PROBLEM SET 11 CBE30355 DUE 12/1/22

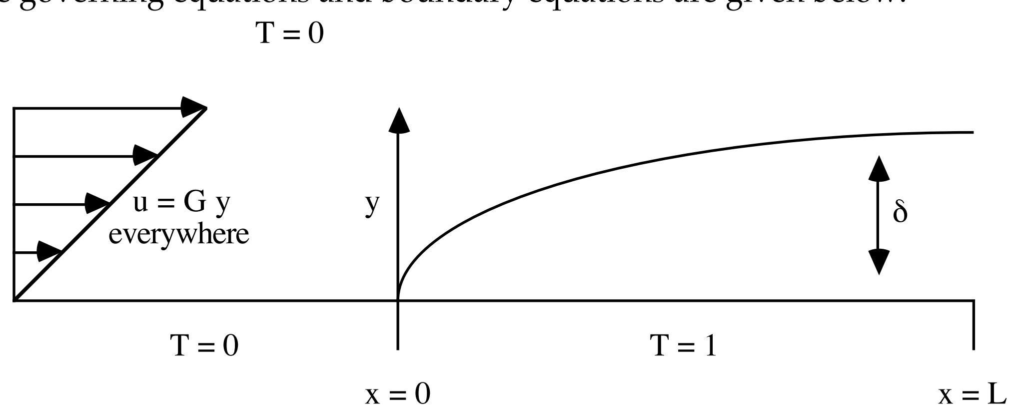
1. Last week you scaled the problem of shear flow past a heated plate. This time you will solve a somewhat simpler version of it!

a). By using the coordinate stretching technique illustrated in class, show that the boundary layer problem described below admits a similarity solution and obtain the similarity rule and similarity variable. Obtain the transformed ODE and boundary conditions. How does the thickness of the thermal boundary layer grow as it moves down the plate?

b). Solve the ODE. Note that $f''/f' = (\ln(f'))'$. You may leave the final result in terms of an explicit integral of a known function, or you may evaluate the integral in terms of the incomplete gamma function (you can look it up in a handbook, or online). Obtain a similar explicit relationship for the heat loss from the plate as a function of the length of the plate. Note that nearly all aspects of the solution except the final numerical value may be learned without explicitly solving the equation.

c). Evaluating the integral above, get the O(1) constant $f'(0)$, and thus the numerical value of the heat loss. You can solve the whole differential equation numerically as discussed in class, evaluate the integral from part b numerically or via mathematica, or evaluate the gamma function integral. All methods work fine, just say how you got your answer!

Recall that the problem was flow past the flat plate y=0 as depicted below. Fluid with a temperature T = 0 flows along the plane in the x direction with velocity u = Gy where G is the shear rate (e.g., plane Couette flow without the upper plane). The plate is maintained at a temperature T = 0 for x < 0 and a dimensionless temperature T = 1 for x > 0. The governing equations and boundary equations are given below.



$$G y \frac{\partial T}{\partial x} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

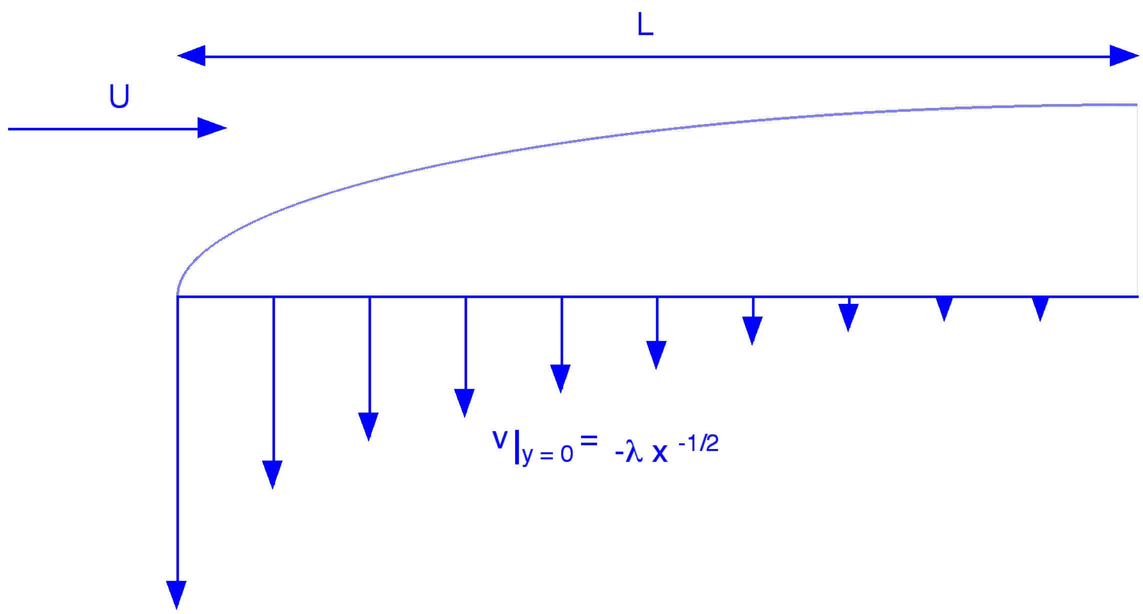
$$T\Big|_{y=0,\, x<0} = 0 \qquad T\Big|_{y>>0} = 0 \qquad T\Big|_{y=0,\, x>0} = 1 \qquad Q = \int_0^L -k \frac{\partial T}{\partial y}\, dx$$

2. Boundary layer growth with suction: One technique used to control the rate of boundary layer growth on airplane wings is suction -- the wing (or plate) is porous, and fluid is sucked out of tiny holes which has the effect of keeping the boundary layer attached and preventing separation. In this problem we will examine the simple case of uniform flow past a flat plate where the vertical suction velocity is given by the power-law relation:

$$v\big|_{y=0} = -\lambda x^{-1/2}$$

a. What should be the characteristic magnitude of λ to affect the boundary layer thickness (e.g., how should it scale with U, μ, ρ, L, etc.) and what should be the magnitude of the total amount of gas withdrawal (the integral of v over the plate)?

b. Solve for the dimensionless displacement thickness and wall shear stress ($f''(0)$) as a function of λ^* (e.g., λ divided by its characteristic scaling) and plot it up. Note that this will require a numerical solution to the Blasius Equation - where your boundary condition $f(0) = 0$ is replaced by one which involves λ^*.



3. An important experimental geometry in electrochemistry is the spinning disk electrode, in which a disk is spun rapidly in a fluid. The diffusion of momentum away from the surface of a disk imparts a centrifugal force which throws the fluid out radially, drawing fluid in axially. All three components of the velocity are non-zero (the coriolis force matters too), however some components are larger than others. This problem is essentially the same as the tea leaf problem you looked at for homework last week, only now you are actually going to solve it!

a. Consider a disk of radius R spinning with angular velocity Ω in an infinite fluid at rest. By scaling the r and θ momentum equations and the continuity equation, estimate the boundary layer thickness and characteristic radial and vertical velocities as a function of the parameters in the problem.

b. We are primarily interested in the total flow rate of fluid into and out of the boundary layer. This is just the vertical velocity times the area of the disk! Based on your scalings, estimate the flow rate for a 5cm radius disk spinning in water at an angular velocity of 100 radians/s. What is the characteristic boundary layer thickness for these values?

c. As could be demonstrated from either the scaling process or via simple affine stretching, this problem admits the classic von Karman similarity solution: that $u_\theta^* = r^* f(z^*)$, $u_r^* = r^* g(z^*)$, and $u_z^* = h(z^*)$. Plug these into the r and θ momentum equations and the continuity equation to obtain a set of three ODE's for f, g, & h. Determine the five boundary conditions.

d. By breaking this problem into a set of five coupled first order ODE's such as described in class it is pretty straightforward to solve the problem numerically using ode23.m (or your favorite integrator) and fsolve.m (or your favorite multidimensional root finder) to determine the two unknown initial conditions via the shooting method. Do this and plot up f, g, and h.

(Note: you can't directly integrate "to infinity" for this problem due to stability issues: the integration to large z^* is *very* strongly dependent on the unknown initial conditions and both the integrator and root finder tend to hang up. An easy way of fixing this is to iteratively increase your limit of integration (starting at some small initial value, such as 1) until the problem converges, using the last best fit values at the current limit as the initial guess for the next larger limit of integration. An example of such a code (and a bit fancier than required here) can be found in the senior lab experiment near the bottom of the page:

http://www.nd.edu/~dtl/cheg459/pivexperiment/)

e. You are primarily interested in the flow rate of fluid into the boundary layer. This is just the limit of u_z^* at large z^* times its scaling and the disk area. What is this value from your numerical solution? How much does the exact flow rate deviate from the answer you got for part b, where you took the unknown O(1) constant to be one?

What you should learn from these problems:

Problem 1: Heat Loss from a plate in shear flow.

a. This problem, as well as the next two, demonstrate the use of *similarity transforms* to convert PDE's to ODE's. Lots of practice using Morgan's Theorem and affine stretching!
b. Practice actually "getting the number" - the O(1) unknown constant left over after scaling.
c. Practice analytically solving a linear second order ODE, this time yielding a function a bit more complicated than a parabola… It is interesting to note the asymptotic form of the solution to this sort of problem. The solutions decay away to zero exponentially as you move away from the plate, typical for boundary layer problems. That's why "infinity" doesn't have to be all that large numerically: it's a fairly abrupt transition as you move out of the boundary layer.

Problem 2: Flow past a flat plate with suction.

a. This is a nice scaling problem, demonstrating that you don't actually have to suck all that much fluid out of a boundary layer to affect its size. This is one of the three simple approaches to controlling boundary layer growth and delaying separation, the others being injection (why flaps on wings have small gaps between wing elements), and vortex generation (promoting turbulence in boundary layers, also delaying separation). Suction has been used, but not much: imagine what would happen to a plane if the holes get plugged!
b. Practice breaking a problem into a set of first order ODE's and solving them numerically using the shooting method.

Problem 3: Rotating Disk Flow:

a. This is the classic von Karman similarity problem. The similarity transform used here actually works even for parallel plate flow, although then the pressure gradient in the radial direction wouldn't be zero (instead you would have a no net radial flow condition that fixes the radial pressure gradient).
b. A bit more practice solving a problem numerically, somewhat more involved than the flat plate problem.
c. Once again demonstrating that the scaling (the easiest part) provides an excellent estimate of the desired quantity (total flux into the boundary layer and boundary layer thickness).