1. Flow inside a 90° trough. Consider the flow depicted below. A trough with an internal angle of 90° (e.g., the walls are at $\theta = \pm\theta_0 = \pm\pi/4$) is leaking out through a slit in the center with flow rate per unit extension into the paper Q/W. We want to solve for the velocity distribution. If we can ignore the inertial terms (Re << 1), the flow is governed by the Biharmonic Equation.

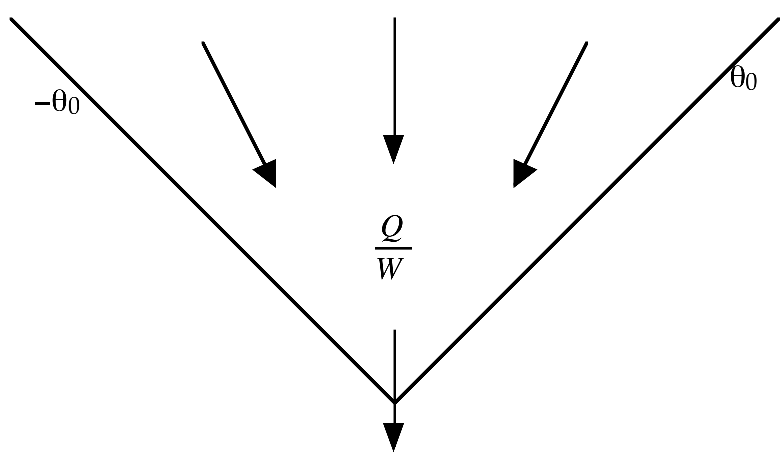
a. Write down the boundary conditions which govern the streamfunction for this problem.

b. We anticipate that the streamfunction will be of the form $\psi = \frac{Q}{W} r^{\lambda} f_{\lambda}(\theta)$. From the boundary conditions, what must λ be?

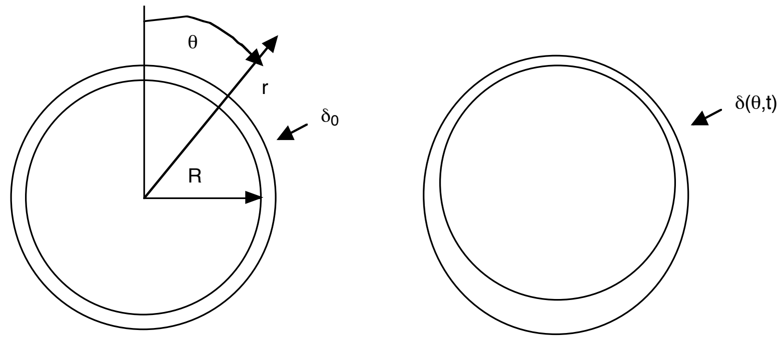
c. Solve for $f_{\lambda}(\theta)$. It is helpful to make maximum use of symmetry!

d. What is the radial velocity as a function of r and θ? Is there any flow in the θ direction?

Hint: The four homogeneous solutions for f(θ) are 1, θ, sin(2θ), and cos(2θ).



2. Film Drainage Flows: In this problem we examine the problem of drainage from a cylinder of radius R coated with a layer of fluid of density ρ and viscosity μ, and with initial thickness δ_0<<R. We wish to determine the evolution of the thickness of the layer as a function of θ and t.



a. Redraw your coordinates for some value of θ in the flat earth limit (e.g., Cartesian coordinates!) and solve for the velocity profile in the draining film. Remember that g is now a function of θ! This should otherwise be identical to the falling film problem we solved in class. Note that $\delta \neq \delta_0$ as the film drainage evolves, rather that is just its initial condition!

b. Recognizing that the time derivative of the film thickness is just the radial velocity (or y velocity in your "flat earth" coordinate system) evaluated at δ(θ,t), use the continuity equation to determine the timescale of the drainage problem t_c.

c. Now integrate the continuity equation to obtain the dimensionless partial differential equation that the film thickness must obey.

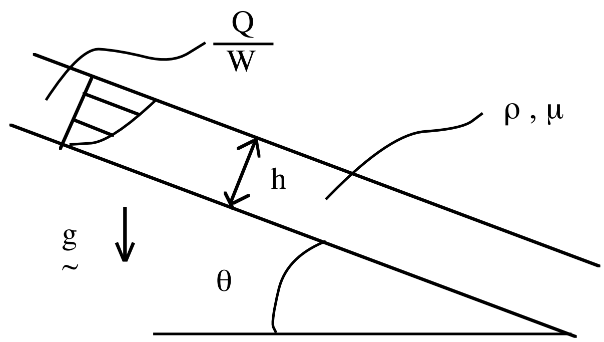
d. Solve for the time for the first drip to form (e.g., the solution blows up) at θ = π (the bottom of the cylinder). Note that the sin(θ) term disappears at θ = π, thus at the bottom (and at the top) the thickness is just a function of time.

3. Dimensional Analysis of Flow Down an Inclined Plane: Consider the inclined plane depicted below. The plane is inclined by an angle θ from the horizontal, and a fluid (viscosity μ, density ρ) is flowing down the plane with flow rate Q/W per unit width of the plane (normal to the plane of the paper - this is a two-dimensional problem). We wish to determine the thickness of this fluid layer as a function of the parameters of the problem using dimensional analysis.

a. Form a dimensional matrix and prove that the problem involves just three dimensionless groups. Determine three independent dimensionless groups (Hint: one will be the Reynolds number).

b. At low Reynolds numbers we anticipate that the flow rate will be proportional to gravity. Use this to strengthen the result of the dimensional analysis, and determine the relationship between the flow rate and fluid thickness to within some unknown function of the angle of inclination. The thickness is proportional to what power of Q/W?

c. At very high Reynolds numbers we expect the flow to be fully turbulent and to no longer depend on the viscosity. Use this to determine a new relationship between the thickness and the flow rate in the high Re limit.



4). The sliding block:

a. For the sliding block considered in class with dimensions L = 20cm, W = 30cm, and d_2-d_1 = 0.01cm (e.g., a fixed inclination angle), determine the velocity U at which the block can support a weight of 100kg while maintaining a minimum separation d_1 = 0.01cm. Take the fluid viscosity to be 1 poise.

b. Calculate the drag on the block when it is fully lubricated at the conditions specified above (e.g., the lubrication equations apply!) and compare this to the drag when the lubrication fails (e.g., the drag is solely due to friction). Assume a coefficient of friction of 0.11, a typical value. Hint: it is *much* easier to determine the lubrication drag by calculating the shear stress on the plate – if you evaluate it on the block, you would also have to consider the contribution of pressure forces due to the slight incline angle of the block. Doing it right yields the same result either way, though.

c. The dimensionless lift is solely a function of the ratio $\Delta d/d_1$, as derived in class. We know that it has an asymptotic value of 0.5 $\Delta d/d_1$, but there should also be a maximum! Plot up the dimensionless lift as a function of this ratio, compare it to the asymptotic relationship, and determine the optimum value of $\Delta d/d_1$ as well as the maximum lift. Use the "grid on" and "axis" commands to make the plot clear.

What you should learn from these problems:

Problem 1: Streamfunction: Trough drainage.

a. This problem gives you practice in using the streamfunction in cylindrical coordinates.
b. Demonstration that allowing the boundary conditions to suggest the form of the solution results in a dramatic simplification of the solution procedure.
c. Introduction of the general separable solution for the streamfunction in cylindrical geometries.

Problem 2: This is a nice example of film drainage, relevant to many coating problems.

a. Demonstration that the "parabolic velocity distribution" can lead to complex behavior in the presence of a free surface.
b. Practice simplifying problems using (once again) the flat earth limit.
c. Analytic solution to a blow-up problem (the drip on the bottom).

Problem 3: Dimensional Analysis:

a. Here you use dimensional analysis to determine the dimensional groups a problem depends on.
b. Practice using additional physical insight to get much stronger dimensional analysis results, estimates of the film thickness in two very dissimilar flow regimes.

Problem 4: This is an application of the Reynolds Lubrication Equation (defined pressure BC's):

a. Fish out the solution from the notes and apply them with numbers.
b. Calculate the drag and compare to frictional resistance: it should be -much- smaller!
c. Demonstrate that the "unknown O(1) constant" left over from scaling isn't always all that close to 1 - but scaling still gets you most of the way there!