1. Dimensional analysis in cooking a turkey:

a. If you’ve ever looked at the cooking timetables for large things like turkeys or roasts, you may have noticed that they are pretty complex: so much per pound if the weight is in one range, and different length in another range, etc. In general, the cooking time per pound goes down the larger the bird. If we assume that 1) all turkeys are geometrically similar, 2) the physical properties such as thermal diffusivity, density, etc., are all constant, and 3) the cooking time depends only on the mass, the density, and the thermal diffusivity (for idealized turkeys, anyway), use dimensional analysis to show how cooking time should vary with the mass of the bird.

b. Now we values for cooking time values obtained from the website http://www.butterball.com/how-tos/roast-a-turkey. The convection oven times are given by weights:

<table>
<thead>
<tr>
<th>Weight</th>
<th>Cook Time (Unstuffed)</th>
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<tbody>
<tr>
<td>6-10 lbs.</td>
<td>1½ -2 hrs.</td>
</tr>
<tr>
<td>10-18 lbs.</td>
<td>2-2½ hrs.</td>
</tr>
<tr>
<td>18-22 lbs.</td>
<td>2½-3 hrs.</td>
</tr>
<tr>
<td>22-24 lbs.</td>
<td>3-3½ hrs.</td>
</tr>
</tbody>
</table>

Since the weights are in a range, you can get a total of 5 “exact” weights and times. Using regression analysis, fit the data to a power law and determine if the exponent matches that predicted from part a to within the 90% confidence level. Graphically present your results showing the data, the best fit power law, and the “theoretical prediction” based on assuming the exponent obtained in part a.

2. Consider the damped pendulum depicted below. The pendulum consists of a ball of mass m and radius a suspended by a fine wire of length L below a pivot. The ball is moving through a fluid of viscosity μ (the fluid is assumed massless for convenience) which retards its motion. It is assumed that the ball moves sufficiently slowly that Stokes Law applies. Under these conditions, the equation describing the motion of the sphere is given by:

\[ M L \frac{d^2 \theta}{dt^2} = -Mg \sin(\theta) - 6\pi \mu a L \frac{d\theta}{dt} \]

\[ \theta_1 = 0 = \theta_0 ; \quad \frac{d\theta}{dt} \bigg|_{t=0} = 0 \]
a. By choosing an appropriate reference time scale, render the equation and initial condition dimensionless so that the dependent and independent variables are of O(1). Show that the dimensionless solution depends on only two dimensionless groups, one of which is just the initial angular displacement. What is the physical meaning of the other dimensionless group?

b. You are assigned the problem of adjusting the fluid viscosity so as to bring the sphere to rest as rapidly as possible. Recognizing that if $\mu = 0$ the pendulum will oscillate forever, and if $\mu$ is very large the sphere will take a long time to get to the equilibrium ($\theta = 0$) position, estimate the correct value of the viscosity for critical damping. Estimate from dimensional analysis (e.g., assume all unknowns are “O(1)”) how long it will take the sphere to approach to within, say, 25% of the equilibrium position.

c. Given that $\theta_0$ is very small, we may make the approximation $\sin(\theta) = \theta$. Solve the resulting differential equation, and compare the exact solution to that estimated by dimensional analysis above.

3. In an interesting paper (Davies & Stokes, J. Rheol 2005) a fundamental problem with current parallel-plate viscometers was identified. As you know, the viscosity of a fluid is measured in a parallel-plate viscometer by rotating one plate at a known angular velocity and measuring the resultant torque on the other plate. The viscosity calculation, however, requires accurate knowledge of the gap width between the plates. In a modern instrument it is possible to set the gap accurately automatically -provided-you have the reference point of where the two plates are in contact: e.g., where the gap is zero. For an instrument such as this, you just push a button, the upper plate descends at a constant velocity of $100 \mu m/s$, and the gap is considered to be zero when the measured upward thrust on the upper plate exceeds $10^4$ dynes, providing the reference point for all subsequent gaps - very convenient! This would be fine if there were no fluid in the gap (e.g., a vacuum), but as you know the viscosity of air is non-zero ($1.8 \times 10^{-4}$ poise). In this problem we analyze what the effect of air is on the gap zeroing of 3cm radius parallel-plates.

a. For a normal force detection threshold $F$, approach velocity $V$, air viscosity $\mu$, and plate radius $R$, calculate the error in the gap set zero.

b. For the numbers given above, calculate the numerical value of the error in microns. Is it practical to reduce this error by an order of magnitude by decreasing $V$? Be specific!

c. If we set the gap to 250$\mu m$ using this incorrect zero (e.g., actual gap differs from 250$\mu m$ due to the zero error), what is the corresponding error in the viscosity measurement $\mu_{exp}/\mu$ of a fluid?

d. How can you use the output of the instrument to get the correct viscosity? Be specific! (Hint: It will require analysis of a couple of measurements.)
4. Stokes Flow and Index Notation. The Stokes Flow equations are just:

\[ \mu \nabla^2 \mathbf{u} = \nabla \mathbf{p} \]

a. By taking the divergence of the equations of motion and applying the equation of continuity, prove that \( \nabla^2 \mathbf{p} = 0 \) for an incompressible fluid undergoing flow at zero Reynolds number. Use index notation only, and note that the \( \nabla \) and \( \nabla^2 \) operators commute.

b. Complex problems in Stokes Flow can be solved by breaking the solution for the velocity into the sum of a particular solution and a homogeneous solution, e.g.,

\[ \mathbf{u}_i = \mathbf{u}_i^p + \mathbf{u}_i^h \]

where \( \mathbf{u}_i^p \) and \( \mathbf{u}_i^h \) satisfy:

\[ \frac{\partial^2 \mathbf{u}_i^p}{\partial x_j^2} = \frac{1}{\mu} \frac{\partial \mathbf{p}}{\partial x_i} \quad \text{and} \quad \frac{\partial^2 \mathbf{u}_i^h}{\partial x_j^2} = 0 \]

but neither \( \mathbf{u}_i^p \) or \( \mathbf{u}_i^h \) (in general) satisfy the continuity equation individually (only the sum). Normally this wouldn’t do any good, but you can get a general solution for \( \mathbf{u}_i^p \) if you know the pressure! Using the result from part a, prove that this particular solution is just:

\[ \mathbf{u}_i^p = \frac{x_i}{2\mu} \mathbf{p} \]

In the graduate transport class we make extensive use of this to solve complicated creeping flow problems, as harmonic functions (e.g., those for which \( \nabla^2 \phi = 0 \)) are well known, so getting both \( \mathbf{p} \) and \( \mathbf{u}_i^h \) are relatively simple!
What you should learn from these problems:

Problem 1: There are a couple of things you should get from this problem.

a. Practice using the Buckingham P theorem and straight-up dimensional analysis.

b. A bit of practice using graphical comparison, linear regression, error calculation, etc.

Problem 2: This is a scaling problem, and you should:

a. Learn how to render a differential equation and boundary condition dimensionless.

b. Use scalings to “estimate” quantities without actually having to solve a differential equation. Such estimates are often useful for “back of the envelope” calculations that let you quickly recognize what is going on in an experiment or problem you are studying – hopefully in time to determine if things are occurring which are contrary to expectations, which lets you examine it more closely while “things are going on”. This can be a very important time saver in the long run as experiments are expensive!

c. Practice solving a second order ODE.

Problem 3: Lubrication flow between two disks:

a. This makes you work through the notes on defined flow rate lubrication problems.

b. Demonstrating how you can use the results of one problem to provide information on a rather different one (e.g., tying the results together).

c. Recognizing that sometimes “push button instruments” can yield systematic errors due to unanticipated effects, and thinking through how you could fix the results by proper recognition of why the results were in error…

Problem 4: The Stokes Flow Equations:

a. Practice manipulating PDE’s in index notation (pretty much straight math / multivariable calculus).

b. This is a set-up for solving for flow past a sphere using spherical harmonics. We’ll play with this a bit, but I don’t really regard it as “core”. It is the fastest way to get the “6π” factor for the drag on a sphere, though, as well as the starting point for all really complicated Stokes flow problems.