PROBLEM SET 7 CBE 30355 DUE 10/27/22

1. Dimensional analysis in cooking a turkey:

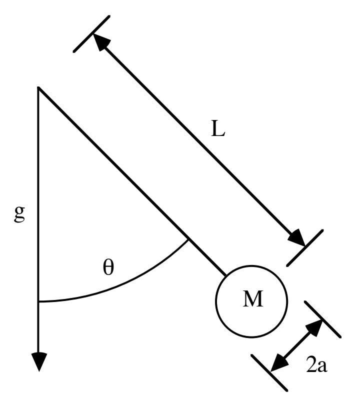
a. If you've ever looked at the cooking timetables for large things like turkeys or roasts, you may have noticed that they are pretty complex: so much per pound if the weight is in one range, and different length in another range, etc. In general, the cooking time per pound goes down the larger the bird. If we assume that 1) all turkeys are geometrically similar, 2) the physical properties such as thermal diffusivity, density, etc., are all constant, and 3) the cooking time depends only on the mass, the density, and the thermal diffusivity (for idealized turkeys, anyway), use dimensional analysis to show how cooking time should vary with the mass of the bird.

b. Now we have values for cooking time values obtained from the website http://www.butterball.com/how-tos/roast-a-turkey. The convection oven times are given by weights:

Weight	Cook Time (Unstuffed)
6-10 lbs.	1½ -2 hrs.
10-18 lbs.	2-2½ hrs.
18-22 lbs.	2½-3 hrs.
22-24 lbs.	3-3½ hrs.

Since the weights are in a range, you can get a total of 5 "exact" weights and times. Using regression analysis, fit the data to a power law and determine if the exponent matches that predicted from part a to within the 90% confidence level. Graphically present your results showing the data, the best fit power law, and the "theoretical prediction" based on assuming the exponent obtained in part a.

2. Consider the damped pendulum depicted below. The pendulum consists of a ball of mass m and radius a suspended by a fine wire of length L below a pivot. The ball is moving through a fluid of viscosity μ (the fluid is assumed massless for convenience) which retards its motion. It is assumed that the ball moves sufficiently slowly that Stokes Law applies. Under these conditions, the equation describing the motion of the sphere is given by:



$$M L \frac{d^2\theta}{dt^2} = - M g \sin(\theta) - 6\pi \mu a L \frac{d\theta}{dt}$$

$$\theta_{t=0} = \theta_0 \; ; \; \frac{d\theta}{dt}_{t=0} = 0$$

a. By choosing an appropriate reference time scale, render the equation and initial condition dimensionless so that the dependent and independent variables are of O(1). Show that the dimensionless solution depends on only two dimensionless groups, one of which is just the initial angular displacement. What is the physical meaning of the other dimensionless group?

b. You are assigned the problem of adjusting the fluid viscosity so as to bring the sphere to rest as rapidly as possible. Recognizing that if $\mu = 0$ the pendulum will oscillate forever, and if μ is very large the sphere will take a long time to get to the equilibrium ($\theta = 0$) position, estimate the correct value of the viscosity for critical damping. Estimate from dimensional analysis (e.g., assume all unknowns are "O(1)") how long it will take the sphere to approach to within, say, 25% of the equilibrium position.

c. Given that θ_o is very small, we may make the approximation $\sin(\theta) = \theta$. Solve the resulting constant coefficient differential equation, and compare the exact solution to that estimated by dimensional analysis above.

3. Mixing two fluids of different density in a very large tank can be quite challenging: the denser fluid tends to stay on the bottom, and it is hard to get it to mix in. You are in charge of developing a scale model to simulate a mixer for the Hanford cleanup problem. The full-scale system has a tank that is 10m high, and the two fluids to be mixed have a density of 1.4 g/cm^3 and 1.05 g/cm^3, and a viscosity of 4 cp and 2 cp, respectively. You have constructed a 1:8 scale model to test mixing strategies.

a. If you were to preserve strict dynamic similarity, what should be the densities and viscosities of the fluids in the model system, and what should be the scaling factor for the velocities between the model and full-scale systems?

b. You discover that there simply isn't any reasonable way to satisfy the conditions in (a), thus you go to the concept of approximate dynamic similarity. What fluids and scaling factors should you use in this case?

c. For the conditions determined in (b), how would the mixing times scale between the two systems?

4. Stokes Flow and Index Notation. The Stokes Flow equations are just:

$$\mu \nabla^2 \underset{\sim}{u} = \underset{\sim}{\nabla} p$$

a. By taking the divergence of the equations of motion and applying the equation of continuity, prove that $\nabla^2 p = 0$ for an incompressible fluid undergoing flow at zero Reynolds number. Use index notation only, and note that the ∇ and ∇^2 operators commute.

b. Complex problems in Stokes Flow can be solved by breaking the solution for the velocity into the sum of a particular solution and a homogeneous solution, e.g.,

$$u_i = u_i^p + u_i^h$$

where u_i^p and u_i^h satisfy:

$$\frac{\partial^2 u_i^p}{\partial x_j^2} = \frac{1}{\mu}\frac{\partial p}{\partial x_i} \quad \text{and} \quad \frac{\partial^2 u_i^h}{\partial x_j^2} = 0$$

but neither u_i^p or u_i^h (in general) satisfy the continuity equation individually (only the sum). Normally this wouldn't do any good, but you can get a general solution for u_i^p if you know the pressure! Using the result from part a, prove that this particular solution is just:

$$u_i^p = \frac{x_i}{2\mu} p$$

In the graduate transport class we make extensive use of this to solve complicated creeping flow problems, as harmonic functions (e.g., those for which $\nabla^2\phi = 0$) are well known, so getting both p and u_i^h are relatively simple!

What you should learn from these problems:

Problem 1: There are a couple of things you should get from this problem.

a. Practice using the Buckingham Π theorem and straight-up dimensional analysis.
b. A bit of practice using graphical comparison, linear regression, error calculation, etc.

Problem 2: This is a scaling problem, and you should:

a. Learn how to render a differential equation and boundary condition dimensionless.
b. Use scalings to "estimate" quantities without actually having to solve a differential equation. Such estimates are often useful for "back of the envelope" calculations that let you quickly recognize what is going on in an experiment or problem you are studying – hopefully in time to determine if things are occurring which are contrary to expectations, which lets you examine it more closely while "things are going on". This can be a *very important* time saver in the long run as experiments are expensive!
c. Practice solving a second order ODE.

Problem 3: There are several things you should get from this problem.

a. Practice using the relationships for strict dynamic similarity.
b. Recognition that strict similarity is (usually) not practical.
c. Coming up with a "best choice" scale model for approximate dynamic similarity.
d. Figuring out how to scale things (like mixing time) based on the other scaling relationships.

Problem 4: The Stokes Flow Equations:

a. Practice manipulating PDE's in index notation (pretty much straight math / multivariable calculus).
b. This is a set-up for solving for flow past a sphere using spherical harmonics. We'll play with this a bit, but I don't really regard it as "core". It *is* the fastest way to get the "6π" factor for the drag on a sphere, though, as well as the starting point for all really complicated Stokes flow problems.