Linear Stability Theory: a bomb!

In linear stability theory we examine the exponential growth or decay of an infinitesimal disturbance.

We begin with the <u>base state</u>: a solution (often steady) to the equations whose stability we are examining.

We perturb the base state with a <u>normal mode</u> disturbance of $O(\varepsilon)$ (e.g., really small).

We collect terms of $O(\varepsilon)$ and figure out whether the perturbation grows or decays in time.

If <u>all</u> modes decay, the base state is stable

If <u>any</u> mode grows exponentially in time the base state is <u>unstable</u>.

The shape factor of this (first) growing mode is the <u>most unstable mode</u>.

Let's apply this to a bomb: an auto catalytic reaction!

For a sphere, we have:

$$\frac{\partial n}{\partial t} = D \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial n}{\partial r}\right) + \alpha n$$

where neutrons of dimensionless conc. n diffuse w/ mass dif. D and reproduce via fission at a rate αn (per vol * time).

We have B.C.'s: $n\big|_{r=0}$ = finite

$n\big|_{r=R} = 0$ (all escape, no reflection)

Let's render dimensionless:

$r^* = r/R \qquad t^* = t/t_c$ (n is dimensionless)

$$\therefore \frac{1}{t_c}\frac{\partial n}{\partial t^*} = \frac{D}{R^2}\frac{1}{r^{*2}}\frac{\partial}{\partial r^*}\left(r^{*2}\frac{\partial n}{\partial r^*}\right) + \alpha n$$

Divide out:

$$\frac{\partial n}{\partial t^*} = \left[\frac{t_c D}{R^2}\right]\frac{1}{r^{*2}}\frac{\partial}{\partial r^*}\left(r^{*2}\frac{\partial n}{\partial r^*}\right) + \alpha t_c n$$

Let's let $t_c = R^2/D$ (diff^n time)

and Define $\alpha^* = \alpha t_c = \frac{\alpha R^2}{D}$

So:
$$\frac{\partial n}{\partial t^*} = \frac{1}{r^{*2}}\frac{\partial}{\partial r^*}\left(r^{*2}\frac{\partial n}{\partial r^*}\right) + \alpha^* n$$

$$n\Big|_{r^*=0} = \text{finite} \qquad n\Big|_{r^*=1} = 0$$

We seek the stability of the base state

$n_0 = f^n(r^*)$

This is just $n_0 = 0$!

Now we perturb it:

$n = n_0 + \varepsilon n'$

Plug in:
$$\varepsilon\frac{\partial n'}{\partial t^*} = \frac{\varepsilon}{r^{*2}}\frac{\partial}{\partial r^*}\left(r^{*2}\frac{\partial n'}{\partial r^*}\right) + \alpha^*\varepsilon n'$$

Dividing by ε just gives the same eq'n!

It is much more interesting for non-linear eq'ns where you would throw out terms of $O(\varepsilon^2)$ - making the problem linear!

We seek the normal mode disturbance:

$$n' = e^{st^*} f(r^*)$$

This is unstable if the real part of s is positive (it may be complex!)

Plugging in:

$$s e^{st^*} f = e^{st^*} \frac{1}{r^{*2}} \frac{\partial}{\partial r^*}\left(r^{*2} \frac{\partial f}{\partial r^*}\right) + e^{st} \alpha^* f$$

Dividing out and rearranging:

$$\frac{1}{r^{*2}}\left(r^{*2} f'\right)' = (s - \alpha^*) f \equiv -\lambda f$$

$$\text{where } \lambda \equiv \alpha^* - s, \quad s = \alpha^* - \lambda$$

We have the B.C.'s $f(0) = \text{finite}$, $f(1) = 0$

This is a Sturm-Liouville eigenvalue problem. It has non-trivial (e.g. $f \neq 0$) sol'n only for discrete eigenvalues λ!

To solve, we employ a <u>trick</u>

Let $f \equiv \frac{g}{r^*}$

so $f' = -\frac{1}{r^{*2}} g + \frac{g'}{r}$

$r^{*2} f' = -g + r^* g'$

$(r^{*2} f')' = -\cancel{g'} + \cancel{g'} + r^* g''$

and thus $\frac{g''}{r^*} = -\lambda \frac{g}{r^*}$

or $g'' = -\lambda g$!

w/ B.C.'s $g(0) = 0$ (so $f(0)$ is finite)
and $g(1) = 0$

The solutions are just sines & cos!

$$g = A \sin \sqrt{\lambda}\, r^* + B \cos \sqrt{\lambda}\, r^*$$

from $g(0) = 0$ we have $B = 0$

from $g(1) = 0$ we <u>require</u> $\sqrt{\lambda} = n\pi$

where $n = 1, 2, 3 \ldots$

So

$$f = \frac{\sin n\pi r^*}{r^*}$$

and $s = \alpha^* - \lambda = \alpha^* - n^2\pi^2$

We get exponential growth if $s > 0$

The most unstable mode is the $n=1$ case (usually true), so things blow up if

$$\alpha^* > \pi^2$$

In this case, $n' = e^{(\alpha^* - \pi^2)t^*} \frac{\sin \pi r^*}{r^*}$

with a <u>growth rate</u> of $\alpha^* - \pi^2$

This same approach can be used for <u>much</u> more complicated problems!