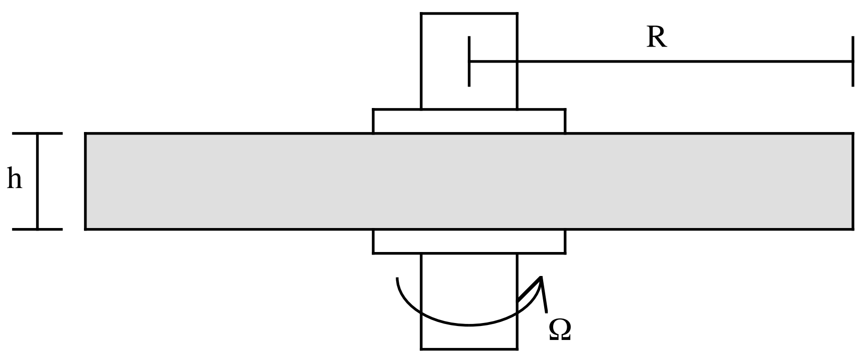
PROBLEM SET 6 CBE 30355 DUE 10/10/24

1. Consider the parallel plate viscometer depicted below. The viscometer consists of two parallel disks of radius R separated by a gap h. In operation, the gap between the disks is filled with a viscous fluid and the lower plate is rotated with some angular velocity Ω, resulting in some torque on the upper plate. The gap width is quite small ($h/R \ll 1$), so the fluid is confined to the space between the plates by surface tension. The ratio of the torque to the angular velocity is proportional to the fluid viscosity (at least for Newtonian fluids).

a. If we may neglect the non-linear inertia terms (i.e., low Reynolds number flow), show that the equations of motion are satisfied by a velocity $u_\theta = f(r,z)$ with u_r, $u_z = 0$. Determine the velocity profile and calculate the torque on the upper plate as a function of the experimental parameters.

b. By examining the equations of motion in the r and z directions, demonstrate that the above solution <u>will not</u> satisfy the full Navier-Stokes equations. Identify which terms give rise to difficulties and provide a short physical explanation for what is occurring. Sketch the velocity profile you expect to see in the r-z plane (Don't try to solve for this secondary current velocity profile unless you like <u>a lot</u> of extra work). In our lab we have used this secondary current for all sorts of experiments.



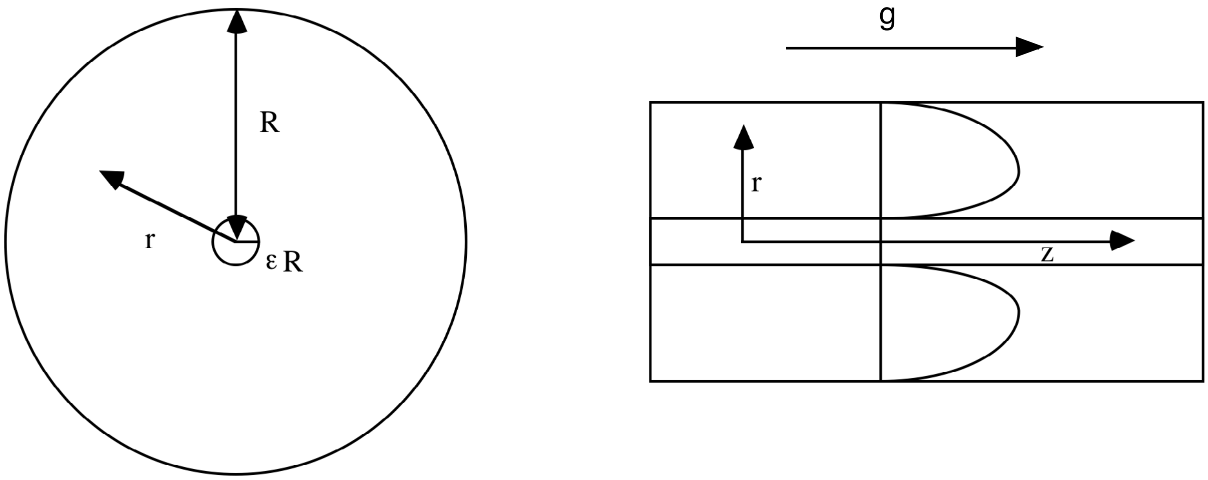
2. In class last Tuesday we discussed the flow of a fluid through a pipe driven by a pressure gradient. In this problem, consider a pipe of radius R with a small cylindrical wire of radius εR running axially down the center. If the remaining space in the tube is filled with a viscous fluid of density ρ and viscosity μ and there is no pressure gradient (the tube is open at both ends), calculate the resulting flow rate due to gravity as a function of ε and compare its magnitude to that when the wire is absent. The velocity profile needs to be obtained analytically, and isn't too bad, but the flow rate gets a bit messy. It *really really* helps to do this in dimensionless form! You may do the flow rate numerically (plotting it up as a function of ε) if you wish, or get it using Wolfram Alpha.

a. What is the flow rate as a function of ε? By how much is the flow rate reduced if $\varepsilon = 0.05$? Is this surprising? (Note: this is actually relevant to the reduction in flow rate/increase in pressure drop when you thread a catheter through an artery or vein.)

b. Determine the force per unit length exerted by the fluid on the wire for this value of ε.

c. What is the ratio of the force on the wire to the force on the outer wall at $r = R$ (again per unit length)?

d. Show that the sum of the force per unit length on the wire and wall calculated above is equal to the weight per unit length of the fluid in the tube.

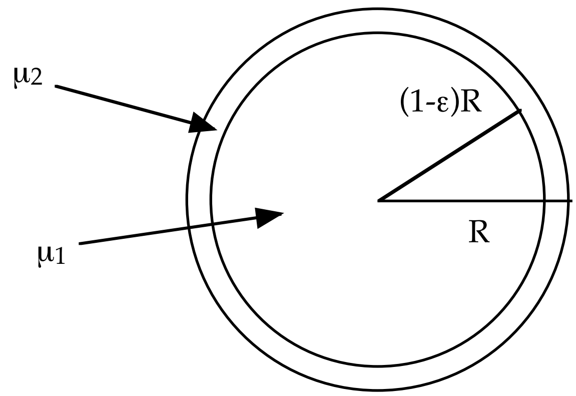


3. Core-Annular flow is an important technique for reducing the pressure drop for pumping a viscous fluid such as a heavy oil. The idea is that the viscous oil is surrounded by a narrow sheath of fluid with much lower viscosity (such as water). Because most of the shear would occur in this narrow region next to the wall, the stress and pressure drop is much lower for a given flow rate. Here we solve this problem.

a. Consider the geometry depicted below. A tube of radius R is filled by two fluids. The inner fluid has a viscosity μ_1, and occupies the center of the tube up to a radius $(1-\varepsilon)R$, while the outer fluid has a viscosity μ_2 and occupies the annulus between this radial position and the wall. Calculate the pressure-drop flow rate relationship for the first fluid (we don't care about the sheath fluid!) as a function of the parameters of the problem.

b. The ratio of the flow rate/pressure drop relationship in core-annular flow to that of the case $\varepsilon = 0$ (e.g., no sheath fluid) is just a dimensionless function of ε and the viscosity ratio $\lambda = \mu_1 / \mu_2$. What is the value of this ratio (e.g., $Q/Q|_{\varepsilon=0}$) for $\varepsilon = 0.01$ and $\lambda = 1000$?

c. Since ε is usually very small, it is actually much easier to solve the problem in the "flat earth limit". Essentially this reduces to the flow of the viscous inner fluid filling the entire pipe, but with the usual no-slip boundary condition at the wall modified to include a wall slip velocity that is proportional to the shear stress at the wall. Determine what this boundary condition looks like and re-solve the problem in the small ε limit. Compare the solution obtained this way to your answer for part b.

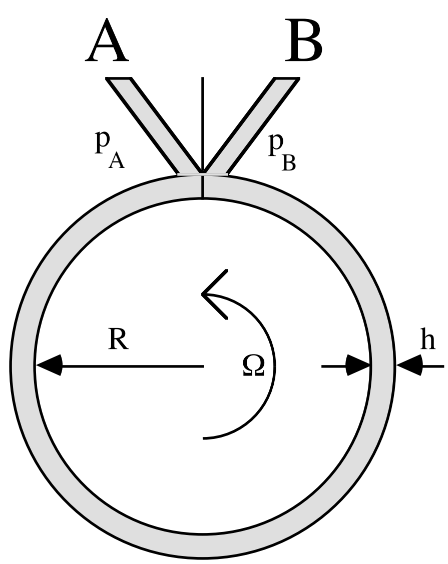


4. A viscosity pump is depicted below. Fluid is pumped from inlet A to outlet B by the rotating drum of radius R. Note that $p_A < p_B$, and that this pressure gradient will induce some backflow from B to A. The gap width h is considered to be much less than R, so that the flow in the gap may be modeled as flow between parallel planes.

a. Neglecting all inertial effects, calculate the flow rate per unit width of this pump Q/W (it is assumed to extend out of the paper a distance W) as a function of Ω, p_B-p_A, R, and h.

b. What is the maximum Δp it can pump against?

c. Calculate the resulting torque on the shaft that drives the drum and the mechanical energy input to the system. If the useful work done on the fluid is given by Q·Δp, what is the energy efficiency of the pump? Where does the extra energy go?



What you should learn from these problems:

Problem 1: This one has three key features:

a. Solving the velocity profile in a geometry that is very important to rheology.
b. *Allowing the boundary conditions to suggest the form of the solution*. Most fluid mechanics problems are rendered *much* easier using this concept.
c. Recognizing that inertial effects can produce secondary flows – which are quite large and significant in this geometry.

Problem 2: This one has several things in it:

a. Solving unidirectional flow in cylindrical coordinates.
b. Showing that small obstructions can have large effects on pressure drop/flow rate relationships.
c. Showing that the forces calculated from detailed velocity distributions *must and do* match what you would get from integral force balances.
d. Practice in rendering a problem dimensionless to simplify calculations.

Problem 3: This problem focuses on two-fluid flows:

a. Solving an industrially relevant problem – at least for a simple model system.
b. Practice rendering a problem dimensionless (unless you want to do lots of extra work!).
c. Recognizing that *velocity and shear stress are continuous at an interface!*
d. Showing that the "flat earth limit" provides a useful and accurate approximation if the separation of length scales is sufficiently large.

Problem 4: The viscosity pump:

a. Demonstrating that, once again, the "flat earth limit" makes a problem a lot simpler to solve!
b. Application of the principle that (for uni-directional flows) you can regard a flow as the linear superposition of two simpler problems. This works for any linear problem, actually, which is one reason they're much easier to solve.
c. Connection between pressure drop, flow rate, and rate of work.