\begin{array}{l} y^*=0 \text{ surface} \\ \text{is stationary} \end{array}\right)$$

and for the moving surface:

$$h^*=h/H$$

Let $u^*\Big|_{y^*=h^*}=U^*=\frac{U(x,t)}{U}$

In our case $U^*=1$ (uniform velocity), but in general it could be a function of both x & t.

Likewise: $v^*\Big|_{y^*=h^*}=V^*=\frac{V(x,t)}{\varepsilon U}$

For our example problem $V^*=0$

To solve this problem we <u>integrate</u> the x-momentum eq'n over y! We can do this because P^* <u>isn't</u> a function of y! (e.g., $\frac{\partial P^*}{\partial y^*}=0$)

So:

$$u^*=\frac{1}{2}\left(\frac{\partial P^*}{\partial x^*}\right)y^{*2}+C_1(x,t)\,y^*+C_2(x,t)$$

If we apply B.C. at $y^*=0$ we get $\underline{\underline{C_2(x,t)=0}}$

Applying B.C. at $y^*=h^*$ gives:

$$u^*=\underbrace{\frac{1}{2}\left(\frac{\partial P^*}{\partial x^*}\right)y^*(y^*-h^*)}_{\text{Channel flow}}+\underbrace{U^*\frac{y^*}{h^*}}_{\text{simple Shear}}$$

So u^* is just the sum of channel & shear flow!

We <u>still</u> need an equation for P^*. To get it we look at the C.E.:

$$\frac{\partial v^*}{\partial y^*}=-\frac{\partial u^*}{\partial x^*}$$

We can integrate this to get v^*:

$$v^*=\int_0^{y^*}\left(-\frac{\partial u^*}{\partial x^*}\right)dy^*$$

The lower limit is zero to satisfy the B.C. $v^*\Big|_{y^*=0}=0$

We can evaluate this at $y^*=h^*$:

$$v^*\Big|_{y^*=h^*}=V^*=\int_0^{h^*}\left(-\frac{\partial u^*}{\partial x^*}\right)dy^*$$

This gives us an equation for the pressure gradient!

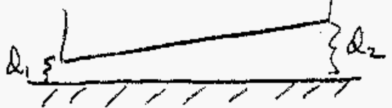
$$V^*=\frac{1}{12}\frac{\partial^2 P^*}{\partial x^{*2}}h^{*3}+\frac{1}{4}\frac{\partial P^*}{\partial x^*}h^{*2}\frac{\partial h^*}{\partial x^*}$$
$$-\frac{1}{2}\frac{\partial U^*}{\partial x^*}h^*+\frac{1}{2}U^*\frac{\partial h^*}{\partial x^*}$$

We can rearrange this:

$$\frac{\partial}{\partial x^*}\left(h^{*3}\frac{\partial P^*}{\partial x^*}\right)=6\Big[h^*\frac{\partial U^*}{\partial x^*}-U^*\frac{\partial h^*}{\partial x^*}+2V$$

This is known as the <u>Reynolds Lubrication Equation</u>. Together w/ the B.C.'s $P^*\Big|_{x^*=0}=P^*\Big|_{x^*=1}=0$ we can calculate the pressure!

OK, let's apply this:



$H\equiv d_1$, $\quad h=\frac{d_2-d_1}{L}x+d_1$

In dimensionless form,

$$h^* = 1 + \frac{d_2 - d_1}{d_1}\, x^*$$

we also have: ↘ let $d_2 - d_1 = \Delta d$

$$U^* = 1,\quad V^* = 0$$

$$\frac{d}{dx^*}\left(h^{*3}\frac{dp^*}{dx^*}\right) = -6\,\frac{\Delta d}{d_1}$$

Integrating once:

$$h^{*3}\,\frac{dp^*}{dx^*} = -6\,\frac{\Delta d}{d_1}\,x^* + C_1$$

Dividing by h^{*3} and integrating again:

$$p^* = -6\,\frac{\Delta d}{d_1}\int_0^{x^*}\frac{x^*}{\left(1+\frac{\Delta d}{d_1}x^*\right)^3}\,dx^* + C_1\int_0^{x^*}\frac{dx^*}{\left(1+\frac{\Delta d}{d_1}x^*\right)^3}\,dx^*$$

where the second constant of

integration vanishes because $p^*|_{x^*=0} = 0$.

Evaluating this at $x^* = 1$ and applying the $p^*|_{x^*=1} = 0$ B.C. yields

$$C_1 = \frac{6\,\frac{\Delta d}{d_1}\int_0^1 \frac{x\,dx}{h^3}}{\int_0^1 \frac{dx}{h^3}} = 6\left(\frac{\Delta d}{d_1 + d_2}\right)$$

So:

$$p^* = 6\left(\frac{\Delta d}{d_1 + d_2}\right)\frac{x^* - x^{*2}}{\left(1+\frac{\Delta d}{d_1}x^*\right)^2}$$

The force is just the integral of this:

$$F^* \equiv \frac{F}{\frac{U\mu L^2 W}{H^2}} = \int_0^1 p^*\,dx^*$$

$$= 6\left(\frac{d_1}{\Delta d}\right)\left[\left(\frac{d_1}{\Delta d}\right)\ln\left(1+\frac{\Delta d}{d_1}\right) - \frac{1}{1+\frac{1}{2}\frac{\Delta d}{d_1}}\right]$$

which looks pretty complex! In the limit $\Delta d / d_1 \ll 1$, however, we get:

$$F^* = \frac{1}{2}\,\frac{\Delta d}{d_1} + O\left(\left(\frac{\Delta d}{d_1}\right)^2\right)$$

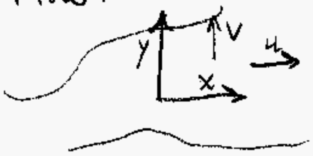
which is a pretty simple result! Note that everything *except* the numerical value of the coefficient could have been obtained *without* solving the equations! This is the importance of knowing how to *scale the equations*!

The Stream Function

Lubrication flows were an example of quasi-parallel flows: flows where the characteristic length scales were sufficiently different that the 2-D flow was essentially 1-D.

If the length scales are *not* different, a 2-D flow *remains* 2-D & we must use a different approach!

Suppose we have an incomp. 2-D flow:



(205)

We have the C.E:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

If we define the scalar function $\psi(x,y)$ by:

$$u=\frac{\partial \psi}{\partial y}\ ,\quad v=-\frac{\partial \psi}{\partial x}$$

this has the property that the CE is satisfied automatically:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\equiv\frac{\partial^2 \psi}{\partial x\partial y}-\frac{\partial^2 \psi}{\partial y\partial x}=0$$

Basically, by doing this stream function substitution we are reducing the number of dependent variables (e.g. u,v to ψ) while increasing the order of the differential equation

(206)

The stream function has many useful properties! First, it is constant along a streamline.

Remember the material derivative?

$$\frac{D\phi}{Dt}\equiv\frac{\partial \phi}{\partial t}+\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\phi$$

the 2nd term is the change in the direction of motion! For the stream f^{n} $\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\psi=0$

We can prove this:

$$\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\psi\equiv u\frac{\partial \psi}{\partial x}+v\frac{\partial \psi}{\partial y}$$

$$=\frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x}-\frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y}=0$$

So curves of constant ψ are streamlines: they follow the motion of fluid elements! That's why ψ is called the stream function!

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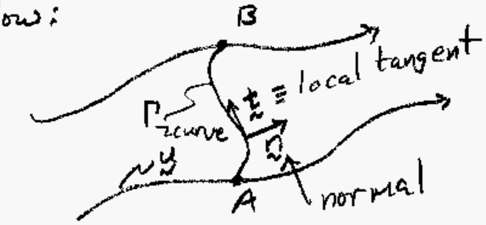
Another property: suppose we want to calculate the flowrate through any segment of the flow:



$$\underset{\substack{\uparrow\\ \text{extension in}\\ \text{3rd direction}}}{\frac{Q}{W}}=\int_{\Gamma}(\underset{\sim}{u}\cdot\hat{n})\,\underset{\substack{\uparrow\\ \text{path integral}}}{ds}$$

We can evaluate this for any Γ connecting A & B using the stream function!

(208)

For a unit normal:

$$\hat{n}=(n_x,n_y)$$

the tangent is $(-n_y,n_x)$

So:

$$\frac{Q}{W}=\int_{\Gamma}(u,v)\cdot(n_x,n_y)ds$$

$$=\int_{\Gamma}\left(\frac{\partial \psi}{\partial y},-\frac{\partial \psi}{\partial x}\right)\cdot(n_x,n_y)ds$$

$$=\int_{\Gamma}(-n_y,n_x)\cdot\left(\frac{\partial \psi}{\partial x},\frac{\partial \psi}{\partial y}\right)ds$$

$$=\int_{\Gamma}\underbrace{\underset{\sim}{t}\cdot\underset{\sim}{\nabla}\psi}_{\hookrightarrow\ \text{variation of }\psi\text{ along }\Gamma}ds\equiv\psi(B)-\psi(A)$$

So the flowrate through any arc from A to B is just the difference in the streamfunction at these points!

OK, how do you get ψ? Let's plug into the N-S eq'ns:

$$(1)\quad \rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]=-\frac{\partial P}{\partial x}+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]+\rho g_x$$

$$(2)\quad \rho\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right]=-\frac{\partial P}{\partial y}+\mu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]+\rho g_y$$

Let's just look at the RHS of these eq'ns:

$$RHS_1=-\frac{\partial P}{\partial x}+\mu\left[\frac{\partial^3\psi}{\partial x^2\partial y}+\frac{\partial^3\psi}{\partial y^3}\right]+\rho g_x$$

$$RHS_2=-\frac{\partial P}{\partial y}+\mu\left[-\frac{\partial^3\psi}{\partial x^3}-\frac{\partial^3\psi}{\partial x\partial y^2}\right]+\rho g_y$$

We can <u>eliminate</u> the P terms by

the operation:

$$\frac{\partial RHS_1}{\partial y}-\frac{\partial RHS_2}{\partial x}\quad !$$

$$=\mu\left[\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}\right]$$

$$\equiv\mu\nabla^4\psi$$

$$\hookrightarrow \nabla^4\psi\equiv\nabla^2(\nabla^2\psi)$$

$\hookrightarrow$ Biharmonic operator

The LHS is rather nasty:

$$\rho\left[\frac{\partial}{\partial t}(\nabla^2\psi)+\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)}\right]$$

where:

$$\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)}\equiv\begin{vmatrix}\frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y}\\ \frac{\partial}{\partial x}(\nabla^2\psi) & \frac{\partial}{\partial y}(\nabla^2\psi)\end{vmatrix}$$

$\hookrightarrow$ Determinant

Because the LHS is so nasty, we usually use this eq'n only for Re << 1 when we can <u>ignore</u> the LHS!

For <u>low</u> <u>Re</u>, we have the Biharmonic Equation:

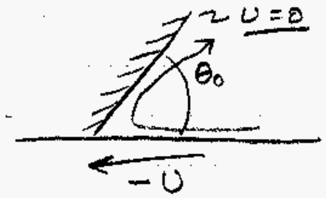
$$\underline{\nabla^4\psi=0}$$

with appropriate B.C.'s

This equation appears in other physical problems too - particularly in the deflection of thin elastic plates! The streamfunction is <u>identical</u> to the deflection of an elastic plate (like a thin sheet of glass) with the same B.C.'s

such as the value of ψ or its derivatives on the boundary. This provides a good way of visualizing the spatial dependence of ψ => just visualize the corresponding deflection problem!

OK, let's work an example! Suppose we have the <u>wiper</u> scraping fluid off a plate. What does the flow look like?



We have $\underline{\nabla^4\psi=0}$

(213)

We'll use cylindrical coordinates:

$$u_\theta = -\frac{\partial \psi}{\partial r}, \quad u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

Now in cylindrical coords, we have: $\nabla^2 \equiv \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

w/ $\nabla^4 \psi \equiv \nabla^2(\nabla^2 \psi)$

and B.C.'s:

$$u_r\Big|_{\theta=0} = -U, \quad u_\theta\Big|_{\theta=0} = 0$$

$$u_r\Big|_{\theta=\theta_0} = u_\theta\Big|_{\theta=\theta_0} = 0 \quad \Big\} \text{ wiper}$$

In terms of ψ these become:

$$\frac{\partial \psi}{\partial \theta}\Big|_{\theta=0} = -Ur, \quad \frac{\partial \psi}{\partial r}\Big|_{\theta=0} = 0$$

$$\frac{\partial \psi}{\partial r}\Big|_{\theta=\theta_0} = \frac{\partial \psi}{\partial \theta}\Big|_{\theta=\theta_0} = 0$$

(214)

The <u>inhomogeneous</u> B.C. suggests a solution of the form:

$$\psi = U r f(\theta)$$

where $f(\theta)$ has the B.C.'s:

$$f'(0) = -1, \quad f(0) = 0$$

$$f'(\theta_0) = f(\theta_0) = 0$$

Let's see if this works!

$$\nabla^2 \psi = \nabla^2 (U r f)$$

$$= \frac{U}{r}\left(f + f''\right)$$

$$\nabla^4 \psi = \nabla^2\left[\frac{U}{r}\left(f + f''\right)\right]$$

$$= \frac{2U}{r^3}\left(f + f''\right) - \frac{U}{r^3}\left(f + f''\right) + \frac{1}{r^3}\left(f'' + f^{IV}\right)$$

$$\underline{\underline{= 0}}$$

This reduces to:

$$f^{IV} + 2f'' + f = 0$$

(215)

This has the general solution:

$$f = A\sin\theta + B\cos\theta + C\theta\sin\theta + D\theta\cos\theta$$

Where the constants are det. from the B.C.'s:

$$f'(0) = -1, \quad f(0) = 0; \quad f(\theta_0) = f'(\theta_0) = 0$$

Now from $\underline{f(0) = 0}$ we get $\underline{B = 0}$

the others are harder!

After some algebra:

$$f(\theta) = \frac{-1}{\theta_0^2 - \sin^2\theta_0}\Big[\theta_0^2 \sin\theta - (\theta_0 - \sin\theta_0\cos\theta_0)\,\theta\sin\theta - (\sin^2\theta_0)\,\theta\cos\theta\Big]$$

OK, what is the pressure distribution in the fluid (and on the wiper)?

(216)

In cylindrical coords, we have the N-S eq'ns (RHS only!):

$$\frac{\partial p}{\partial r} = \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right]$$

Recall:

$$u_\theta = -\frac{\partial \psi}{\partial r} = -Uf$$

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = Uf'$$

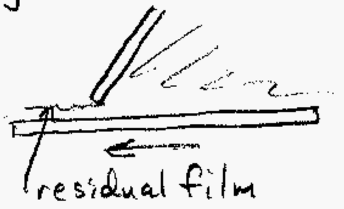
So:

$$\frac{\partial p}{\partial r} = \frac{\mu U}{r^2}\left[-f' + f''' + 2f'\right]$$

$$= \frac{\mu U}{r^2}\left[f' + f'''\right]$$

Thus $p \sim \frac{\mu U}{r}$

Note that this is *singular* (blows up) as $r \to 0$! This isn't even an *integrable* singularity, as the *total force* on the wiper diverges as $\log(r)$ as $r \to 0$! Basically, this huge force pushes the wiper off the surface, leaving a thin film behind!

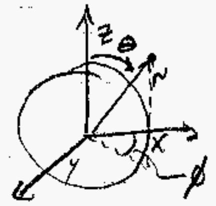


The details of the flow near the tip is fairly nasty—it requires a technique called *matched asymptotic expansions*.

A classic stream function problem is *creeping flow* ($Re \ll 1$) *past a sphere*.

Suppose a sphere of radius a is moving w/ velocity U in the z-direction. The flow is fully *3-D*, but it is axisymmetric.

We choose a *spherical coord system* such as that given below:



Basically, θ is the *latitude* & ϕ is the *longitude*!

Thus:

$z = r\cos\theta$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

For this problem $u_\phi = 0$ and there is no ϕ dependence! We have the B.C.'s:

$$u_r\Big|_{r=a} = U\cos\theta; \quad u_\theta\Big|_{r=a} = -U\sin\theta$$

$u_r, u_\theta \to 0$ as $r \to \infty$

In *spherical coordinates* we have the C.E.:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta) + \frac{1}{r\sin\theta}\underbrace{\cancel{\frac{\partial}{\partial\phi}}(u_\phi)}_{0} = 0$$

The structure of the CE suggests:

$$u_r = \frac{-1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$$

$$u_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

Which *automatically satisfies* the CE!

This is *not* the same as ψ for 2-D flow & leads to a different equation! For axisymmetric flows at $Re = 0$:

$$E^4\psi = 0$$

where:

$$E^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$

(221)

OK, how do we solve this? We look at the B.C.'s:

$$u_\theta\Big|_{r=a} = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\Big|_{r=a} = -U\sin\theta$$

thus $\frac{\partial\psi}{\partial r}\Big|_{r=a} = -Ua\sin^2\theta$

and $u_r\Big|_{r=a} = \frac{-1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\Big|_{r=a} = U\cos\theta$

Thus: $\frac{\partial\psi}{\partial\theta}\Big|_{r=a} = -Ua^2\sin\theta\cos\theta$

The structure of these B.C.'s <u>suggests</u> a solution of the form:

$$\psi = \sin^2\theta\, f(r)$$

We'll <u>try</u> this and see if it works!

(222)

Plug into B.C.'s:

$$\frac{\partial\psi}{\partial\theta}\Big|_{r=a} = 2\sin\theta\cos\theta\, f(a) = -Ua^2\sin\theta\cos\theta$$

Thus $f(a) = -\frac{1}{2}Ua^2$

$$\frac{\partial\psi}{\partial r}\Big|_{r=a} = \sin^2\theta\, f'(a) = -Ua\sin^2\theta$$

Thus $f'(a) = -Ua$

So far, so good! Now for the B.C.'s at $r\to\infty$:

$$u_\theta\Big|_{r\to\infty} = 0 = \sin\theta\,\frac{f'(r)}{r}\Big|_{r\to\infty}$$

so $\lim_{r\to\infty}\frac{f'(r)}{r} = 0$

and $u_r\Big|_{r\to\infty} = 0 = -\cos\theta\,\frac{f(r)}{r^2}\Big|_{r\to\infty}$

so $\lim_{r\to\infty}\frac{f(r)}{r^2} = 0$

(223)

Now we have to derive a DE. for $f(r)$:

$$E^4\psi \equiv E^2\left(E^2\psi\right) \equiv E^2\left(E^2\left(\sin^2\theta f(r)\right)\right)$$

Recall:

$$E^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)$$

$$= \sin^2\theta f'' + \frac{f}{r^2}\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\sin^2\theta}{\partial\theta}\right)$$

$$= \left(f'' - 2\frac{f}{r^2}\right)\sin^2\theta$$

Similarly,

$$E^4\psi = \left[f^{IV} - \frac{4}{r^2}f'' + \frac{8}{r^3}f' - \frac{8}{r^4}f\right]\sin^2\theta$$

$$= \underline{\underline{0}}$$

Thus we get the 4th order ODE:

$$r^4 f^{IV} - 4r^2 f'' + 8rf' - 8f = 0$$

w/ B.C.'s: $f(a) = -\frac{1}{2}Ua^2$, $f'(a) = -Ua$

(224)

$\lim_{r\to\infty}\frac{f(r)}{r^2} = 0$, $\lim_{r\to\infty}\frac{f'(r)}{r} = 0$

Now since all the terms in the ODE have the form $r^i f^{\underline{ith}}$, the general solution is of the form:

$$f(r) = r^n$$

Plugging in yields the polynomial:

$$n(n-1)(n-2)(n-3) - 4n(n-1) + 8n - 8 = 0$$

This has 4 roots:

$$n = -1,\ 1,\ 2,\ 4$$

Thus:

$$f(r) = \frac{e}{r} + br + cr^2 + dr^4$$

where the constants are determined from the B.C.'s!

The condition that $f(r)$ die off at $r\to\infty$ <u>requires</u> $c = d = 0$

Thus:

$$f(r) = \frac{e}{r} + br$$

Now at $r = a$:

$$f(a) = \frac{e}{a} + ba = -\frac{1}{2}Ua^2$$

and

$$f'(a) = -\frac{e}{a^2} + b = -Ua$$

Solving for e & b we get:

$$e = \frac{1}{4}Ua^3, \quad b = -\frac{3}{4}Ua$$

$$\psi = Ua^2\left(\frac{1}{4}\,\frac{a}{r} - \frac{3}{4}\,\frac{r}{a}\right)\sin^2\theta$$

which gives the velocities:

$$u_r = -\frac{U\cos\theta}{2}\left\{\left(\frac{a}{r}\right)^3 - 3\,\frac{a}{r}\right\}$$

$$u_\theta = -\frac{U\sin\theta}{4}\left\{\left(\frac{a}{r}\right)^3 - 3\,\frac{a}{r}\right\}$$

We can also obtain the pressure distribution:

$$p = p_0 + \frac{3}{2}\mu a U\,\frac{\cos\theta}{r^2}$$

It's important to note that the velocity dies off only as $O(a/r)$ for large r. This means that as Re → 0, the disturbance produced by a sphere is felt at very large distances! You have to go ~100 radii for the velocity to drop to 1% of the value at the sphere! This means that boundaries (e.g., vessel walls) have a strong influence on the motion of objects — an important result in low Re flows!

Ok, now we have the velocity and the pressure. What about the drag? (force exerted by the fluid

on the sphere!)

$$\underset{\sim}{F} = \int_{r=a} \underset{\approx}{\sigma}\cdot\underset{\sim}{n}\,dA$$

Recall that:

$$\underset{\approx}{\sigma} = -p\,\underbrace{\underset{\approx}{\delta}}_{\text{isotropic part}} + \overbrace{\underset{\approx}{\tau}}^{\text{Deviatoric stress}}$$

Thus:

$$\underset{\sim}{F} = \underbrace{\int_{\partial D} -p\,\underset{\sim}{n}\,dA}_{\substack{\text{normal forces}\\ \text{(form drag)}}} + \underbrace{\int_{\partial D} \underset{\approx}{\tau}\cdot\underset{\sim}{n}\,dA}_{\substack{\text{shear forces}\\ \text{(skin friction)}}}$$

At high Re, form drag is large, while skin friction is negligible! At low Re, both are comparable!

The integrals are a bit messy to evaluate, but eventually you get:

$$\underset{\sim}{F} = -\hat{\underset{\sim}{e}}_z\Big\{\underbrace{2\pi\mu a U}_{\text{form drag}} + \underbrace{4\pi\mu a U}_{\text{skin friction}}\Big\}$$

$$= -6\pi\mu a U\,\hat{\underset{\sim}{e}}_z$$

This is known as Stokes' Law and is of fundamental importance in the study of suspensions at low Re. You should remember this!!

Note that from pure dimensional analysis we had:

$$\frac{F}{\mu U a} = cst \quad \text{for } Re \ll 1$$

Getting the value of the constant took all the effort!

There's an alternative way to calculate the Drag: Do an Energy Balance

Since there's no accumulation of momentum (kinetic energy) all of the work done by the sphere on the fluid is Dissipated in heat! The work done by the sphere on the fluid is just:

$$\frac{\text{Work}}{\text{Time}} = \underset{\sim}{U} \cdot \underset{\sim}{F} \; \swarrow \text{force on fluid}$$

$= $ Total viscous dissipation

The viscous dissipation per unit volume is $\underset{\approx}{\tau} : \underset{\sim}{\nabla} \underset{\sim}{u}$

or in index notation:

$$\tau_{ij} \frac{\partial u_i}{\partial x_j}$$

Thus:

$$\underset{\sim}{F} \cdot \underset{\sim}{U} = \int_{r>a} \underset{\approx}{\tau} : \underset{\sim}{\nabla} \underset{\sim}{u} \, dV$$

↳ all volume exterior to sphere

This yields the same result!

Before we leave creeping flow, (e.g. $Re \ll 1$) let's look at another special property: Minimum Dissipation Theorem. Proving this is beyond this course (it's covered in 544), but we can use the result!

Among the set of all vector fields $\underset{\sim}{u}$ which satisfy:

1) the no-slip conditions on a body (e.g. $\underset{\sim}{u}\big|_{\partial D} = \underset{\sim}{U}(\underset{\sim}{x})$)

and

2) satisfy $\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = 0$ (continuity)

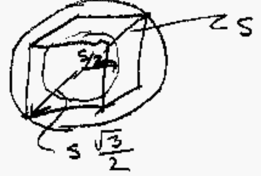
then the velocity field which also satisfies the creeping flow equations results in the minimum viscous dissipation!

Since Dissipation $\equiv \underset{\sim}{F} \cdot \underset{\sim}{U}$, this provides a means of estimating the drag on a complex shape!

Example: what is the Drag on a cube w/ sides of length s?

A corollary to the minimum dissip. theorem is that the drag on any object is less than that on one which completely encloses it! This is only true for $Re \ll 1$

Ok, how about the cube? It's drag is greater than that of a sphere of radius $s/2$ (which it encloses) but less than that of a sphere of radius $s \cdot \frac{\sqrt{3}}{2}$ which encloses it!



Thus:

$$6\pi\mu U \frac{s}{2} < F_{cube} < 6\pi\mu U s \frac{\sqrt{3}}{2}$$

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These are *rigorous* *bounds* provided Re<<1 (higher Re is *very* different!). We can also *estimate* the drag by just taking the geometric mean:

$$F_{cube} \approx 6\pi\mu U s \,\frac{(3)^{1/4}}{2}$$

Another consequence of the minimum dissipation theorem is that streamlining an object by *enclosing* it in a smooth shell only *increases* the drag! This is certainly *not* true for higher Re!

(234)

Ok, we've looked at low Re flows. Now let's look at high Re limit.

Recall the high Re scaling:

$$\underset{\sim}{x}^* = \frac{\underset{\sim}{x}}{\ell}, \quad \underset{\sim}{u}^* = \frac{\underset{\sim}{u}}{U}, \quad t^* = \frac{t}{(\ell/U)}$$

$$p^* = \frac{p - p_\infty}{(\rho U^2)} \leftarrow \text{inertial scaling}$$

Thus:

$$\left(\frac{\partial \underset{\sim}{u}^*}{\partial t^*} + \underset{\sim}{u}^* \cdot \underset{\sim}{\nabla}^* \underset{\sim}{u}^*\right) = -\underset{\sim}{\nabla}^* p^* + \frac{1}{Re}\nabla^{*2}\underset{\sim}{u}^* + \frac{1}{Fr}\underset{\sim}{g}^*$$

For *low* Re we threw out inertial terms. For *high* Re we throw out viscous terms $(O(\frac{1}{Re}))$! This yields the *inviscid* (zero viscosity) *Euler eq'ns*:

$$\rho \frac{D\underset{\sim}{u}}{Dt} = -\underset{\sim}{\nabla} p + \rho \underset{\sim}{g}$$

(235)

We can *eliminate* the $\rho \underset{\sim}{g}$ term by defining an *augmented pressure*

$$\mathcal{P} \equiv p - \rho \underset{\sim}{g} \cdot \underset{\sim}{x}$$

Thus $\underset{\sim}{\nabla}\mathcal{P} = \underset{\sim}{\nabla} p - \rho \underset{\sim}{g}$

provided ρ is *cst*

So:

$$\rho \frac{\partial \underset{\sim}{u}}{\partial t} + \rho \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \underset{\sim}{u} = -\underset{\sim}{\nabla}\mathcal{P}$$

We have the *vector identity*:

$$\underset{\sim}{u} \cdot \underset{\sim}{\nabla} \underset{\sim}{u} = \underbrace{\tfrac{1}{2}\underset{\sim}{\nabla}(\underset{\sim}{u} \cdot \underset{\sim}{u})}_{\overset{\parallel}{\underset{\sim}{\nabla}(\frac{1}{2}u^2)}} - \underset{\sim}{u} \times \underbrace{(\underset{\sim}{\nabla} \times \underset{\sim}{u})}_{\underset{\text{(vorticity)}}{\underset{\sim}{\omega}}}$$

Thus:

$$\rho \frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{\nabla}\left(\tfrac{1}{2}\rho u^2\right) + \underset{\sim}{\nabla}\mathcal{P} = \rho \underset{\sim}{u} \times \underset{\sim}{\omega}$$

or

$$\rho \frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{\nabla}\left(\tfrac{1}{2}\rho u^2 + p - \rho \underset{\sim}{g} \cdot \underset{\sim}{x}\right) = \rho \underset{\sim}{u} \times \underset{\sim}{\omega}$$

(236)

These equations are most useful for *irrotational flow* (e.g., $\underset{\sim}{\omega} = 0$)

$$\underset{\sim}{\omega} \equiv \underset{\sim}{\nabla} \times \underset{\sim}{u}$$

If a flow starts out *irrotational*, then *only* the viscous term can *produce* vorticity! Thus, if the flow is *inviscid*, it *stays* irrotational!

You can prove this by taking the curl of the N-S equations, but it gets a little messy!

Anyway, if $\underset{\sim}{\omega} = \underline{\underline{0}}$ we get:

$$\rho \frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{\nabla}\left(\tfrac{1}{2}\rho u^2 + p - \rho \underset{\sim}{g} \cdot \underset{\sim}{x}\right) = 0$$

If the flow is also *steady*:

$$\underset{\sim}{\nabla}\left(\tfrac{1}{2}\rho u^2 + p - \rho \underset{\sim}{g} \cdot \underset{\sim}{x}\right) = 0$$

How does this vary along a streamline? From Lagrangian perspective, time rate of change (for steady flow) along streamline is just:

$\underset{\sim}{u}\cdot\underset{\sim}{\nabla}$ (← whatever you're interested in!

Thus:

$$\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\left(\frac{1}{2}\rho u^2+p-\rho\underset{\sim}{g}\cdot\underset{\sim}{x}\right)=0$$

or $\frac{1}{2}\rho u^2+p+\rho g z=\text{cst}$ along a streamline!

(Note: $-\underset{\sim}{g}\cdot\underset{\sim}{x}=(-\underset{\sim}{\hat{e}}_z g\cdot\underset{\sim}{x})=gz$ if $\underset{\sim}{g}$ is in $-z$ direction!)

This is known as Bernoulli's Eq'n, valid for steady, inviscid, irrotational flows!

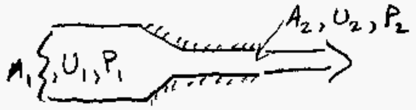
What is the physical interp. of Bernoulli's eq'n? Conservation of Mechanical Energy! If we have no frictional losses (e.g., $\mu=0\Rightarrow$ inviscid flow) then mechanical energy is conserved along a streamline!

$\frac{1}{2}\rho u^2\equiv$ Kinetic Energy/volume

$p+\rho g z\equiv$ "Potential Energy"/volume

Thus, if one goes up, the other goes down!

How can we use this? Look at a jet of water at high Re:



Neglecting losses, what is the velocity of the jet, the force on the nozzle?

Conservation of Mass: $U_1A_1=U_2A_2$

Conservation of mech. Energy (e.g., Bernoulli's eq'n):

$$\frac{1}{2}\rho U_1^2+P_1=\frac{1}{2}\rho U_2^2+P_2$$

Thus

$$P_1-P_2=\frac{1}{2}\rho\left(U_2^2-U_1^2\right)=\frac{1}{2}\rho U_1^2\left(\frac{U_2^2}{U_1^2}-1\right)=\frac{1}{2}\rho U_1^2\left(\left(\frac{A_1}{A_2}\right)^2-1\right)$$

So

$$U_1=\left[\frac{2(P_1-P_2)}{\rho\left(\left(\frac{A_1}{A_2}\right)^2-1\right)}\right]^{1/2}$$

and $U_2=\frac{A_1}{A_2}U_1$

This assumes that the flow field is uniform across inlet & outlet, & that there are no frictional losses.

What about the force on the nozzle? We did this sort of problem before!

$$\int_{\partial D}(\rho\underset{\sim}{u})\,\underset{\sim}{u}\cdot\underset{\sim}{n}\,dA=\Sigma\underset{\sim}{F}$$ (force exerted on fluid)

We are interested in x-component (flow direction), thus:

$$\Sigma F_x=\int_{\partial D}(\rho u_x)\,\underset{\sim}{u}\cdot\underset{\sim}{n}\,dA=\rho U_1(-U_1A_1)+\rho U_2(U_2A_2)$$

$$=\rho\left(U_2^2A_2-U_1^2A_1\right)=\rho U_1^2A_1\left(\frac{U_2^2}{U_1^2}\frac{A_2}{A_1}-1\right)$$

$$=\rho U_1^2A_1\left(\frac{A_1}{A_2}-1\right)$$

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But from Bernoulli's eq'n:

$$\rho U_1^2 = \frac{2(P_1 - P_2)}{\left(\frac{A_1}{A_2}\right)^2 - 1}$$

So:

$$\Sigma F_x = 2A_1(P_1 - P_2)\,\frac{\left(\frac{A_1}{A_2} - 1\right)}{\left(\frac{A_1}{A_2}\right)^2 - 1}$$

$$= \frac{2A_1(P_1 - P_2)}{\frac{A_1}{A_2} + 1} = \frac{2A_1A_2(P_1 - P_2)}{A_1 + A_2}$$

Now $\Sigma F_x = \overset{\text{force exerted by fluid on nozzle}}{-F_N} + P_1A_1 - P_2A_2$

So $F_N = P_1A_1 - P_2A_2 - \frac{2A_1A_2(P_1 - P_2)}{A_1 + A_2}$

Now if $P_2 = 0$ (atmospheric pressure forces on nozzle are neglected) then:

$$F_N = P_1A_1\left(1 - \frac{2A_2}{A_1 + A_2}\right)$$

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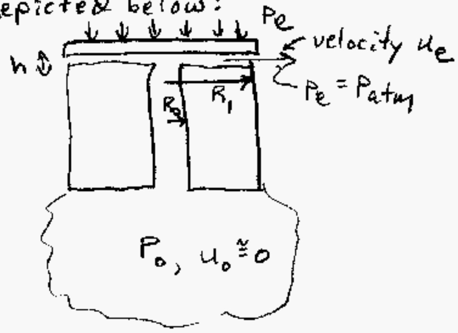
Let's look at a more complicated problem: what are the forces on a plate near a spool of thread as depicted below:



What happens? Can we blow the plate off the spool of thread?

First, what is u_e? We shall <u>assume</u> inviscid, irrotational flow. Thus:

$$\tfrac{1}{2}\rho u_e^2 + P_e = \tfrac{1}{2}\rho \underset{0}{\cancel{u_0^2}} + P_0$$

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So we have:

$$\tfrac{1}{2}\rho u_e^2 = P_0 - P_e$$

To solve, we need the radial velocity everywhere under the plate! This, in turn, gives us the pressure! By conservation of mass:

$$2\pi r h\, u(r) = 2\pi R_1 h\, u_e$$

$\therefore\ u(r) = \frac{R_1}{r} u_e$, at least for $r > R_0$. We can take $u = 0$ for $r < R_0$ (stagnation flow – it's a bit approximate!)

So: $\tfrac{1}{2}\rho u_e^2 + P_e = \tfrac{1}{2}\rho u^2 + P$

$$\text{or } P = P_e + \tfrac{1}{2}\rho u_e^2\left(1 - \frac{u^2}{u_e^2}\right)$$

$$= \begin{cases} P_e + (P_0 - P_e)\left(1 - \left(\frac{R_1}{r}\right)^2\right) & R_0 < r < R_1 \\ P_0 & 0 < r < R_0 \text{ (stagnation)} \end{cases}$$

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To get the <u>net</u> force on the plate, we need to integrate:

$$F = \int_0^{R_1} (P - P_e)\, 2\pi r\, dr$$

$$= (P_0 - P_e)\pi R_0^2 + \int_{R_0}^{R_1} (P_0 - P_e)\left(1 - \frac{R_1^2}{r^2}\right) 2\pi r d$$

$$= \pi R_1^2 (P_0 - P_e)\left(1 - 2\ln\left({}^{R_1}/_{R_0}\right)\right)$$

So if $2 \ln {}^{R_1}/_{R_0} > 1$ the net force drives the plate <u>towards</u> the spool! The harder you blow, the tighter it sticks! The critical ratio is

$${}^{R_1}/_{R_0} > 1.65$$

Bernoulli problems offer lots of interesting, counter-intuitive examples like this!

Ok, so far we've just looked at the case of <u>Uniform Flow</u>. What happens when the flow is non-uniform? Bernoulli's equation still applies, but now $\underset{\sim}{u}$ will be more complex!

If a flow is <u>irrotational</u> (e.g., $\underset{\sim}{\nabla} \times \underset{\sim}{u} = 0$), then $\underset{\sim}{u}$ <u>must</u> be able to be represented by the <u>gradient</u> of a <u>scalar potential</u>!

We take:

$$\underset{\sim}{u} = -\underset{\sim}{\nabla} \phi$$

What does ϕ satisfy? Remember the C.E.:

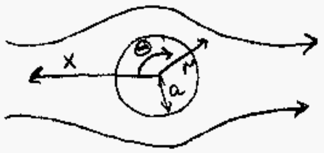
$$\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = 0$$

Thus: $$\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = \underline{\underline{-\nabla^2 \phi = 0}}$$

So ϕ satisfies Laplace's eq'n! Such problems are easy to solve for many geometries!

Problems for which $\underset{\sim}{u} = -\underset{\sim}{\nabla}\phi$, $\nabla^2\phi = 0$ are known as <u>ideal potential flow</u>, and occur for <u>steady</u>, <u>inviscid</u> <u>irrotational</u> flow fields!

Let's work a classic example - flow past a cylinder



$$\underset{\sim}{u} = -\underset{\sim}{\nabla}\phi, \quad \nabla^2\phi = 0$$

In cylindrical coordinates:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial \phi}{\partial \theta^2} = 0$$

What are the B.C.'s?

$$\left.\begin{aligned} u_\theta\Big|_{r\to\infty} &\equiv -\frac{1}{r}\frac{\partial \phi}{\partial \theta}\Big|_{r\to\infty} = U \sin\theta \\ u_r\Big|_{r\to\infty} &\equiv -\frac{\partial \phi}{\partial r}\Big|_{r\to\infty} = -U\cos\theta \end{aligned}\right\} \underset{\sim}{\nabla}\phi\Big|_{r\to\infty} = \underset{\sim}{\hat{e}}_x$$

What about the B.C.'s <u>on</u> the cylinder?? We've thrown out viscosity (inviscid flow), so the no-slip eq'n no longer applies! Instead, we have <u>no flow thru the object</u>!

$$\underset{\sim}{u} \cdot \underset{\sim}{n}\Big|_{\partial D} = 0$$

Thus $$-\underset{\sim}{\nabla}\phi \cdot \underset{\sim}{n}\Big|_{r=a} = 0 \equiv u_r\Big|_{r=a}$$

How do we solve this? Look at <u>inhomogeneous B.C.'s</u> (those at $r \to \infty$). They suggest a solution of the form:

$$\phi = f(r)\cos\theta$$

We plug into B.C.'s:

$$-\frac{1}{r}\frac{\partial \phi}{\partial \theta}\Big|_{r=\infty} = \frac{f}{r}\Big|_{r\to\infty} \sin\theta = U\sin\theta$$

$$\therefore \frac{f}{r}\Big|_{r\to\infty} = U$$

$$-\frac{\partial \phi}{\partial r}\Big|_{r\to\infty} = -f'\cos\theta = -U\cos\theta$$

$$\therefore f'\Big|_{r\to\infty} = U$$

Both are satisfied if

$$f \sim U r \text{ as } r \to \infty$$

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Plugging into $\nabla^2\phi = 0$:

$$\cos\theta f'' + \frac{\cos\theta}{r} f' - \frac{f}{r^2}\cos\theta = 0$$

or

$$f'' + \frac{f'}{r} - \frac{f}{r^2} = 0$$

w/ B.C.'s:

$$f\Big|_{r\to\infty} = Ur \; ; \; f'\Big|_{r=a} = 0$$

f is of the form:

$f = r^n$ which yields:

$$n(n-1) + n - 1 = 0$$

or

$$(n+1)(n-1) = 0$$

$$\therefore \; n = 1, -1$$

$$f(r) = \frac{c_1}{r} + C_2 r$$

From condition as $r\to\infty$, $C_2 = U$

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At $r=a$ we have

$$f'\Big|_{r=a} = \left[-\frac{c_1}{r^2} + U\right]\Big|_{r=a} = 0$$

Thus $c_1 = Ua^2$

And hence:

$$\phi = U\left(\frac{a^2}{r} + r\right)\cos\theta$$

This yields the velocity distribution:

$$u_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = U\left(1+\frac{a^2}{r^2}\right)\sin\theta$$

$$u_r = -\frac{\partial\phi}{\partial r} = -U\left(1-\frac{a^2}{r^2}\right)\cos\theta$$

A couple of things to note. First, $u_\theta\Big|_{r=a} \neq 0$! Thus the tangential velocity violates the no-slip condition, as expected! This leads to the development of a very thin

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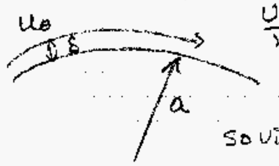
boundary layer next to the surface where both *viscosity* and *no-slip* condition must apply! A Reynolds number based on the *thickness* of the boundary layer is of $O(1)$!



$$\frac{Ua}{\nu} >> 1$$

but $\frac{U\delta}{\nu} = O(1)$

so viscous effects are important in the B.L.

We'll look at B.L. problems in *much* more detail in a bit!

Second, $u_\theta\Big|_{r=a} = 2U\sin\theta$, which varies from *zero* at the *leading* and *trailing* *stagnation points* to **twice** the free stream velocity at

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$\theta = \frac{\pi}{2}$! This means the fluid is *accelerated* going around the cylinder, and thus the pressure is *lowest* at $\theta = \frac{\pi}{2}$! Let's calculate this:

We have Bernoulli's eq'n:

$$\frac{1}{2}\rho u^2 + p + \rho g z = cst$$

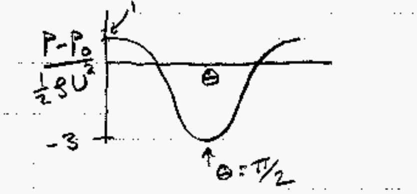
We neglect gravity! Far upstream we have $p = p_0$, $u = U$ on *all* streamlines. Thus:

$$\frac{1}{2}\rho u^2 + p = p_0 + \frac{1}{2}\rho U^2$$

at $r=a$ $u \equiv u_\theta$ $(u_r = 0)$, thus:

$$p\Big|_{r=a} = p_0 + \frac{1}{2}\rho U^2\left(1 - 4\sin^2\theta\right)$$

We can plot this up:



We can use this to calculate the Drag ($\underset{\sim}{F} \cdot \underset{\sim}{e}_x$) on the cylinder!

There is no skin friction (no viscosity), thus:

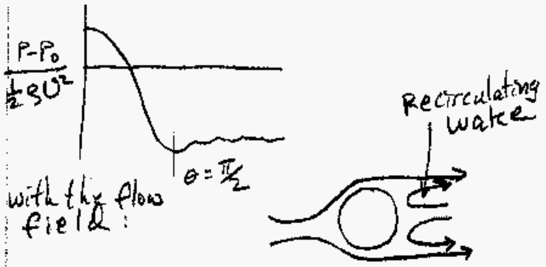
$$\underset{\sim}{F} = -\int_{\partial D} p\, \underset{\sim}{n}\, dA$$

$$F_x = \underset{\substack{\uparrow \\ \text{length} \\ \text{of cylinder}}}{L} \int_0^{2\pi} (-p)\, \underset{\sim}{n} \cdot \underset{\sim}{\hat{e}}_x\, a\, d\theta$$

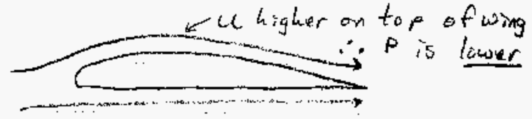
$$= L \int_0^{2\pi} \tfrac{1}{2}\rho U^2 (1 - 4\sin^2\theta) \cos\theta\, d\theta = \underline{\underline{0}}$$

Thus the Drag for ideal potential flow around a cylinder is zero! This is known as D'Alembert's Paradox, and arises because the pressure distribution is symmetric – there is high pressure on both the front and back sides, which cancel out! What really happens? ⇒ You don't get pressure recovery on the back side!

with the flow field:



The Boundary Layer separates, & no longer is attached to the boundary! This results in a much higher Drag! ⇒ Separation is critical for high Re flows! Consider flow past a wing:



The ΔP from top to bottom provides Lift which makes the plane fly!

If there is no separation, the Drag is quite low! It's the Drag that the engines have to overcome to keep the plane moving! A commercial airliner has a max L/D ratio of ~20!

What happens if the B.L. separates? This will happen if the plane moves too slowly, or at too large an angle of attack:

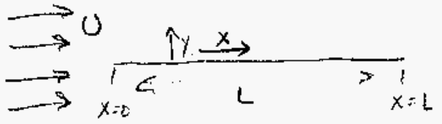
Separation does two things. First, it greatly increases Drag, decreasing the L/D ratio and, since engines aren't designed to overcome this force, the plane slows down! Since $L \sim \rho U^2$, slowing down the plane kills the lift, and the plane falls! Second, wing control surfaces (e.g. elevators) are on the trailing edge of the wing. If the flow separates, these surfaces are now in a separation bubble and can no longer control the motion of the plane. This whole process is called stall and a huge part of wing design is figuring out how to avoid it!

Boundary Layer Theory

The scaling of the N-S eq'ns at high Re suggests that viscous terms are unimportant on a length scale comparable to the size of a body. The Euler flow eq'ns which result require eliminating the no-slip condition! This leads to discontinuities in the velocity at the surface, thus viscous forces must be important in this region, termed the boundary layer: the region where inertia and viscosity are equally significant! We can determine the thickness of the B.L. δ by rescaling the

N-S eq'ns in this region!

Let's look at a simple problem: high Re flow past a plate of length L at zero incidence (e.g., edge on into flow):



If we have $Re_L = \frac{UL}{\nu} >> 1$

(Re_L = plate Reynolds #, based on length L)

we get the Euler flow eq'ns:

$$\frac{D\underset{\sim}{u}}{Dt} = -\underset{\sim}{\nabla} p \quad \left(+\cancel{\underset{\text{small}}{\frac{1}{Re}}\nabla^2 \underset{\sim}{u}}\right) \; ; \; \underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 0$$

The B.C. is just $\underset{\sim}{u}\cdot\underset{\sim}{n}\Big|_{y=0} = 0$ (no normal flow)

Because we've eliminated viscosity, we've also eliminated the No-slip Condition!

Far from the plate ($y \to \infty$) we have the undisturbed flow:

$$\underset{\sim}{u}\Big|_{y\to\infty} = U\hat{\underset{\sim}{e}}_x$$

This set of equations has the simple solution:

$$\underset{\sim}{u} = U\hat{\underset{\sim}{e}}_x \quad \text{everywhere!}$$

But this leads to a step change in the velocity at $y=0$ (the plate).

Since viscous forces are proportional to velocity derivatives, they must become important in this region!

Suppose viscous forces are important over some region $y = O(\delta)$. We shall rescale the N-S equations to preserve the viscous term & the No Slip condition.

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Let : $x^* = x/L$, $u^* = u/U$

$v^* = \frac{v}{\mathcal{V}}$, $y^* = y/\delta$, $p^* = \frac{p - p_\infty}{\rho U^2}$

To determine δ & $\mathcal{V}$ we must look at the equations. First (<u>always</u>) we do the C.E. :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\therefore \quad \frac{U}{L}\frac{\partial u^*}{\partial x^*} + \frac{\mathcal{V}}{\delta}\frac{\partial v^*}{\partial y^*} = 0$$

$$\text{or} \quad \frac{\partial u^*}{\partial x^*} + \underset{\longrightarrow\, =1}{\left(\frac{L\mathcal{V}}{U\delta}\right)}\frac{\partial v^*}{\partial y^*} = 0$$

Thus we require:

$$\mathcal{V} = \frac{\delta}{L}U$$

Which is the same scaling we got in lubrication theory!

Now for the x-momentum eq'n:

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$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Let $t^* = Ut/L$, now we scale:

$$\rho\frac{U^2}{L}\left(\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right)$$

$$= -\underset{\substack{\uparrow \\ \text{inertial scaling} \\ \text{for pressure}}}{\frac{\rho U^2}{L}}\frac{\partial p^*}{\partial x^*} + \frac{\mu U}{\delta^2}\left(\underset{\substack{\uparrow \\ \text{dominant viscous} \\ \text{term!}}}{\frac{\partial^2 u^*}{\partial y^{*2}}} + \frac{\delta^2}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}}\right)$$

Dividing through:

$$\left(\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial x^*} + \frac{\mu L}{\rho U\delta^2}\left(\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\delta^2}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}}\right)$$

We want to <u>keep</u> a viscous term! Thus we <u>require</u>:

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$$\frac{\mu L}{\rho U\delta^2} = \underline{\underline{1}} \quad \text{or} \quad \frac{\delta^2}{L^2} = \frac{\mu}{\rho U L} = \frac{1}{Re_L}$$

where $Re_L \equiv$ plate Reynolds number!

So $\frac{\delta}{L} = \left(\frac{1}{Re_L}\right)^{1/2} \ll 1$ for high Re_L and we get a boundary layer!

We thus have the Boundary <u>Layer Eq'ns</u> derived by Prandtl in 1904:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u^*}{\partial y^{*2}} + \underset{\swarrow\ \text{small}}{O\left(\frac{1}{Re_L}\right)}$$

What about the <u>pressure</u>? We need another eq'n. Let's look at the y-momentum eq'n:

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

(264)

$$\rho\frac{U^2\delta}{L^2}\left(\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right) = -\frac{\rho U^2}{\delta}\frac{\partial p^*}{\partial y^*} + \frac{\mu U}{\delta L}\left(\frac{\partial^2 v^*}{\partial y^{*2}} + \frac{\delta^2}{L^2}\frac{\partial^2 v^*}{\partial x^{*2}}\right)$$

Dividing through and rearranging:

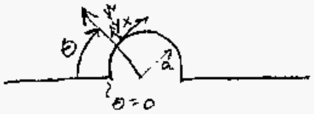
$$\frac{\partial p^*}{\partial y^*} = -\frac{\delta^2}{L^2}\left(\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right) + \frac{1}{Re_L}\left(\frac{\partial^2 v^*}{\partial y^{*2}} + \frac{\delta^2}{L^2}\frac{\partial^2 v^*}{\partial x^{*2}}\right)$$

Or:

$$\frac{\partial p^*}{\partial y^*} = \underset{\swarrow\ \text{small !}}{O\left(\frac{1}{Re_L}\right)}$$

What does this mean?? Basically, a boundary layer is <u>too thin</u> to support a <u>pressure gradient</u> in the y-direction! The pressure <u>distribution</u> due to the external Euler (inviscid) flow is <u>impressed</u> <u>on</u> <u>the</u> boundary <u>layer</u>!

This applies equally well to other boundary layer problems, such as flow past a cylinder, etc. In these flows we take x to be the coordinate along the surface (e.g., $x \equiv a\theta$ for a cylinder of radius a) and y to be the coordinate normal to the surface (e.g., $y \equiv r - a$ for the same geometry):



Ok, let's return to the flat plate problem. We have the B.L. eq'ns:

C.E.: $\dfrac{\partial u^*}{\partial x^*} + \dfrac{\partial v^*}{\partial y^*} = 0$

x-mom:

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial P^*}{\partial x^*} + \frac{\partial^2 u^*}{\partial y^{*2}}$$

y-mom:

$$\frac{\partial P^*}{\partial y^*} = 0$$

Where we have ignored terms of $O(\frac{1}{Re_L})$.

The B.C.'s are:

$u^*\big|_{y^*=0} = v^*\big|_{y^*=0} = 0$ (no-slip)

$P^* = P^*\Big|^{(EF)}_{y/L \to 0} \hookrightarrow$ Euler flow solution at $y/L = 0$

$u^*\Big|_{y^* \to \infty} = u^*\Big|^{(EF)}_{y/L \to 0}$ again, EF sol'n at the surface

These latter matching conditions work provided $\delta/L \ll 1$

Outer limit of BL ≡ Inner limit of EF

For the flat plate problem,

$u^{*\,EF} = 1$ (const) & $P^{*\,EF} = 0$

For steady state flow $\frac{\partial u^*}{\partial t^*} = 0$ so:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{\partial^2 u^*}{\partial y^{*2}}$$

$$u^*\big|_{y^*=0} = v^*\big|_{y^*=0} = 0 \qquad u^*\big|_{y^* \to \infty} = 1$$

How do we solve this set of eq'ns? The flow is 2-D, so it is natural to define a stream function:

$$u^* = \frac{\partial \psi^*}{\partial y^*}\,, \qquad v^* = -\frac{\partial \psi^*}{\partial x^*}$$

Substituting in:

$$\frac{\partial \psi^*}{\partial y^*}\frac{\partial^2 \psi^*}{\partial x^* \partial y^*} - \frac{\partial \psi^*}{\partial x^*}\frac{\partial^2 \psi^*}{\partial y^{*2}} = \frac{\partial^3 \psi^*}{\partial y^{*3}}$$

With B.C.'s:

$$\frac{\partial \psi^*}{\partial x^*}\Big|_{y^*=0} = \frac{\partial \psi^*}{\partial y^*}\Big|_{y^*=0} = 0\,; \quad \frac{\partial \psi^*}{\partial y^*}\Big|_{y^*=\infty} = 1$$

We still have a 3rd order non-linear PDE. What can we do with it??

This sort of problem often admits a similarity transform which allows us to convert a PDE to an ODE, a tremendous simplification! How do we know if this will happen? Apply Morgan's Theorem:

1) If a problem, including B.C.'s, is invariant to a one-parameter group of continuous transformations then the number of independent variables may be reduced by one.

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2) The reduction is accomplished by choosing as new dependent and independent variables combinations which are invariant under the transformations.

The techniques for applying this theorem can be quite messy, but we'll stick to the simplest one: simple affine stretching.

Let's stretch all of the dep. & indep. variables! Let:

$$\psi^* = A\bar{\psi}, \quad x^* = B\bar{x}, \quad y^* = C\bar{y}$$

where A, B, C are a group of stretching parameters. If the problem can be made invariant while leaving one of these undetermined,

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it will satisfy Morgan's Theorem! Let's do this. Plugging in:

$$\frac{A^2}{BC^2}\left(\bar{\psi}_{\bar{y}}\,\bar{\psi}_{\bar{x}\bar{y}} - \bar{\psi}_{\bar{x}}\,\bar{\psi}_{\bar{y}\bar{y}}\right) = \frac{A}{C^3}\,\bar{\psi}_{\bar{y}\bar{y}\bar{y}}$$

where subscripts denote derivatives.

Dividing thru:

$$\bar{\psi}_{\bar{y}}\,\bar{\psi}_{\bar{x}\bar{y}} - \bar{\psi}_{\bar{x}}\,\bar{\psi}_{\bar{y}\bar{y}} = \frac{B}{AC}\,\bar{\psi}_{\bar{y}\bar{y}\bar{y}}$$

Thus the equation is invariant if $\frac{B}{AC} = 1$ (e.g., A, B & C disappear!)

We also have to look at the B.C.'s

$$\frac{A}{B}\frac{\partial\bar{\psi}}{\partial\bar{x}}\bigg|_{C\bar{y}=0} = 0 \;\Rightarrow\; \frac{\partial\bar{\psi}}{\partial\bar{x}}\bigg|_{\bar{y}=0} = 0$$

(no restrictions)

Similarly,

$$\frac{\partial\bar{\psi}}{\partial\bar{y}}\bigg|_{\bar{y}=0} = 0$$

(271)

Now for the inhomogeneous B.C.:

$$\frac{A}{C}\frac{\partial\bar{\psi}}{\partial\bar{y}}\bigg|_{C\bar{y}=\infty} = 1$$

Now $\frac{\infty}{C} = \infty$, so the location doesn't add a restriction, but we get:

$$\frac{\partial\bar{\psi}}{\partial\bar{y}}\bigg|_{\bar{y}=\infty} = \frac{C}{A}$$

which is invariant only if $\frac{C}{A} = 1$

In general, homogeneous B.C.'s don't lead to restrictions on the stretching parameters, but inhomogeneous ones do!

In this problem we only had two restrictions, but we had 3 parameters! Thus we satisfy Morgan's Theorem!

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What will work? In general, any combination of variables which is invariant under the transformations will work, but some are better than others!!

For example, we have the transf:

$$\psi^* = A\bar{\psi}, \quad x^* = B\bar{x}, \quad y^* = C\bar{y}$$

and the restrictions:

$$\frac{B}{AC} = 1, \quad \frac{C}{A} = 1$$

Thus one possibility is:

$$\frac{x^*}{\psi^* y^*} = f(\eta); \quad \eta = \frac{y^*}{\psi^*}$$

which is clearly invariant! This would work, but would be extremely messy to use, with lots of implicit differentiation required! A better

(273)

choice is to <u>recast</u> the restrictions so that the variable ζ only involves <u>independent</u> variables!

We had:

$$\frac{B}{AC} = 1, \quad \frac{C}{A} = 1$$

A more convenient pair of restrictions is obtained by division:

$$\frac{A}{C} = 1, \quad \frac{B}{C^2} = 1$$

Which yields the transform:

$$\frac{\psi^*}{y^*} = f(\zeta), \quad \zeta = \frac{x^*}{y^{*2}}$$

This works better, but it's still not the best choice! The problem is that we are taking 3rd derivatives with respect to y^*, but only 1st derivatives w.r.t. x^*.

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It thus makes sense to put all the complexity in x^*:

$$\frac{A}{B^{1/2}} = 1, \quad \frac{C}{B^{1/2}} = 1$$

yields:

$$\frac{\psi^*}{(2x^*)^{1/2}} = f(\zeta); \quad \zeta = \frac{y^*}{(2x^*)^{1/2}}$$

(the factor of 2 in ζ and ψ are there for historical reasons – it gets rid of a constant in the transformed DE – and has <u>no</u> <u>significance</u>! What matters is the dependence on y^* & x^*!)

<u>This is known as</u> <u>Canonical Form</u>: <u>Put all the complexity in the variable with the lowest highest derivative.</u>

<u>There can be exceptions to this for special</u> problems, but it usually works pretty well!

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Ok, now let's get the transformed ODE:

$$\frac{\partial \psi^*}{\partial y^*} = \frac{\partial (2x^*)^{1/2} f}{\partial y^*} = (2x^*)^{1/2} \frac{df}{d\zeta} \frac{\partial \zeta}{\partial y^*}$$

<u>but</u> $\frac{\partial \zeta}{\partial y^*} = \frac{\partial}{\partial y^*}\left(\frac{y^*}{(2x^*)^{1/2}}\right) = \frac{1}{(2x^*)^{1/2}}$

so: $\frac{\partial \psi^*}{\partial y^*} = f'$!

Similarly:

$$\frac{\partial^2 \psi^*}{\partial y^{*2}} = \frac{1}{(2x^*)^{1/2}} f''$$

$$\frac{\partial^3 \psi^*}{\partial y^{*3}} = \frac{1}{2x^*} f'''$$

These were simple, because we put all the complexity in x^*. Now we pay for it!

$$\frac{\partial \psi^*}{\partial x^*} = \frac{\partial}{\partial x^*}\left((2x^*)^{1/2} f\right) = \frac{1}{(2x^*)^{1/2}} f + (2x^*)^{1/2} f' \frac{\partial \zeta}{\partial x^*}$$

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But!

$$\frac{\partial \zeta}{\partial x^*} = \frac{\partial}{\partial x^*}\left(\frac{y^*}{(2x^*)^{1/2}}\right) = -\frac{1}{2}\frac{y^*}{x^*(2x^*)^{1/2}} = -\frac{1}{2}\frac{\zeta}{x^*}$$

Thus:

$$\frac{\partial \psi^*}{\partial x^*} = \frac{1}{(2x^*)^{1/2}}\left(f - \zeta f'\right)$$

and finally:

$$\frac{\partial^2 \psi^*}{\partial x^* \partial y^*} = \frac{\partial}{\partial x^*}(f') = -\frac{1}{2}\frac{\zeta}{x^*} f''$$

Ok, now we plug back into the DE:

$$\frac{\partial \psi^*}{\partial y^*}\frac{\partial^2 \psi^*}{\partial x^* \partial y^*} - \frac{\partial \psi^*}{\partial x^*}\frac{\partial^2 \psi^*}{\partial y^{*2}} = \frac{\partial^3 \psi^*}{\partial y^{*3}}$$

so:

$$-\frac{1}{2}\frac{\zeta}{x^*} f'' f' - \frac{1}{(2x^*)^{1/2}}\left(f - \zeta f'\right)\frac{1}{(2x^*)^{1/2}} f'' = \frac{1}{2x^*} f'''$$

Which simplifies to:

$$f''' + f f'' = 0 \quad !$$

This is known as the <u>Blasius Equation</u> for flow past a flat plate!

We also have the B.C.'s:

$$u^*\Big|_{y^*=0} = 0 \equiv \frac{\partial \psi^*}{\partial y^*}\Big|_{y^*=0} = f'(0)$$

$$\therefore f'(0) = 0$$

$$v^*\Big|_{y^*=0} = 0 \equiv -\frac{\partial \psi^*}{\partial x^*}\Big|_{y^*=0} = \frac{-1}{(2x^*)^{1/2}}\left(f - \eta f'\right)\Big|_{\eta=0}$$

Now since $\eta f'\big|_{\eta=0} = 0$ we get

$$f(0) = 0$$

Finally,

$$u^*\Big|_{y^*\to\infty} = 1 \equiv f'(\infty)$$

$$\therefore f'(\infty) = 1$$

So the complete problem reduces to the non-linear ODE:

$$u^* = \frac{\partial \psi^*}{\partial y^*} = f'(\eta)$$

$$v^* = -\frac{\partial \psi^*}{\partial x^*} = \frac{-1}{2x^{*1/2}}\left(f - \eta f'\right)$$

where

$$\underbrace{\psi^* = (2x^*)^{1/2} f(\eta)}_{\text{Similarity Rule}} \; ; \; \underbrace{\eta = \frac{y^*}{(2x^*)^{1/2}}}_{\text{Similarity Variable}}$$

and:

$$f''' + ff'' = 0$$

$$f(0) = f'(0) = 0 \quad f'(\infty) = 1$$

What can we learn from all this? First, that the thickness of the boundary layer grows as $x^{*1/2}$. Since the profile is self-similar (same shape for all x), we approach the free stream velocity for some constant value of η. We expect, for example, that we reach 50% of the free stream velocity at some $\eta = \eta_{50\%} = O(1)$:

$$f'(\eta_{50\%}) = \tfrac{1}{2} \quad (\text{free stream was } f' = 1)$$

To get the value of $\eta_{50\%}$ we'd

have to solve the ODE, which can be done numerically, but we know $\eta_{50\%}$ will be some O(1) constant (the actual value is $\eta_{50\%} = 1.096$).

What y value is this?

$$\eta_{50\%} = \frac{y^*_{50\%}}{(2x^*)^{1/2}} = \frac{y_{50\%}}{\left(2\frac{\nu x}{U}\right)^{1/2}}$$

Thus

$$y_{50\%} = \left(2\frac{\nu x}{U}\right)^{1/2} \eta_{50\%}$$

So, within some O(1) number, we reach 50% of the free stream velocity at $y \sim \left(\frac{\nu x}{U}\right)^{1/2}$ – and we get this without solving the equation!

What about the drag on the plate? Remember that we can break drag into two pieces:

$$\underset{\sim}{F} = \underbrace{\underset{\sim}{F}_n}_{\substack{\text{Normal forces}\\ \text{(form drag)}}} + \underbrace{\underset{\sim}{F}_t}_{\substack{\text{Shear forces}\\ \text{(skin friction)}}}$$

In this case normal forces are zero, thus we just get skin friction! The skin friction is the shear stress:

$$\underbrace{\tau_w}_{\text{wall shear stress}} = \tau_{yx}\Big|_{y=0} = \mu\left[\frac{\partial u}{\partial y} + \underbrace{\cancel{\frac{\partial v}{\partial x}}}_{0 \text{ at plate}}\right]\Bigg|_{y=0}$$

$$\text{So } \tau_w = \mu\frac{\partial u}{\partial y}\Big|_{y=0} = U\mu\frac{\partial}{\partial y}(f')\Big|_{y=0}$$

$$= \frac{U\mu}{\delta}\frac{\partial f'}{\partial y^*}\Big|_{y^*=0} = \frac{U\mu}{\delta(2x^*)^{1/2}} f''(0)$$

where, plugging in for x^* & δ, yields

$$\tau_w = \frac{\mu}{\sqrt{2}}\left(\frac{U^3}{\nu x}\right)^{1/2} f''(0)$$

Where $f''(0)$ is again some O(1) constant

Which must be calculated numerically. Doing this, we get $f''(0) = 0.4696$, so:

$$\tau_w = 0.332\,\mu\left(\frac{U^3}{\nu x}\right)^{1/2}$$

We may define a local drag coefficient:

$$C_D^{(loc)} = \frac{\tau_w}{\underset{\nwarrow \text{ kinetic E/vol of flow}}{\tfrac{1}{2}\rho U^2}}$$

So: $$C_D^{(loc)} = \frac{0.664}{\left(\frac{\rho U x}{\mu}\right)^{1/2}} \equiv \frac{0.664}{\underset{\uparrow \text{ local plate Re}}{Re_x^{1/2}}}$$

So the drag decreases as we move down the plate. This makes sense because the B.L. is getting thicker, so the shear rate is going down.

What is the total drag?

$$\frac{F}{\frac{1}{2}\rho U^2 \underbrace{LW}_{\text{area of plate}}} = \frac{1}{L}\int_0^L C_D^{(loc)}\,dx$$

$$= \frac{1}{L}\int_0^L \frac{0.664}{\left(\frac{\rho U}{\mu}\right)^{1/2}} x^{-1/2}\,dx = \frac{1.328}{Re_L^{1/2}}$$

or $$\frac{F}{\frac{1}{2}\rho U^2 LW} = \frac{2^{3/2} f''(0)}{Re_L^{1/2}}$$

In which we could have gotten everything to within some unknown O(1) cst without having ever solved the ODE! This is the power of both scaling analysis and similarity transforms. The former tells you how a problem depends on the parameters involved, while the latter tells you alot about the functional form!

OK, for flow past a flat plate we had a uniform Euler Flow. What happens for a more complicated system?

Let's look at Stagnation Flow produced by a jet impinging on a surface (often used in cleaning).

First we look at the Euler Flow: the flow is inviscid and irrotational, so:

$$\underset{\sim}{u} = -\underset{\sim}{\nabla}\phi \ , \quad \nabla^2\phi = 0$$

In this coordinate system we have

$$u = -\frac{\partial\phi}{\partial x}, \quad v = -\frac{\partial\phi}{\partial y}$$

With B.C. $v\big|_{y=0} = 0$ (zero normal velocity)

Now for inviscid stagnation flow the solution is very simple:

$$u = \lambda x \ , \quad v = -\lambda y$$

which yields the potential:

$$\phi = -\tfrac{1}{2}\lambda(x^2 - y^2)$$

We will also need the pressure at the surface $y = 0$. Let the pressure at the origin be p_0. Since the flow is inviscid we have Bernoulli's eq'n:

$$p + \tfrac{1}{2}\rho(\underset{\sim}{u}\cdot\underset{\sim}{u}) = \text{cst along a streamline}$$

The surface $y = 0$ is a streamline, and at $x = y = 0$ the velocity vanishes.

Thus: $$p\big|_{y=0} = p_0 - \tfrac{1}{2}\rho u^2$$

All this is for Euler Flow. What about

the flow in the boundary layer?

We have the B.L. eq'ns:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

where we have divided by ρ & dropped the $\frac{\partial^2 u}{\partial x^2}$ term. We also have the B.C.'s:

$$u, v\Big|_{y=0} = 0 \quad ; \quad u\Big|_{y/\delta \to \infty} = u^{EF}\Big|_{y/L \to 0} = \lambda x$$

and p is given by Bernoulli's eq'n <u>outside</u> the BL.:

$$p + \frac{1}{2}\rho\, u^{(EF)^2} = cst$$

$$\therefore \frac{\partial p}{\partial x} + \rho\, u^{EF}\frac{\partial u^{EF}}{\partial x} = 0$$

$$\text{or } \frac{1}{\rho}\frac{\partial p}{\partial x} = -\lambda^2 x$$ (pressure decreases in x-direction)

so in the B.L.:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \lambda^2 x + \nu\frac{\partial^2 u}{\partial y^2}$$

Let: $x^* = x/L$, $y^* = y/\delta$, $u^* = \frac{u}{\lambda L}$, $v^* = \frac{v}{\lambda\delta}$

↳ driven by ⊄

Thus:

$$\frac{\lambda^2 L^2}{L}\left(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) = \lambda^2 L x^* + \frac{\nu\lambda L}{\delta^2}\frac{\partial^2 u^*}{\partial y^{*2}}$$

or:

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = x^* + \underbrace{\frac{\nu}{\lambda\delta^2}}_{1}\frac{\partial^2 u^*}{\partial y^{*2}}$$

So $\delta = \left(\frac{\nu}{\lambda}\right)^{1/2}$ which is indep. of L!

Physically, the <u>negative</u> pressure gradient acts as a <u>source</u> of momentum in the B.L., which retards its growth!

So:

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = x^* + \frac{\partial^2 u^*}{\partial y^{*2}}$$

$$u^*\Big|_{y^* \to \infty} = x^*$$ (inner limit of EF)

$$u^*\Big|_{y^*=0} = v^*\Big|_{y^*=0} = 0$$

We define the streamfunction ψ:

$$u^* = \frac{\partial \psi^*}{\partial y^*}, \quad v^* = -\frac{\partial \psi^*}{\partial x^*}$$

Thus:

$$\psi^*_{y^*}\,\psi^*_{x^*y^*} - \psi^*_{x^*}\,\psi^*_{y^*y^*} = x^* + \psi^*_{y^*y^*y^*}$$

$$\psi^*_{y^*}\Big|_{y^* \to \infty} = x^*, \quad \psi^*_{x^*}\Big|_{y^*=0} = \psi^*_{y^*}\Big|_{y^*=0} = 0$$

Let's look for a similarity transform!

$$\psi^* = A\bar{\psi}, \quad x^* = B\bar{x}, \quad y^* = C\bar{y}$$

So:

$$\frac{A^2}{BC^2}\left[\overline{\psi_y\,\psi_{xy} - \psi_x\,\psi_{yy}}\right] = B\bar{x} + \frac{A}{C^3}\bar{\psi}_{\overline{yyy}}$$

Dividing <u>through by</u> B:

$$\frac{A^2}{B^2C^2}\left[\psi_y\,\psi_{xy} - \psi_x\,\psi_{yy}\right] = \bar{x} + \frac{A}{BC^3}\bar{\psi}_{\overline{yyy}}$$

Thus the DE is <u>invariant</u> if <u>both</u>

$$\frac{A^2}{B^2C^2} = 1 \quad \underline{\text{and}} \quad \frac{A}{BC^3} = 1$$

What about i<u>nhomogeneous</u> B.C.?

$$\frac{A}{C}\frac{\partial\bar{\psi}}{\partial\bar{y}}\Big|_{\bar{y}\to\infty} = B\bar{x}$$

or $\frac{A}{BC} = 1$ which is the <u>same</u> restriction as we already had!

So we only have <u>2</u> restrictions, and we'll get a similarity transform!

What is it?

$$\frac{A}{BC} = 1, \quad \frac{A}{BC^3} = 1 \quad \therefore \boxed{C = 1, \frac{A}{B} = 1}$$

Thus:

$$\frac{\psi^*}{x^*} = f(\eta)\ ; \quad \underline{\underline{\eta = y^*}}$$

η is <u>not</u> a function of x^*! We could have guessed this because δ wasn't a function of L either!

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So $\psi^* = x^* f(y^*)$

$\frac{\partial \psi^*}{\partial x^*} = f(y^*)$; $\frac{\partial \psi^*}{\partial y^*} = x^* f'$, etc.

We get the transformed D.E.:

$$(x^* f')(f') - (f)(x^* f'') = x^* + x^* f'''$$

or, rearranging,

$$f''' = f'^2 - ff'' - 1$$

and $f(0) = f'(0) = 0$, $f'(\infty) = 1$

The shear stress (which is what leads to cleaning the surface!) is just:

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{\mu \lambda x}{\delta} f''(0)$$

where $f''(0)$ is some constant!

It can be shown that any B.L. flow where $u^{EF}\big|_{y/L \to 0} \sim x^{\alpha}$ will admit a similarity solution!

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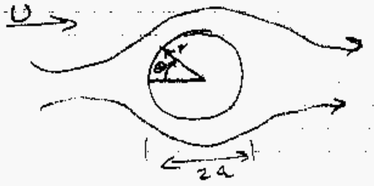
OK, what about B.L. flows in more complex geometries? Consider a cylinder:



From Euler Flow equations, we have the pressure distribution:

$$(p - p_0)\big|_{r=a} = \frac{1}{2} \rho U^2 (1 - 4 \sin^2\theta)$$

To look at this problem, we define Boundary Layer coordinates: we let:

$x = \theta a$ (distance along bdy from leading stagnation point)

$y = r - a$ (distance normal to bdy)

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Thus in the boundary layer:

$$\frac{\partial p}{\partial x} = \frac{1}{a} \frac{\partial}{\partial \theta} \left(p^{EF}\big|_{r=a} \right) = -\frac{4 \rho U^2}{a} \sin\theta \cos\theta$$

Thus for $0 < \theta < \frac{\pi}{2}$ the pressure gradient is negative. This means it is a source of momentum in the BL, and retards BL growth!

For $\theta > \pi/2$ we have $\frac{\partial p}{\partial x} > 0$, so it is a sink of momentum in the BL. This leads to rapid growth, and ultimately to BL separation!

To drive a BL against an adverse pressure gradient ($\frac{\partial p}{\partial x} > 0$) you have to get momentum in somehow. For laminar BL's this occurs only due to viscous diffusion ($\nu \frac{\partial^2 u}{\partial y^2}$), which is weak. A more efficient method

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is by promoting turbulence, since (as we'll see next lecture!) this leads to an eddy viscosity many times that of the molecular viscosity. This is done on airplane wings by vortex generators: tiny little fins that stick up out of the wing surface. These have the effect of increasing skin friction (which is small) but decreasing form drag by delaying or preventing separation.

Another example: baseballs! For a smooth sphere, the EF drag is zero because of complete pressure recovery on the back side! In practice, BL separation kills off the

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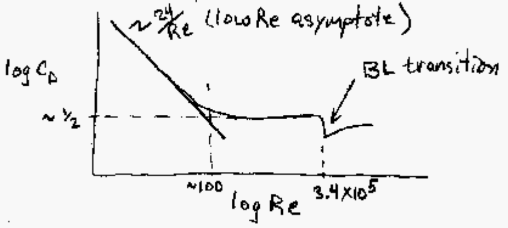
recovery and leads to a drag which scales as:

$$F \sim C_D \left(\underbrace{\tfrac{1}{2}\rho U^2}_{\text{inertial scaling for pressure}} \underbrace{\pi a^2}_{\text{cross-section area}} \right)$$

We can plot up C_D vs. Re:



The abrupt transition at $Re \sim 3.4\times10^5$ results from the transition of the BL to turbulence, delaying separation, giving an increase in pressure recovery and reducing drag ~ 6 fold! On a baseball this transition is triggered at a lower Re

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by the seams. If the ball is thrown without rotation, it can cause it to dart sideways in an unpredictable manner due to recovery on one side, and not the other!

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What about boundary layer flow on a more complex shape such as a wing? Again, define boundary layer coordinates:

$x \equiv$ distance along surface from leading stagnation point

$y \equiv$ distance normal to surface

If $\delta/L \ll 1$ we may ignore curvature in the boundary layer! We thus get the B.L. eq'ns in Cartesian coordinates:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

where p is obtained from Bernoulli's eq'n applied to the Euler (inviscid) flow outside the B.L. Let u_0, p_0 be the velocity & pressure far upstream

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and let u_∞ be the inner limit of the EF solution (e.g., the EF velocity evaluated at the surface). Thus:

$$p = p_0 + \tfrac{1}{2}\rho u_0^2 - \tfrac{1}{2}\rho u_\infty^2$$

and thus:

$$\frac{\partial p}{\partial x} = -\rho u_\infty \frac{d u_\infty}{d x}$$

We also have the B.C.'s:

$$u\Big|_{y=0} = v\Big|_{y=0} = 0, \quad u\Big|_{y/\delta\to\infty} = u_\infty$$

and the C.E.:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

we may eliminate v by integrating over the B.L.:

$$v = -\int_0^y \frac{\partial u}{\partial x}\,dy \qquad \text{since } v\Big|_{y=0} = 0$$

Thus:

$$u\frac{\partial u}{\partial x} - \left(\int_0^y \frac{\partial u}{\partial x}\,dy\right)\frac{\partial u}{\partial y} = u_\infty \frac{du_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

with B.C.'s:

$$u\big|_{y=0} = 0 \;, \quad u\big|_{y/\delta \to \infty} = u_\infty$$

Even with knowledge of $u_\infty(x)$ (e.g., the EF solution) we still have to solve this numerically. For anything other than power law forms $u_\infty \sim x^\alpha$ we won't get a similarity solution either!

We can develop a more useful expression, known as the integral BL eq'n by integrating over the BL thickness in the y-direction!

$$\int_0^{y/\delta\to\infty} \left\{ u\frac{\partial u}{\partial x} - \left(\int_0^y \frac{\partial u}{\partial x}\,dy\right)\frac{\partial u}{\partial y} - u_\infty\frac{du_\infty}{dx}\right\} dy$$

$$= \int_0^{y/\delta\to\infty} \nu\frac{\partial^2 u}{\partial y^2}\,dy$$

Let's work on the LHS:

$$LHS = \int_0^{y/\delta\to\infty} \left\{ u\frac{\partial u}{\partial x} - u_\infty\frac{du_\infty}{dx}\right\} dy - \int_0^{y/\delta\to\infty} \left(\int_0^y \frac{\partial u}{\partial x}\,dy\right)\frac{\partial u}{\partial y}\,dy$$

The second term may be integrated by parts to yield:

$$LHS = \int_0^{y/\delta\to\infty} \left\{ u\frac{\partial u}{\partial x} - u_\infty\frac{du_\infty}{dx}\right\} dy - \left[u\int_0^y \frac{\partial u}{\partial x}\,dy\right]_0^{y/\delta\to\infty} + \int_0^{y/\delta\to\infty} u\frac{\partial u}{\partial x}\,dy$$

$$= \int_0^{y/\delta\to\infty} \left\{ u\frac{\partial u}{\partial x} - u_\infty\frac{du_\infty}{dx}\right\} dy + \int_0^{y/\delta\to\infty} \left\{ u\frac{\partial u}{\partial x} - u_\infty\frac{\partial u}{\partial x}\right\} dy$$

where we have made use of the B.C. $u\big|_{y=0} = 0$ and that $u\big|_{y/\delta\to\infty} = u_\infty(x)$ which isn't a $f^n(y)$ and can be pulled

inside the integral. The whole LHS becomes:

$$LHS = \int_0^{y/\delta\to\infty} \left\{ 2u\frac{\partial u}{\partial x} - u_\infty\frac{du_\infty}{dx} - u_\infty\frac{\partial u}{\partial x}\right\} dy$$

$$= \int_0^{y/\delta\to\infty} \left\{ \frac{\partial u^2}{\partial x} - u_\infty\frac{du_\infty}{dx} - \frac{\partial (u u_\infty)}{\partial x} + u\frac{du_\infty}{dx}\right\} dy$$

$$= \frac{d}{dx}\int_0^{y/\delta\to\infty} (u^2 - u u_\infty)\,dy + \frac{du_\infty}{dx}\int_0^{y/\delta} (u - u_\infty)\,dy$$

$$= -\frac{d}{dx}\left\{ u_\infty^2 \int_0^{y/\delta\to\infty} \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right) dy \right\} - u_\infty\frac{du_\infty}{dx}\int_0^{y/\delta\to\infty} \left(1 - \frac{u}{u_\infty}\right) dy$$

We thus define two integrals:

$$\delta^* \equiv \int_0^{y/\delta\to\infty} \left(1 - \frac{u}{u_\infty}\right) dy \equiv \text{Displacement thickness}$$

$$\Theta \equiv \int_0^{y/\delta\to\infty} \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right) dy \equiv \text{Momentum thickness}$$

Both have units of length. The

displacement thickness is the distance streamlines outside the B.L. are deflected by the wedge of slow moving fluid in the boundary layer.

The ratio $H \equiv \delta^*/\Theta$ is known as the shape factor and is a dimensionless measure of the shape of the B.L. velocity profile. For laminar flow past a flat plate:

$$H = \delta^*/\Theta = 2.59$$

but this will change for $u_\infty \neq cst$, and if we have turbulent flow!

Ok, what's all this good for? Let's look at the RHS:

$$RHS = \int_0^{y/\delta\to\infty} \nu\frac{\partial^2 u}{\partial y^2}\,dy = \frac{\mu}{\rho}\frac{\partial u}{\partial y}\bigg|_0^{y/\delta\to\infty} = -\frac{1}{\rho}\mu\frac{\partial u}{\partial y}\bigg|_{y=0}$$

But this is just the shear stress at the surface! $\tau_o \equiv \mu \frac{\partial u}{\partial y}\Big|_{y=0}$

So:

$$\frac{\tau_o}{\rho} = \frac{d}{dx}\left(u_\infty^2 \Theta\right) + \delta^* u_\infty \frac{du_\infty}{dx}$$

which is known as the von Kármán boundary-layer momentum balance. In general, it's very difficult to measure a velocity derivative $\frac{\partial u}{\partial y}$ at a surface, so instead we use integrals of u to get Θ & δ^* and then use these to calculate skin friction!

For our flat plate problem $u_\infty \equiv U$ (cst) thus in this case:

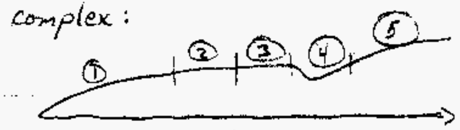
$$\frac{\tau_o}{\rho}\Big|_{\substack{\text{Blasius}\\ \text{problem}}} = U^2 \frac{d\Theta}{dx}$$

and the total drag is just:

$$F = w\int_0^L \tau_o \, dx = w\rho U^2 \Theta\Big|_{x=L}$$

which is very convenient! This technique is used in Senior Lab.

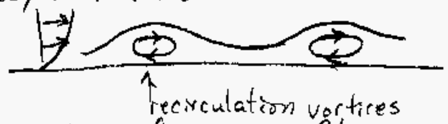
So far we've focused on Laminar BL flows (although the von Kármán balance works pretty well in turbulent flow too). This is valid up to $Re_x \sim 3\times10^5$. Beyond this point life gets more complex:



① Laminar flow (Blasius sol'n)

② Unstable laminar flow - 2-D Tollmein - Schlicting waves which can be predicted via an instability analysis

from the Navier-Stokes equations. They look like this:



superimposed on mean flow

③ Unstable laminar flow: 3-D waves and vortex formation

④ Bursting of vortices and growth of fixed turbulent spots

⑤ Fully developed turbulent boundary layer flow

which, like flow along a sufficiently long flat plate, brings us to a discussion of turbulence!

Turbulence

Turbulent flow is chaotic and time dependent, so it is difficult to describe directly using the N-S equations. Instead, we look at the time-average of the motion!

Let $\underset{\sim}{u} = \overline{\underset{\sim}{u}} + \underset{\sim}{u}'$

where $\overline{\underset{\sim}{u}} \equiv \frac{1}{\delta t}\int_t^{t+\delta t} \underset{\sim}{u}\, dt$

e.g., we average $\underset{\sim}{u}$ over some small interval of time. By definition, then, fluctuations average out:

$$\frac{1}{\delta t}\int_t^{t+\delta t} \underset{\sim}{u}'\, dt \equiv \underline{\underline{0}}$$

The objective is to develop a set of averaged equations for $\overline{\underset{\sim}{u}}$, $\overline{p}$!

First, we look at the C.E.:

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 0$$

$$\therefore \frac{1}{\delta t}\int_t^{t+\delta t}(\underset{\sim}{\nabla}\cdot\underset{\sim}{u})\,dt = \underset{\sim}{\nabla}\cdot\left\{\frac{1}{\delta t}\int_t^{t+\delta t}\underset{\sim}{u}\,dt\right\} \equiv \underset{\sim}{\nabla}\cdot\underset{\sim}{\bar{u}}$$

In general, the linear terms don't give us any trouble! It's the non-linear ones that cause problems!

Let's look at the N-S eq'ns:

$$\rho\frac{\partial\underset{\sim}{u}}{\partial t} + \rho\,\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{u} = -\underset{\sim}{\nabla}\,p + \mu\nabla^2\underset{\sim}{u}$$

Let's time average each term:

$$\frac{1}{\delta t}\int_t^{t+\delta t}\rho\frac{\partial\underset{\sim}{u}}{\partial t}\,dt = \frac{\rho}{\delta t}\Big[\underset{\sim}{u}(t)\Big]_t^{t+\delta t}$$

$$= \rho\,\frac{\underset{\sim}{\bar{u}}(t+\delta t) - \underset{\sim}{\bar{u}}(t)}{\delta t} + \rho\,\frac{\underset{\sim}{u}'(t+\delta t) - \underset{\sim}{u}'(t)}{\delta t}$$

Now the second term may be non-zero, but it will have zero mean on average and

shouldn't contribute to the flow. If the time scale for turb. fluctuations is short with respect to the time scale for mean variations (e.g., the time scale of increasing or decreasing flow rates through a pipe) then the first term reduces to:

$$\frac{1}{\delta t}\rho\left[\bar{u}(t+\delta t) - \bar{u}(t)\right] \approx \rho\frac{\partial\underset{\sim}{\bar{u}}}{\partial t}$$

Next look at pressure:

$$\frac{1}{\delta t}\int_t^{t+\delta t}\underset{\sim}{\nabla}p\,dt \equiv \underset{\sim}{\nabla}\bar{P}$$

and the viscosity term:

$$\frac{1}{\delta t}\int_t^{t+\delta t}\mu\nabla^2\underset{\sim}{u}\,dt = \mu\nabla^2\underset{\sim}{\bar{u}}$$

So the linear terms didn't cause any trouble. Now for the non-linear

convection term:

$$\frac{1}{\delta t}\int_t^{t+\delta t}\rho\,\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{u}\,dt = \frac{\rho}{\delta t}\int_t^{t+\delta t}(\underset{\sim}{\bar{u}}+\underset{\sim}{u}')\cdot\underset{\sim}{\nabla}(\underset{\sim}{\bar{u}}+\underset{\sim}{u}')\,dt$$

$$= \frac{\rho}{\delta t}\int_t^{t+\delta t}\Big[\underset{\sim}{\bar{u}}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{\bar{u}} + \underbrace{\underset{\sim}{\bar{u}}\cdot\underset{\sim}{\nabla}\cancel{\underset{\sim}{u}'} + \cancel{\underset{\sim}{u}'}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{\bar{u}}}_{\text{avg to zero!}} + \underset{\sim}{u}'\cdot\underset{\sim}{\nabla}\,\underset{\sim}{u}'\Big]\,dt$$

$$= \rho\,\underset{\sim}{\bar{u}}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{\bar{u}} + \underbrace{\rho\,\frac{1}{\delta t}\int_t^{t+\delta t}\underset{\sim}{u}'\cdot\underset{\sim}{\nabla}\,\underset{\sim}{u}'\,dt}_{\text{non-zero!}}$$

Let's write this as

$$\rho\left\langle\underset{\sim}{u}'\cdot\underset{\sim}{\nabla}\,\underset{\sim}{u}'\right\rangle \equiv \underset{\sim}{\nabla}\cdot\left\langle\rho\,\underset{\sim}{u}'\underset{\sim}{u}'\right\rangle$$

for $\rho = cst$ (and $\underset{\sim}{\nabla}\cdot\underset{\sim}{u}' = 0$) where $\langle\,\rangle$ denote time averaging

Thus we get the time-averaged eq'ns:

$$\rho\frac{\partial\underset{\sim}{\bar{u}}}{\partial t} + \rho\,\underset{\sim}{\bar{u}}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{\bar{u}} = -\underset{\sim}{\nabla}\bar{P} + \mu\nabla^2\underset{\sim}{\bar{u}} - \underset{\sim}{\nabla}\langle\rho\,\underset{\sim}{u}'\underset{\sim}{u}'\rangle$$

This last term may be written as:

$$-\underset{\sim}{\nabla}\cdot\left(\langle\rho\,\underset{\sim}{u}'\underset{\sim}{u}'\rangle\right) \equiv \underset{\sim}{\nabla}\cdot\underset{\approx}{\bar{\tau}}^{turb}$$

where $\underset{\approx}{\bar{\tau}}^{turb} \equiv -\langle\rho\,\underset{\sim}{u}'\underset{\sim}{u}'\rangle \equiv$ Reynolds Stress

It's the added momentum flux due to turbulent fluctuations!

To solve these equations we need a way of modelling $\underset{\approx}{\bar{\tau}}^{turb}$ in terms of velocity gradients, much like Newton's Law of Viscosity for laminar stresses! Unfortunately, this is hard to do, and only approximate models exist!

Let's look at the simplest model: Prandtl mixing length theory

In gases, mass, momentum & energy transport rates are calculated by looking at the rate with which molecules cross streamlines ⇒ since they physically carry momentum, mass & energy, if they cross streamlines you get a flux of these quantities! You can use this to estimate the viscosity of a gas, for example!

In turbulence, Prandtl's idea was that eddies do the same thing! As two eddies exchange places (across streamlines) they also lead to momentum transfer (e.g., the Reynolds stress). In a channel, these arguments lead to:

$$\bar{\tau}_{yx}^{turb} = \underbrace{\left\{\rho \ell^2 \left|\frac{\partial u}{\partial y}\right|\right\}}_{\to \bar{\mu}^t} \frac{\partial u}{\partial y}$$

The quantity above is the <u>eddy</u> <u>viscosity</u>

by analogy with Newton's law of viscosity! The quantity "ℓ" is the length scale of the eddies, and the shear rate $\left|\frac{\partial u}{\partial y}\right|$ is the <u>rate</u> with which such exchanges take place!

Prandtl made the further approximation: Eddies are bigger in the middle of a pipe than near the wall, so let:

$$\ell \equiv \alpha y$$

where the wall is at $y=0$. This const. α is known as the von Kármán const, and is about $\alpha = 0.36$ by fitting to empirical data!

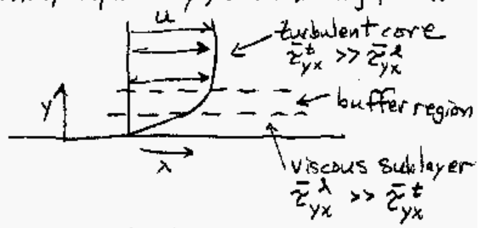
Ok, now let's apply this to flow near a wall. If the shear stress

is <u>constant</u>, we get:

$$\bar{\tau}_{yx} = \tau_0 = \bar{\tau}_{yx}^{laminar} + \bar{\tau}_{yx}^{turb}$$

In general, $\bar{\tau}_{yx}^{t} >> \bar{\tau}_{yx}^{\ell}$ (about 100x!) so we'll <u>ignore</u> the laminar contrib.

We find, empirically, the following picture:



In the turbulent core:

$$\tau_0 \cong \rho \alpha^2 y^2 \left|\frac{\partial \bar{u}}{\partial y}\right| \frac{\partial \bar{u}}{\partial y}$$

So:

$$\frac{\partial \bar{u}}{\partial y} = \frac{1}{y}\frac{1}{\alpha}\left(\frac{\tau_0}{\rho}\right)^{1/2}$$

Let's render this <u>dimensionless</u>! The scaling for <u>velocity</u> is the

<u>Friction</u> <u>velocity</u> $v_* \equiv \left(\frac{\tau_0}{\rho}\right)^{1/2}$

The scaling for y is the <u>viscous</u> <u>length</u> <u>scale</u> $\equiv \frac{\nu}{\left(\frac{\tau_0}{\rho}\right)^{1/2}}$

We thus define scaled coordinates:

$$\bar{u}^+ = \frac{\bar{u}}{\left(\frac{\tau_0}{\rho}\right)^{1/2}} \qquad y^+ = \frac{y}{\nu/\left(\frac{\tau_0}{\rho}\right)^{1/2}}$$

So:

$$\frac{\partial \bar{u}^+}{\partial y^+} = \frac{1}{\alpha}\frac{1}{y^+}$$

and

$$\bar{u}^+ = \frac{1}{\alpha}\ln y^+ + C$$

or the velocity profile should be logarithmic near the wall! The constants α and C are obtained by fitting the model to the data. For flow through tubes we get $\alpha \approx 0.36$, $C \approx 3.8$ for $y^+ \gtrsim 26$ (e.g., $y^+ \approx 26$ is the edge of the turbulent core). This works for $Re \geq 20{,}000$ in smooth pipes. For $y^+ < 26$ you need to use other correlations which include $\bar{\tau}_{yx}^{\ell}$ (e.g., viscosity). For very small y^+ (e.g., $y^+ \leq 5$) we may ignore $\underline{\underline{\bar{\tau}_{yx}^{t}}}$ rather than $\bar{\tau}_{yx}^{\ell}$! This is the viscous sublayer, which yields:

$$\bar{u}^+ = y^+ \quad 0 < y^+ \leq 5$$

So we get:

$$\bar{u}^+ = \begin{cases} y^+ & 0 < y^+ < 5 \\ \frac{1}{0.36} \ln y^+ + 3.8 & y^+ \geq 26 \end{cases}$$

and a more complicated expression in the buffer region $5 \leq y^+ \leq 26$

What are the physical dimensions of the friction velocity and viscous length scale?

Suppose we are pumping water through a 4" (10 cm) Dia pipe at $\langle u \rangle = 1\, m/s$.

We have:

$$Re = \frac{\langle u \rangle D}{\nu} \approx 10^5$$

At this Re we are well into the turbulent regime! Empirical correlations suggest that for $2.1 \times 10^3 < Re < 10^5$ the wall shear stress is about:

$$\frac{\tau_0}{\frac{1}{2}\rho \langle u \rangle^2} \approx \frac{0.0791}{Re^{1/4}}$$

Thus $\tau_0 \cong 22\, dyne/cm^2$ - about 27 x greater than would be the case for laminar flow! We thus get the friction velocity $V_* = \left(\frac{\tau_0}{\rho}\right)^{1/2} = 4.7\, cm/s$ and the viscous length:

$$\frac{\nu}{V_*} \simeq 0.002\, cm = 20 \mu m!$$

Thus since we reach the turbulent core only 520μm from the wall, virtually the entire tube is turbulent! In general, for smooth tubes:

$$\frac{\nu}{V_*} = \frac{\nu}{\left(\frac{\tau_0}{\rho}\right)^{1/2}} = \frac{\nu}{\langle u \rangle} \frac{1}{\left(\frac{\tau_0}{\rho \langle u \rangle^2}\right)^{1/2}}$$

$$\approx \frac{\nu}{\langle u \rangle} \frac{1}{\left(\frac{\frac{1}{2} \cdot 0.0791}{Re^{1/4}}\right)} = \frac{\nu}{\langle u \rangle} 5 \cdot Re^{1/8}$$

for $2100 < Re < 10^5$, which provides a convenient way of estimating the thickness of the viscous sublayer (about 5 - 26 times this value).

Friction Factors

How do you, as an engineer, determine ΔP and Q (flow rate) in a piping system? Such systems may be very complex networks, and the flow is usually turbulent. The easiest way is to use empirical friction factors!

Let's start with Dimensional Analysis:

$$\Delta P = f^n(\langle u \rangle, L, D, e, \mu, \rho)$$

where e is the surface roughness of a pipe. We can form the Dimensional matrix:

$$\begin{array}{c|ccccccc} & \Delta P & \langle u \rangle & L & D & e & \mu & \rho \\ \hline M & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ L & -1 & 1 & 1 & 1 & 1 & -1 & -3 \\ T & -2 & -1 & 0 & 0 & 0 & -1 & 0 \end{array}$$

The rank of this matrix is $\underline{3}$, thus

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we get $7-3=4$ dimensionless groups! We can pick these a number of ways, but let's look for ones that make sense! We choose the aspect ratios:

$$\frac{L}{D}, \frac{e}{D}$$

And the Reynolds # $Re = \frac{\langle u \rangle D \rho}{\mu}$

The last one is the dimensionless pressure. Usually we're at high Re, so use inertial scaling:

$$\frac{\Delta P}{\rho \langle u \rangle^2} = f^{II}\left(\frac{L}{D}, \frac{e}{D}, Re\right)$$

$\hookrightarrow$ known as the Euler #

It's actually more convenient to define a head loss

$$h_L \equiv \frac{\Delta P}{\rho g}$$

– the loss in hydrostatic head due to fluid friction!

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Thus:

$$\frac{h_L}{(\langle u \rangle^2/g)} = f^{II}\left(\frac{L}{D}, \frac{e}{D}, Re\right)$$

Empirically, we observe that for $L/D >> 1$ we have $h_L \sim L$ (e.g., double the pipe length & you double the pressure drop).

Thus:

$$\frac{h_L}{\langle u \rangle^2/g} = \frac{L}{D} \; f^{II}\left(\frac{e}{D}, Re\right)$$

We can define the Fanning Friction Factor f_f s.t.

$$\frac{h_L}{(\langle u \rangle^2/g)} = \frac{L}{D} \; (2 f_f)$$

where $f_f \equiv f^{II}\left(\frac{e}{D}, Re\right)$

If we determine f_f either theoretically or empirically, it's easy to get the head loss!

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Let's look at low Re first. For laminar flow we get Poiseuille's Law:

$$\Delta P = 32 \frac{\mu \langle u \rangle L}{D^2}$$

Thus:

$$h_L \equiv \frac{\Delta P}{\rho g} = 32 \frac{\mu \langle u \rangle L}{\rho g D^2}$$

or, $f_f = 16 \frac{\mu}{D \langle u \rangle \rho} \equiv \frac{16}{Re}$!

Note that f_f is inversely proportional to Re as $Re \to 0$! This is because we've used inertial scalings for ΔP, whereas at low Re $\Delta P \sim \frac{\langle u \rangle \mu L}{D^2}$ (viscous scaling).

Empirically, for laminar flow f_f isn't a strong function of e/D provided $e/D << 1$. In fact, for $Re \to 0$ we can show that the correction is $O(e/D)$

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using the Minimum Dissipation Theorem. This will not be true at higher Re, where even very small e/D can play a big role by promoting turbulence!

Ok, how about turbulent flow? We start with the Law of the Wall obtained by Prandtl & von Kármán:

$$\bar{u}^+ \equiv \underset{\overset{\uparrow}{\frac{1}{\kappa}}}{2.5} \ln y^+ + 5.5 \quad \text{in the turbulent core}$$

$\frac{1}{\kappa}$, $\kappa \simeq 0.4$ (Kármán's value)

remember $\bar{u}^+ = \frac{\bar{u}}{(\tau_0/\rho)^{1/2}} \swarrow V_* \equiv$ friction velocity

Let's assume that this applies throughout the pipe, and use it to calculate $\langle u \rangle$!

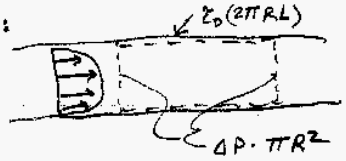
First, we need to relate y^+ to r:

$$y = R - r \; ; \; y^+ = \frac{(\tau_0/\rho)^{1/2}}{\nu} y$$

So:
$$y^+ = \frac{(\tau_0/\rho)^{1/2}}{\nu}(R-r)$$

Now
$$\langle u\rangle = \frac{1}{\pi R^2}\int_0^R u\ 2\pi r\, dr$$
$$= \frac{2}{R^2}\int_0^R \left(\frac{\tau_0}{\rho}\right)^{1/2}\left(5.5+2.5\ln\left(\frac{(\tau_0/\rho)^{1/2}}{\nu}(R-r)\right)\right) r\,dr$$
$$= \left(\frac{\tau_0}{\rho}\right)^{1/2}\left[2.5\ \ln\left(\frac{R\,(\tau_0/\rho)^{1/2}}{\nu}\right)+1.75\right]$$

we need to relate τ_0 to ΔP. We do this with a <u>force balance</u> on the pipe:



Forces must <u>balance</u>, so:
$$\tau_0\ \underbrace{2\pi RL}_{\text{Area of wall}} = \Delta P\ \underbrace{\pi R^2}_{\text{x-section area}}$$

So
$$\frac{\Delta P}{L} = \frac{2\tau_0}{R}$$
which is valid at <u>all</u> Re!

Recall that $\Delta P \equiv 2 f_f \frac{L}{D}\rho\langle u\rangle^2$

Thus $$\frac{\langle u\rangle}{(\tau_0/\rho)^{1/2}} = \frac{1}{\sqrt{f_f/2}}$$

So:
$$\frac{1}{\sqrt{f_f/2}} = 2.5\ \ln\left\{\frac{R\langle u\rangle}{\nu}\sqrt{\frac{f_f}{2}}\right\}+1.75$$

or, as is more usually expressed,
$$\frac{1}{\sqrt{f_f}} = 4.06\log_{10}\left\{Re\sqrt{f_f}\right\} - 0.40$$
as derived by von Kármán. We can get a little better result by <u>fitting</u> this model to empirical ΔP <u>measurements</u>! If we take the <u>constants</u> as <u>adjustable</u>

<u>parameters</u>, we get:
$$\frac{1}{\sqrt{f_f}} = 4.0\log_{10}\left\{Re\sqrt{f_f}\right\} - 0.40$$
which is pretty close to what von Kármán got from mixing-length theory! That was for <u>smooth pipes</u> ($e/D = 0$). For rough pipes, we get empirically:
$$\frac{1}{\sqrt{f_f}} = 4.0\ \log_{10}\left(\frac{D}{e}\right)+2.28$$
<u>provided</u> $\frac{e}{D} \gtrsim \frac{10}{Re\sqrt{f_f}}$

this makes more sense if we recall that $f_f = \frac{\tau_0}{\frac{1}{2}\rho\langle u\rangle^2}$ thus we get:
$$\frac{e}{D} \gtrsim \frac{10}{Re\sqrt{f_f}} = \frac{10}{\left(\frac{\cancel{\langle u\rangle}\rho D}{\nu}\right)\left(\frac{\tau_0}{\frac{1}{2}\rho\cancel{\langle u\rangle^2}}\right)^{1/2}}$$
$$= \frac{10}{\sqrt{2}}\ \frac{\nu/(\tau_0/\rho)^{1/2}}{D}$$
or $e^+ \gtrsim 7$ ⇒ e.g., when the wall roughness sticks up outside the

viscous sublayer!

Many plots of f_f vs Re & $\frac{e}{D}$ are available, but the most useful correlations are:
$$f_f = \frac{16}{Re}\ ,\quad Re<2100$$
$$f_f \approx \frac{0.0791}{Re^{1/4}}\ ,\quad \frac{e}{D}=0,\ 3\times10^3<Re<10^5$$
$$\frac{1}{\sqrt{f_f}} = 4.0\log_{10}\left\{Re\sqrt{f_f}\right\}-0.40,\quad Re>3\times10^3,\ e/D=0$$
$$\frac{1}{\sqrt{f_f}} = 4.0\log_{10}\left(D/e\right)+2.28,\quad \frac{e}{D}\geq\frac{10}{Re\sqrt{f_f}}$$

In a pipe system we don't have just pipe, but we also have fittings! These <u>also</u> contribute to the head loss, we may define, for high Re flow, :
$$h_L = \frac{\Delta P}{\rho g} \equiv K\ \frac{\langle u\rangle^2}{2g}$$

where the "K" values are determined empirically. A table of a few useful values is given below:

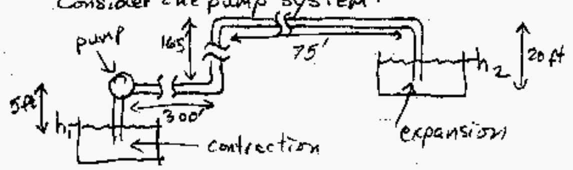
TABLE 14.1 FRICTION LOSS FACTORS FOR VARIOUS PIPE FITTINGS

Fitting	K	L_{eq}/D
Globe valve, wide open	7.5	350
Angle valve, wide open	3.8	170
Gate valve, wide open	0.15	7
Gate valve, $\frac{3}{4}$ open	0.85	40
Gate valve, $\frac{1}{2}$ open	4.4	200
Gate valve, $\frac{1}{4}$ open	20	900
Standard 90° elbow	0.7	32
Short-radius 90° elbow	0.9	41
Long-radius 90° elbow	0.4	20
Standard 45° elbow	0.35	15
Tee, through side outlet	1.5	67
Tee, straight through	0.4	20
180° Bend	1.6	75

(from Welty, Wicks & Wilson)

Ok, how do we use all this? Just add up the headloss on any stream!

Consider the pump system:



Suppose we have all 4" ID smooth pipe, and we want a flow rate $Q = 42\,\mathrm{ft^3/min}$.

What is the required power of the pump?

We have:

565' of 4" pipe	Change in Elevation: 150 ft
3 90° elbows	
1 sudden contraction	
1 sudden expansion	

First we calculate the Re:

$$Re = \frac{\langle u\rangle D}{\nu} \qquad \langle u\rangle = \frac{Q}{A} = \frac{42\,\mathrm{ft^3/min}}{\pi(.167\,\mathrm{ft})^2}$$

$$= 8.0\ \mathrm{ft/s}$$

$Re = 2.47\times10^5$ so flow is <u>turbulent</u>

For this Re, $f_f \approx 0.0038$

Thus for the pipes:

$$(h_L)_{pipes} = (2)(0.0038)\left(\frac{565'}{0.33'}\right)\frac{(8.0\ \mathrm{ft/s})^2}{(32.2\ \mathrm{ft/s^2})}$$

$= 25.4$ ft which is nearly 1 atm!

What about the fittings?

For a 90° elbow, we have $K \approx 0.7$

For a sudden contraction, we have (in general):

$$K_{contraction} \approx 0.45(1-\beta)$$

$$\text{where}\quad \beta \equiv \frac{A_{small}}{A_{large}}$$

Here $\beta = \underline{\underline{0}}$ so $K_{cont} = 0.45$

For an expansion we have:

$$K_{expansion} = (1-\beta)^2 = 1$$

(based on $\langle u\rangle$ in smaller pipe!)

Thus:

$$(h_L)_{fittings} = \left(3\cdot(0.7) + 0.45 + 1\right)\frac{1}{2}\frac{(8.0)^2}{32.2}$$

$$= 3.6\ \mathrm{ft}$$

Ok, so what is the total head loss?

It's just the sum of the change in elevation, $(h_L)_{pipes}$, & $(h_L)_{fittings}$!

$$\Delta h = h_2 - h_1 + (h_L)_{pipes} + (h_L)_{fittings}$$

$$= 150' + 25.4' + 3.6' \simeq \underline{179\ \mathrm{ft}}$$

(dominated by change in elevation)

What is the power requirement?

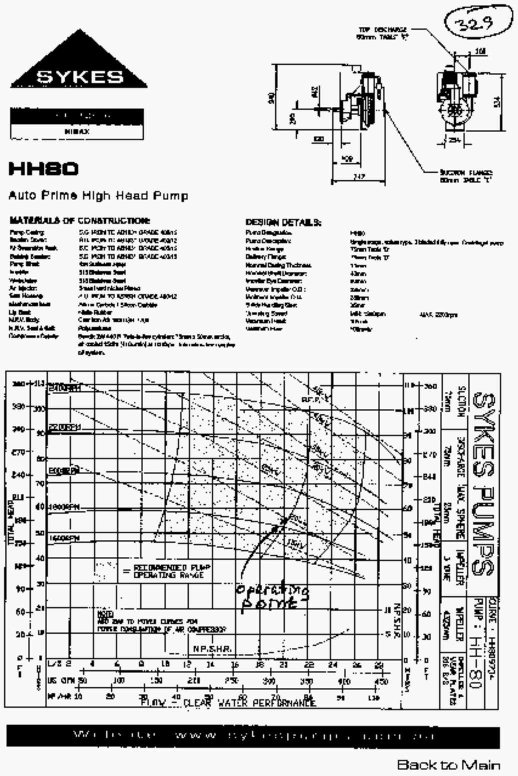
$$W = Q\,\Delta h\,\rho g = 7800\ \mathrm{ft\,lb_f/s} = 14\ \mathrm{hp}$$

So we need a pump output of 14 hp.

The input will be greater due to pump inefficiencies! What pump to use?

We look for a pump that puts out $42\,\mathrm{ft^3/min} \equiv 20\ \ell/s$ with a Δh of $179\,\mathrm{ft} = 54.6\,\mathrm{m}$

The pump curve of a pump which would do the job is on next page:



SYKES

HH80

Auto Prime High Head Pump

MATERIALS OF CONSTRUCTION:

DESIGN DETAILS:

The trick is to find a pump which can provide the <u>required head</u> (179 ft) and the <u>desired flow rate</u> (20 l/s), and where these values <u>also</u> lie in the <u>recommended operating range</u> of the pump! (shaded area). In this case, the HH80 pump operating at ~1820 RPM produces the required head & flow rate. What is the power consumption? This is given by the dashed curves. The consumption is ~19 kW (plus 2 kW for power consumption of air compressor as per footnote). What is the efficiency?

$$\text{Efficiency} = \frac{\text{Work out}}{\text{Work in}} = \frac{14\ hp}{21\ kW} = 0.50$$

$(1\ hp = 0.746\ kW)$

or about 50% efficiency - not too bad!

Note that the operating point was close to the "BEP" curve ⇒ Best Efficiency Point! As you move away from this curve, the efficiency goes down! On the y-axis, the efficiency is <u>zero</u> ⇒ no flow means no work!

As a final note on pump curves, look at the "NPSHR" curve at the bottom. This is the "Net Positive Suction Head" Required at the pump <u>inlet</u> to <u>prevent</u> cavitation in the pump! For our system:

$$h\Big|_{pump} = \underset{\text{1atm pressure}}{\underset{\uparrow}{33.8\,ft}} - \underset{\Delta h\ \text{elevation}}{\underset{\uparrow}{5\,ft}} - \overset{\text{5ft of pipe}}{\overset{\downarrow}{0.22\,ft}} - \overset{\text{contraction}}{\overset{\downarrow}{0.46\,ft}}$$

$$= 28.1\,ft = 8.6\,m$$

This is greater than the NPSHR of about 2 m required at the operating condition (flow rate), so we're fine.

You often want to <u>control</u> output of a pump by throttling it with a valve. <u>Always</u> put the valve on the <u>downstream side</u>! Otherwise the $(h_L)_{valve}$ will <u>reduce</u> the NPSH at the pump inlet, and it will <u>usually cavitate</u>! This is very bad, because cavitation increases wear and can lead to the pump failing.

What about piping networks in a plant? It's easy to account for these using the head loss approach! Consider the reactor with recycle:

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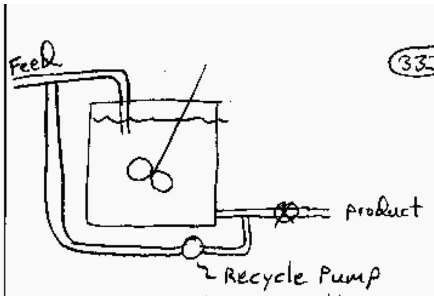
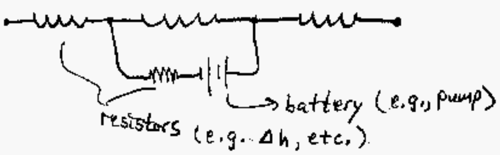


Suppose we want to size the pump for the recycle stream. We can represent the head losses & flow rates as an <u>electrical</u> <u>network</u>!



The head is equivalent to the <u>voltage</u>. The flow rate is equivalent to the <u>current</u>. The <u>only</u> difference is that

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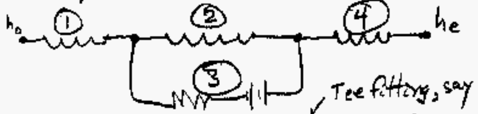
<u>Ohm's</u> <u>Law</u> gets modified due to the <u>non-linear</u> dependence of h_L on Q!

As in circuits: The <u>sum</u> of the head loss along <u>each</u> <u>possible</u> fluid path <u>from</u> a common node <u>to</u> a common node <u>must</u> <u>be</u> <u>the</u> <u>same</u>!

Let's apply this:

First, label the streams:



Tee fitting, say

$$h_L^{(1)} = \left[\left(2 f_f^{(1)} \frac{L^{(1)}}{D} \frac{1}{g}\right) + \frac{0.4}{2g}\right]\left(\frac{Q_1}{A_1}\right)^2$$

$$h_L^{(2)} = \left[\left(2 f_f^{(2)} \frac{L^{(2)}}{D} \frac{1}{g}\right) + \frac{\Sigma K^{(2)}}{2g}\right]\left(\frac{Q_2}{A_2}\right)^2 + \Delta h^{(2)}$$

$$h_L^{(3)} = \left[\left(2 f_f^{(3)} \frac{L^{(3)}}{D} \frac{1}{g}\right) + \frac{\Sigma K^{(3)}}{2g}\right]\left(\frac{Q_3}{A_3}\right)^2 + \Delta h^{(3)} - \Delta h_{pump}$$

$$h_L^{(4)} = \left[\left(2 f_f^{(4)} \frac{L^{(4)}}{D} \frac{1}{g}\right) + \frac{\Sigma K^{(4)}}{2g}\right]\left(\frac{Q_4}{A_4}\right)^2$$

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We combine these definitions with mass and head loss balances:

$$h_o - h_L^{(1)} - h_L^{(2)} - h_L^{(4)} = h_e$$

$$h_L^{(2)} = -h_L^{(3)}$$

$$Q_1 = Q_4 \quad Q_2 = Q_1 + Q_3$$

If we <u>specify</u>, say, Q_1 and the recycle ratio Q_3/Q_2 we could calculate both the total head loss through the system $h_o - h_e$ and the required pump head Δh_{pump}. Note that this is a system of <u>non-linear</u> equations, but it's easy to solve it numerically!

<u>Index</u> <u>Notation</u>

(A)

What is index notation? It is simply a compact & convenient way of representing scalars, vectors, and tensors. It is particularly useful for fluid mechanics, especially (as we shall see) at low Re.

⇒ There is <u>no</u> <u>new</u> <u>physics</u> associated with index notation, however it <u>can</u> reveal symmetries & relations which were <u>already</u> <u>there</u>!

For any tensor, the order of the tensor is given by the

Ⓑ

number of unrepeated indices!

a ⇒ no indices, scalar

x_i, u_j ⇒ one index, both are vectors

σ_{ij} ⇒ two indices, 2nd order tensor

ε_{ijK} ⇒ 3 indices, 3rd order tensor

The letters used as subscripts don't matter, e.g. x_i, x_j, x_p, etc. are equivalent

⇒ exception: in an equation, each term must have the same unrepeated indices, e.g.

$$x_i = y_i \text{ is same as } \underset{\sim}{x} = \underset{\sim}{y}$$

but $x_i = y_j$ is an error!

Ⓒ

A key feature of index notation is the dot product:

⇒ Repeated indices (in a product) implies summation!

$$\text{Thus:} \quad x_i y_i \equiv \underset{\sim}{x} \cdot \underset{\sim}{y} = \sum_i x_i y_i$$

$$(\text{e.g., } x_i y_i = x_1 y_1 + x_2 y_2 + x_3 y_3)$$

Just think of how you would code it up on a computer using loops!

The vector composition or outer product is also simple:

$$\underset{\approx}{A} = \underset{\sim}{x}\,\underset{\sim}{y} \text{ is given by } A_{ij} = x_i y_j$$

Since there are two unrepeated indices, $x_i y_j$ is a 2nd order tensor!

Ⓓ

* You cannot repeat an index in any product more than once:

$$x_i y_j z_i \equiv \underset{\sim}{y}\,(\underset{\sim}{x} \cdot \underset{\sim}{z}) \quad (\text{ok})$$

$$x_i y_i z_i \equiv \text{error}!$$

The order of multiplication (dot product) is preserved by the names/order of the indices!

Remember $\underset{\approx}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$?

In index notation:

$$A_{ij} x_j = b_i$$

To take the transpose, just reverse the order:

$$(A_{ij})^T = A_{ji}$$

Ⓔ

Remember the Normal Equations?

$$\underset{\approx}{A}^T \underset{\approx}{A}\,\underset{\sim}{x} = \underset{\approx}{A}^T \underset{\sim}{b}$$

We would write this as

$$A_{Ki} A_{Kj} x_j = A_{Ki} b_K$$

We could also look at the residual from linear regression:

$$r_i = A_{ij} x_j - b_i$$

$$\underset{\sim}{r} \cdot \underset{\sim}{r} \equiv (A_{ij} x_j - b_i)(A_{iK} x_K - b_i)$$

Note that there are no unrepeated indices in the product, so it's a scalar and that we switched a pair of "j"s to "K"s to avoid repeating j too many times! j was repeated

Ⓔ

already, so this is OK, e.g.

$$\underbrace{x_j x_j = x_k x_k}_{\text{both are scalars}}$$

while

$$\underbrace{x_j \neq x_k}_{\text{unrepeated!}}$$

We define a couple of things:

$\underset{\sim}{\nabla} \equiv \frac{\partial}{\partial x_i}$ ← gradient operator (or j, or k, etc.)

$\underset{\approx}{I} \equiv \delta_{ij}$ ← kronecker δ fn (Identity matrix)

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Note: $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$ (Identity matrix)

$$\frac{\partial x_i}{\partial x_i} = \delta_{ii} = 1 + 1 + 1 = 3$$

Ⓖ

OK, let's use this to solve for the Normal Equations!

Recall we had $\underset{\sim}{\nabla}(\underset{\sim}{r}^T \underset{\sim}{r}) = 0$

In index notation:

$$\frac{\partial}{\partial x_i}\left\{(A_{jk}x_k - b_j)(A_{jl}x_l - b_j)\right\}$$

$$= \frac{\partial}{\partial x_i}\left\{A_{jk}x_k A_{jl}x_l - b_j A_{jl}x_l - A_{jk}x_k b_j + b_j b_j\right\}$$

Or, since we only have to preserve the order of the indices:

$$= \frac{\partial}{\partial x_i}\left\{A_{jk}A_{jl}x_k x_l - 2A_{jk}b_j x_k + b_j b_j\right\}$$

Ⓗ

Note that we combined the two middle terms since

$$A_{jk}x_k \equiv A_{jl}x_l$$

The use of k or l was indeterminate because they were repeated. Only the unrepeated index "j" has to be the same on both sides!

OK, now we take some derivatives. Note that A_{ij} and b_j are constants, so they pop out!

$$\underset{\sim}{\nabla}(\underset{\sim}{r}^T \underset{\sim}{r}) \equiv A_{jk}A_{jl}\frac{\partial}{\partial x_i}(x_k x_l) - 2A_{jk}b_j \underbrace{\frac{\partial x_k}{\partial x_i}}_{\delta_{ki}} + \underbrace{\cancel{\frac{\partial b_j b_j}{\partial x_i}}}_{0}$$

Ⓘ

Now we compute the first term:

$$\frac{\partial}{\partial x_i}(x_k x_l) = x_k \frac{\partial x_l}{\partial x_i} + x_l \frac{\partial x_k}{\partial x_i}$$

(chain rule)

$$= x_k \delta_{il} + x_l \delta_{ik}$$

So: $\underset{\sim}{\nabla}(\underset{\sim}{r}^T \underset{\sim}{r}) \equiv A_{jk}A_{jl}(x_k\delta_{il} + x_l\delta_{ik}) - 2A_{jk}b_j\delta_{ik}$

Taking the dot product of a matrix (or vector) with the identity matrix leaves it unchanged. In index notation this is:

$$A_{ij}\delta_{jk} = A_{ik}$$

(just replace the "j" with a "k")

(J)

So:

$$\underset{\sim}{\nabla}(\underset{\sim}{r}^T\underset{\sim}{r}) \equiv A_{jk}A_{ji}x_k + A_{ji}A_{jl}x_l - 2A_{ji}b_j$$

Now the first two terms are identical since in both cases "l" and "k" are repeated indices and thus indeterminate.

So: $\underset{\sim}{\nabla}(\underset{\sim}{r}^T\underset{\sim}{r}) = 0$ becomes:

$$2A_{ji}A_{jk}x_k - 2A_{ji}b_j = 0$$

or $A_{ji}A_{jk}x_k = A_{ji}b_j$

Which is the same as:

$$\underset{\approx}{A}^T\underset{\approx}{A}\underset{\sim}{x} = \underset{\approx}{A}^T\underset{\sim}{b} \quad !$$

In addition to the δ fn, there is another special beast we'll use

(K)

$\varepsilon_{ijk} \equiv 3^{rd}$ order alternating tensor

We use this in computing the cross-product

$$\varepsilon_{ijk} = \begin{cases} 0 & i=j,\ j=k,\ \text{or } i=k \\ 1 & i,j,k \text{ cyclic} \\ -1 & i,j,k \text{ counter-cyclic} \end{cases}$$

Thus:

$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$

$\varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = -1$

These are the only non-zero elements!

The cross-product is:

$\underset{\sim}{A} \times \underset{\sim}{B} = \underset{\sim}{C}$ is

$c_i = \varepsilon_{ijk}A_jB_k$

(L)

Note that just as

$\underset{\sim}{A} \times \underset{\sim}{B} = -\underset{\sim}{B} \times \underset{\sim}{A}$

In index notation we have

$\varepsilon_{ijk} = -\varepsilon_{jik}$

switching order throws in a (-)!

If ε_{ijk} is cyclic, ε_{jik} must be counter-cyclic & vice versa.

Technically, any matrix for which $A_{ij} = A_{ji}$ is termed symmetric

A matrix for which $B_{ij} = -B_{ji}$ is anti-symmetric

Note: The double dot product (e.g. $A_{ij}B_{ij}$ - no unrepeated indices) of a symmetric & an anti-symmetric

(M)

matrix is zero

$A_{ij}B_{ij} = A_{ji}B_{ij}$ if $\underset{\approx}{A}^T = \underset{\approx}{A}$

$= -A_{ji}B_{ji}$ if $\underset{\approx}{B}^T = -\underset{\approx}{B}$

$\equiv -A_{ij}B_{ij}$ (relabling repeated indices)

Thus since $A_{ij}B_{ij} = -A_{ij}B_{ij}$, both are zero!

We can use this to prove that

$\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla}\phi) = 0$:

$$\varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial x_k}\right) = \underbrace{\varepsilon_{ijk}}_{\text{anti-Symmetric}}\underbrace{\frac{\partial^2}{\partial x_j \partial x_k}}_{\text{symmetric}}(\phi)$$

∴ = 0 !

Ⓝ

Another useful concept is isotropy
Mathematically, a tensor is isotropic if it is invariant under rotation of the coordinate system
Physically, it's isotropic if it looks the same from all directions!

A sphere is isotropic, a football isn't!

All scalars are isotropic
No vectors are isotropic!
The most general 2nd order isotropic tensor is $\lambda \delta_{ij}$
$\hookrightarrow$ const. scala

The most general 3rd order tensor is $\lambda \varepsilon_{ij}$

Ⓞ

The most general 4th order isotropic tensor is:

$$A_{ijkl} = \lambda_1 \delta_{ij}\delta_{kl} + \lambda_2 \delta_{ik}\delta_{jl} + \lambda_3 \delta_{il}\delta_{jk}$$

where $\lambda_1, \lambda_2, \lambda_3$ are scalars

We can use this to prove vector calculus identities

From texts, we have

$$\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{u}) = \underset{\sim}{\nabla} (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) - \nabla^2 \underset{\sim}{u}$$

Let's prove this!

$$\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{u}) \equiv \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left(\varepsilon_{klm} \frac{\partial u_m}{\partial x_l} \right)$$

Note the order of the indices. This is important when working with ε_{ijk}!

Ⓟ

Thus:

$$\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{u}) \equiv \varepsilon_{ijk} \varepsilon_{klm} \frac{\partial^2 u_m}{\partial x_j \partial x_l}$$

What's $\varepsilon_{ijk}\varepsilon_{klm}$??
4 unrepeated indices, so it's a 4th order tensor.
ε_{ijk} is isotropic, so the product is also isotropic

Hence:

$$\varepsilon_{ijk}\varepsilon_{klm} = \lambda_1 \delta_{ij}\delta_{lm} + \lambda_2 \delta_{il}\delta_{jm} + \lambda_3 \delta_{im}\delta_{jl}$$

where λ_1, λ_2 & λ_3 are to be determined

We can calculate these by multiplying both sides by each of the three terms on the RHS (one at a time!) which then yields three eq'ns for the three λ's.

Ⓠ

So:

$$\varepsilon_{ijk} \varepsilon_{klm} \delta_{ij}\delta_{lm} \equiv \underset{0}{\cancel{\varepsilon_{iik}}} \underset{0}{\cancel{\varepsilon_{kll}}}$$

This is zero because ε_{iik} is zero for all i, k. Also, $\varepsilon_{ijk}\delta_{ij} \equiv 0$ because ε_{ijk} is anti-symmetric and δ_{ij} is symmetric, so the double-dot product of the two is likewise zero!

Now for the RHS:

$$(\lambda_1 \delta_{ij}\delta_{lm} + \lambda_2 \delta_{il}\delta_{jm} + \lambda_3 \delta_{im}\delta_{jl}) \times (\delta_{ij}\delta_{lm}) = \lambda_1 \delta_{ii}\delta_{ll} + \lambda_2 \delta_{im}\delta_{im} + \lambda_3 \delta_{im}\delta_{im}$$

$$= \lambda_1 (3)(3) + \lambda_2 (3) + \lambda_3 (3)$$

since $\delta_{ii} = 1 + 1 + 1 = 3$!

Ⓡ

we thus get the first equation:

$$0 = 9\lambda_1 + 3\lambda_2 + 3\lambda_3$$

Now for the second term. We multiply both sides by $\delta_{i\ell}\delta_{jm}$.

we get:

$$\varepsilon_{ijk}\varepsilon_{k\ell m}\delta_{i\ell}\delta_{jm} = 3\lambda_1 + 9\lambda_2 + 3\lambda_3$$

where the RHS was calculated the same way as before.

The LHS is:

$$\varepsilon_{ijk}\varepsilon_{kij}$$

Now if ε_{ijk} is <u>cyclic</u>, so is ε_{kij}. Likewise, if ε_{ijk} is counter-cyclic, so is ε_{kij}. Thus, the product is just $(1)(1) = 1$ or $(-1)(-1) = 1$ for all six non-zero elements!

Ⓢ

This yields:

$$6 = 3\lambda_1 + 9\lambda_2 + 3\lambda_3$$

Likewise, the multiplication by the last term yields:

$$\varepsilon_{ijk}\varepsilon_{k\ell m}\delta_{im}\delta_{j\ell} = 3\lambda_1 + 3\lambda_2 + 9\lambda_3$$

$$= \varepsilon_{ijk}\varepsilon_{kji} = -6$$

These equations have the solution

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = -1$$

Thus:

$$\varepsilon_{ijk}\varepsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$$

and hence:

$$\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{u}) \equiv \varepsilon_{ijk}\varepsilon_{k\ell m}\frac{\partial^2 u_m}{\partial x_j \partial x_\ell}$$

$$= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\frac{\partial^2 u_m}{\partial x_j \partial x_\ell}$$

Ⓣ

$$= \delta_{i\ell}\delta_{jm}\frac{\partial^2 u_m}{\partial x_j \partial x_\ell} - \delta_{im}\delta_{j\ell}\frac{\partial^2 u_m}{\partial x_j \partial x_\ell}$$

$$= \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) - \frac{\partial^2 u_i}{\partial x_j^2}$$

$$\equiv \underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) - \nabla^2 \underset{\sim}{u}$$

which completes the identity!

The last concept we wish to explore is the difference between pseudo-tensors and physical tensors. This distinction arises from the <u>choice</u> of right handed or left-handed coordinate systems. A pseudotensor is one whose <u>sign</u> depends on this choice, a physical tensor is one which doesn't!

Ⓤ

Let's look at some examples:

<u>physical</u>	<u>pseudo</u>
velocity	angular velocity
force	torque
stress	vorticity
δ_{ij}	ε_{ijk}

We go from one to the other via the cross-product!

The vorticity is defined as:

$$\omega_i \equiv \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} \quad (\text{e.g. } \underset{\sim}{\omega} = \underset{\sim}{\nabla} \times \underset{\sim}{u})$$

ω_i is a pseudovector

u_k is a physical vector

Likewise,

$\underset{\sim}{\nabla} \times \underset{\sim}{\omega} \equiv \varepsilon_{ijk}\frac{\partial \omega_k}{\partial x_j}$ is a <u>physical vector</u>. In fact, our vector

(V)

identity yields

$$\underset{\sim}{\nabla} \times \underset{\sim}{\omega} \equiv \varepsilon_{ijk} \frac{\partial \omega_k}{\partial x_j} = \varepsilon_{ijk} \frac{\partial}{\partial x_j}\left(\varepsilon_{klm} \frac{\partial u_m}{\partial x_l}\right)$$

$$= \varepsilon_{ijk}\, \varepsilon_{klm} \frac{\partial^2 u_m}{\partial x_j \partial x_l}$$

$$= \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) - \frac{\partial^2 u_i}{\partial x_j^{\,2}}$$

which is a physical vector

The reason why we make this distinction is that a physical tensor and a pseudotensor can *never* be equal!

How can we use this? Consider the following problem. Suppose we have a body of revolution whose orientation is specified by the unit vector p_i, e.g.

(W)

It's settling under gravity with a net force $\underset{\sim}{F}$ (physical vector). At very low Re, how does its angular velocity (pseudovector) $\underset{\sim}{\Omega}$ depend on $\underset{\sim}{p}$??

At low Re, we can show that $\underset{\sim}{\Omega}$ is *proportional* to $\underset{\sim}{F}$

Thus $\Omega_i = A_{ik} F_k$ where A_{ik} *must* be a pseudotensor which depends *only* on $\underset{\sim}{p}$ and the object's shape!

(X)

There is *only* one way to do this!!

$$A_{ik} = \lambda\, \varepsilon_{ijk}\, p_j$$

where λ is some scalar!

Thus $\Omega_i = \lambda\, \varepsilon_{ijk}\, p_j F_k$ and a *single* experiment can determine λ, which is *constant* for all orientations!

Likewise, if the object has fore-and-aft symmetry (e.g., a football, which looks the same for $\underset{\sim}{p}$ and $-\underset{\sim}{p}$ orientations) we have that A_{ik} *must* be an *even* function of $\underset{\sim}{p}$. Since the *only possible* form of A_{ik} is odd in $\underset{\sim}{p}$, λ *must be zero* for such objects!

(Y)

Thus, in example, rods (fore-aft symmetric cylinders) don't rotate when settling at low Re, *regardless* of orientation.

We can also look at the settling velocity U_i (*physical* vector) for some $\underset{\sim}{F}$:

$$U_i = B_{ij} F_j$$

here B_{ij} is a *physical tensor* which depends on $\underset{\sim}{p}$. The most general form is:

$$B_{ij} = \lambda_1 \delta_{ij} + \lambda_2 p_i p_j$$

②

Thus:

$$U_i = (\lambda_1 \delta_{ij} + \lambda_2 p_i p_j) F_j$$

where λ_1 & λ_2 must be determined from experiment or (nasty) calculation. Actually, by measuring the settling velocity of a rod broadside on and end on, you can get λ_1 & λ_2, allowing you to calculate $\underset{\sim}{U}$ for all orientations - including the lateral velocity for inclined rods! We'll do this experiment later this semester.